GRAVITATIONAL COUPLING OF NEUTRINOS TO MATTER VORTICITY: MICROSCOPIC ASYMMETRIES

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ABSTRACT

We examine the gravitational coupling of neutrinos to matter vorticity; in the context of Einstein theory of gravitation and for technical simplicity, we have considered Gödel model as the cosmological background, whose matter content has a non-null vorticity. The presence of a vorticity field of matter generates, via gravitation, microscopic asymmetries in neutrino physics. At the microscopic level, currents are asymmetric along the direction determined by the vorticity field: neutrino (antineutrino) currents are larger along the direction antiparallel (parallel) to the vorticity field. In case of production of pairs under CP violation, a net number asymmetry is generated between neutrinos and antineutrinos.

1. INTRODUCTION

In the present paper our purpose is to examine the effect of matter vorticity to the physics of neutrinos, the coupling of neutrinos to the vorticity field being realized obviously through gravitation, in the context of the General Theory of Relativity. For illustration we recall that the present observed rotation of galaxies and nebulae could be an indication that the rotation of matter was a remarkable feature of earlier eras, in particular played an important role in the dynamics of the primordial universe. In this sense the results of our investigation could have some interesting applications in the realm of Cosmology and theoretical Astrophysics. Also our paper can be considered as a partial contribution to the question of the cosmological effects on the local laws of physics. A complete program along this line has never been accomplished for several reasons. Among them, technical difficulties in developing a consistent quantum field theory on a general curved space-time, and in describing the geometry of the universe in eras when these cosmological effects should be dominant if not essential.

In the context of Einstein theory of gravitation and for operational simplicity, we take Gödel universe[1] as the cosmological background. It is the simplest know solution of Einstein field equations with rotating incoherent matter. The vorticity field of matter is connected to the property that matter rotates with non-zero angular velocity, in the local inertial frames of its comoving observers. The model is stationary, and the existence of a global time-like Killing vector is decisive for constructing invariant energy modes of the neutrino field.

Neutrinos are introduced as perturbation (test fields) over the cosmological background. They are described by spinorial fields which satisfy Dirac's equation on the curved background.

In section 2 we characterize Gödel universe as the Lie group $H^3 \times R$ with a left invariant metric defined on it. This characterization garantees that all vector fields over $H^3 \times R$ exist globally, and that the invariant decomposition into excitation modes of neutrino - provided by the Killing vectors - is globally defined over the manifold. In section 3 the local dynamics of neutrinos is discussed, with its basis in Dirac's equation, obtainnig as result the local precession of the spin of neutrino, and the conservation of helicity. A complete basis of neutrino solutions is obtained (eigenstates of energy and helicity), which satisfy boundary conditions related to the perturbation character of neutrinos. In section 4 we introduce a normalization for neutrino states, and construct a Fourier space wich allow us to describe, in section 5, the local microscopic asymmetry of neutrino emission which appears in the presence of a vorticity field. We also discuss the symmetry groups involved in our description of neutrinos, and discuss asymmetries between neutrino and antineutrino amplitudes, which could appear due to CP violation and could produce a net asymmetry between the number of neutrinos and antineutrinos.

2. THE STRUCTURE OF GÖDEL UNIVERSE AND THE EXCITATION MODES OF NEUTRINO FIELDS

Gödel universe is characterized here as the simply connected Lie group $H^3 \times R$ - modulo identification of points, with a left invariant metric introduced on $H^3 \times R$ and which is solutions of Einstein field equations for a perfect fluid [2]. Since the invariant vector fields and forms of $H^3 \times R$ are globally defined [3], they are used to construct the invariant modes in which we expand neutrino wave functions yielding a com plete set of solutions which exists globally.

Let E_4 be the 4-dim Euclidean space with Cartesian coordinates $a=(a^0, a^1, a^2, a^3)$. We define H^3 as the set of points of E_4 which satisfy.

$$(a^0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2 = 1 \qquad (2.1)$$

For any $a = (a^0, a^1, a^2, a^3)$, $b = (b^0, b^1, b^2, b^3) \in H^3$ we define the multiplication law(4)

$$ab = (a^0b^0 - a^1b^1 + a^2b^2 + a^3b^3,\ a^0b^1 + a^1b^0 - a^2b^3 + a^3b^2,\ b^1a^3 - b^3a^1 + a^0b^2 + a^2b^0,\ b^2a^1 - b^1a^2 + a^0b^3 + a^3b^0) \qquad (2.2)$$

Under (2.2) H^3 becomes a group, acting on itself by left multiplication; namely for a given $v \in H^3$, a left motion of H into itself is expressed

$$a' = v\,a \qquad (2.3)$$

and from (2.2) we have $a' \in H^3$, for all $a \in H^3$. H^3 is simply transitive since for each $a \neq 0$ these exists only one left motion v from $a \in H^3$ to a given $a' \in H^3$.

H^3 acting on itself by left multiplication (2.3) is a Lie group, with the three independent left invariant vector fields(5) on H^3

$$e_1^\mu(a) = (-a^1, a^0, a^3, -a^2)$$

$$e_2^\mu(a) = (a^2, a^3, a^0, a^1) \qquad (2.4)$$

$$e_3^\mu(a) = (a^3, -a^2, -a^1, a^0)$$

They are obtained by an arbitrary left-motion a of the three independent unit vectors (0,1,0,0), (0,0,1,0), (0,0,0,1), which define the infinitesimal tangent space of H^3 at the identity (1,0,0,0).

We have the analogous picture for right-motions of the Lie group H^3 into itself, namely (cf. (2.3))

$$a' = a\,v \qquad (2.5)$$

With the corresponding independent right-invariant vector fields on H^3,

$$\begin{aligned} f_1^\mu &= (-a^1,\ a^0,\ -a^3,\ a^2) \\ f_2^\mu &= (a^2,\ -a^3,\ a^0,\ -a^1) \\ f_3^\mu &= (a^3,\ a^2,\ a^1,\ a^0) \end{aligned} \qquad (2.6)$$

We have obviously that[6]

$$\left[e_i\ ,\ f_j\right] = 0 \qquad (2.7)$$

Bases (2.4) and (2.6), expressed as

$$X_0 = -\frac{1}{2}\, e_1^\mu \frac{\partial}{\partial a^\mu},\quad X_1 = -\frac{1}{2}\, e_3^\mu \frac{\partial}{\partial a^\mu},\quad X_2 = -\frac{1}{2}\, e_2^\mu \frac{\partial}{\partial a^\mu} \qquad (2.8)$$

$$Y_0 = -\frac{1}{2}\, f_1^\mu \frac{\partial}{\partial a^\mu},\quad Y_1 = -\frac{1}{2}\, f_3^\mu \frac{\partial}{\partial a^\mu},\quad Y_2 = -\frac{1}{2}\, f_2^\mu \frac{\partial}{\partial a^\mu} \qquad (2.9)$$

yield the representations of the algebra of H^3

$$\left[X_0, X_1\right] = X_2\ ,\quad \left[X_1, X_2\right] = -X_0\ ,\quad \left[X_2, X_0\right] = X_1 \qquad (2.10)$$

$$\left[Y_0, Y_1\right] = -Y_2\ ,\quad \left[Y_1, Y_2\right] = Y_0\ ,\quad \left[Y_2, Y_0\right] = -Y_1 \qquad (2.11)$$

We introduce on H^3 the coordinate system (t, x^1, x^2) by the substitutions

$$a^0 = \frac{1}{2} e^{-\frac{x^1}{2}} x^2 \cos \frac{t}{2} + \cosh \frac{x^1}{2} \sin \frac{t}{2} ,$$

$$a^1 = -\frac{1}{2} e^{\frac{x^1}{2}} x^2 \sin \frac{t}{2} + \cosh \frac{x^1}{2} \cos \frac{t}{2} ,$$

$$a^2 = \frac{1}{2} e^{-\frac{x^1}{2}} x^2 \sin \frac{t}{2} + \sinh \frac{x^1}{2} \cos \frac{t}{2} ,$$

$$a^3 = - \frac{1}{2} e^{\frac{x^1}{2}} x^2 \cos \frac{t}{2} + \sinh \frac{x^1}{2} \sin \frac{t}{2} ,$$

Where $-\infty < x^1, x^2 < \infty$ and $0 \leq t < 4\pi$, and the left invariant fields (2.8) become

$$X_0 = \frac{\partial}{\partial t}$$

$$X_1 = -\sin t \frac{\partial}{\partial t} + \cos t \frac{\partial}{\partial x^1} + e^{-x^1} \sin t \frac{\partial}{\partial x^2} \qquad (2.12)$$

$$X_2 = -\cos t \frac{\partial}{\partial t} - \sin t \frac{\partial}{\partial x^1} + e^{-x^1} \cos t \frac{\partial}{\partial x^2}$$

with the corresponding invariant 1-forms

$$\sigma^0 = dt + e^{x^1} dx^2$$

$$\sigma^1 = \cos t \, dx^1 + e^{x^1} \sin t \, dx^2 \qquad (2.13)$$

$$\sigma^2 = -\sin t \, dx^1 + e^{x^1} \cos t \, dx^2$$

Taking on R the coordinate x^3, with vector field $X_3 = \frac{\partial}{\partial x^3}$ and dual 1-form $\sigma^3 = dx^3$, the group $H^3 \times R$ can be characterized by the left invariant vector fields (X_0, X_1, X_2, X_3), which satisfy (2.10) and

$$[X_i, X_3] = 0 \quad , \qquad i = 1,2,3 \quad , \tag{2.14}$$

and which are a basis for the vector fields on H^3 X R; correspondingly the invariant dual 1-forms $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ are a basis for the 1-forms on H^3 x R. The manifold H^3 x R is the covering group of the algebra (2.10), (2.14).

Gödel universe is obtained by introducing on H^3 x R the left invariant metric [7]

$$ds^2 = \frac{1}{\omega^2} \left\{ (\sqrt{2}\, \sigma^0)^2 - (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2 \right\} \tag{2.15}$$

where ω is a positive constant. (2.15) is a solution of Einstein field equations [8,9] with cosmological constant Λ, and incoherent matter whose density ρ must satisfy

$$k\rho = \omega^2 = 2\Lambda \tag{2.16}$$

The four velocity of matter is $\partial/\partial t$. To express metric (2.15) in its original form, we use new coordinates $\tilde{x}^2 = \sqrt{2}\, x^2$, $\tilde{t} = \sqrt{2}\, t$.

The model is stationary because (2.15) admits a time-like Killing vector. The velocity field of matter has zero expansion and shear, but a non-null vorticity [10]

$$\Omega = \sqrt{2}\, \omega \frac{\partial}{\partial x^3} \tag{2.17}$$

We finally remark that Gödel universe is locally isometric to (2.15), but concerning connectivity in the large the above model is obtained from Gödel model by identification of certain point sets, namely by identifying the points $(t+4n\pi, x^1, x^2, x^3)$, n = integer. In Gödel universe the geodesic congruence determined by $\partial/\partial t$ are <u>open</u> time-likes lines.

From (2.7), (2.15) we have obviously that Gödel's model admits the five Killing vectors

$$(Y_0, Y_1, Y_2, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial t}) \tag{2.18}$$

(cf. (2.6), (2.9)). All these vector fields are globally defined on the group manifold(3). From (2.18) we select the Killing vector fields(11)

$$\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial t}$$

to construct the invariant modes $\phi_{(i)}$ globally defined by

$$\pounds_{\frac{\partial}{\partial x^2}} \phi_{(2)} = ik_2 \phi_{(2)} \quad , \quad \pounds_{\frac{\partial}{\partial x^3}} \phi_{(3)} = ik_3 \phi_{(3)} \tag{2.19}$$

$$\pounds_{\frac{\partial}{\partial t}} \phi_{(0)} = -i\, \varepsilon\, \phi_{(0)} \tag{2.20}$$

with respective solutions $\phi_{(2)} \sim e^{ik_2 x^2}$, $\phi_{(3)} \sim e^{ik_3 x^3}$;

$\phi_{(0)} \sim e^{-i\varepsilon t}$. $\partial/\partial t$ is a globally defined time-like Killing vector, generates time translations and we interpret (2.20) as the definition of invariant energy modes; as we shall see in section 3, $\partial/\partial t$ actually defines the Hamiltonian operator which describes the local dynamics of neutrinos. We use the invariant modes $\phi_{(i)}$ to separate neutrino equations and to obtain a complete set of solutions for neutrino amplitudes in the modes (ε, k_2, k_3).

3. LOCAL DYNAMICS OF NEUTRINOS AND SOLUTIONS OF DIRAC'S EQUATION

Neutrinos in interaction with gravitational fields are described by spinorial fields in the curved space-time. For a general review of spinors on a Riemannian space-time, see ref.(12). Here we use four-component spinors from the point of view of tetrad formalism. We choose a tetrad field $e^{(A)}_{\alpha}(X)$ such that the line element is expressed[13]

$$ds^2 = \eta_{AB}\,\theta^A\theta^B \tag{3.1}$$

Where $\theta^A = e^{(A)}_{\alpha}dx^{\alpha}$. The definition of the neutrino wave-function ψ in a curved space-time involves two group structures. Its spinor character is defined with respect to the local Lorentz structure (3.1), that is, it provides a spinorial representation of the local Lorentz group

$$\tilde{\theta}^A = L^A{}_B(x)\,\theta^B \tag{3.2}$$

with

$$L^A{}_D(x)\,\eta_{AB}\,L^B{}_F(x) = \eta_{DF} \tag{3.3}$$

These transformations, which can be made independently at each space-time point, leave (3.1) invariant. Under (3.2), (3.3) the spinors ψ transform as

$$\tilde{\psi}(x) = S(x)\psi \tag{3.4}$$

where the 4x4 matrix S(x) must satisfy[14]

$$(L^{-1})^{A}{}_{B}(x)\gamma^{B} = S(x)\gamma^{A}S^{-1}(x) \qquad (3.5)$$

By other hand, spinors ψ transform as scalar functions with respect to general coordinate transformations of the space-time, and thus provide a scalar representation of the isometry group of the space-time.

The Lagrangian for neutrinos is

$$i\sqrt{-g}\ (\overline{\psi}\gamma^{A}\nabla_{A}\psi - \nabla_{A}\psi\gamma^{A}\psi) \qquad (3.6)$$

In the above formalism $\overline{\psi} = \psi^{+}\gamma^{0}$, where γ^{0} is the constant Dirac matrix. The spinor covariant derivatives are given by

$$\nabla_{A}\psi = e^{\alpha}{}_{(A)}\partial_{\alpha}\psi - \Gamma_{A}\psi$$

$$\nabla_{A}\overline{\psi} = e^{\alpha}{}_{(A)}\partial_{\alpha}\overline{\psi} + \overline{\psi}\,\Gamma_{A} \qquad (3.7)$$

where the Fock-Ivanenko coefficients Γ_{A} have the form

$$\Gamma_{A} = -\frac{1}{4}\gamma_{BCA}\gamma^{B}\gamma^{C} \qquad (3.8)$$

The Ricci rotation coefficients γ_{ABC} are defined by

$$\gamma_{ABC} = -\,e^{\alpha}{}_{(A)||\beta}\,e_{\alpha(B)}e^{\beta}{}_{(C)} \qquad (3.9)$$

and Dirac equation for neutrinos coupled to gravitation is expressed

$$\gamma^A \nabla_A \psi = \gamma^A (e^{\alpha}_{(A)} \partial_\alpha - \Gamma_A) \psi = 0 \qquad (3.10)$$

For (2.15) we take

$$\begin{aligned} \theta^0 &= dt + e^{\omega x^1} dx^2 \\ \theta^1 &= dx^1 \\ \theta^2 &= \frac{\sqrt{2}}{2} e^{\omega x^1} dx^2 \\ \theta^3 &= dx^3 \end{aligned} \qquad (3.11)$$

With this choice, we have the Fock-Ivanenko coefficients (3.8) expressed as

$$\begin{aligned} \Gamma_0 &= - \frac{\sqrt{2}}{4} \omega \gamma^1 \gamma^2 \\ \Gamma_1 &= - \frac{\sqrt{2}}{4} \omega \gamma^2 \gamma^0 \\ \Gamma_2 &= - \frac{\sqrt{2}}{4} \omega \gamma^0 \gamma^1 - \frac{\omega}{2} \gamma^1 \gamma^2 \\ \Gamma_3 &= 0 \end{aligned} \qquad (3.12)$$

For a neutrino field in invariant energy excitation modes (2.20), and eigenstate of γ^5,

$$\gamma^5 \psi = L\psi \quad , \quad L^2 = 1$$

We have in the representation used (14)

$$\psi = \begin{pmatrix} \phi\ (\vec{x}) \\ L\phi\,(\vec{x}) \end{pmatrix} e^{-i\varepsilon t} \qquad (3.13)$$

and, using (3.11) and (3.12) Dirac equation (3.10) yields

$$-i\varepsilon\gamma^0\psi = +\vec{\gamma}.\vec{e}^{\,\alpha}\,\partial_\alpha\psi - \frac{\sqrt{2}}{4}\,\omega\gamma^0\gamma^1\gamma^2\psi - \frac{\omega}{2}\gamma^1\psi \qquad (3.14)$$

where we use the notation $\vec{A}.\vec{B} = \sum_{k=1}^{3} A^k B^k$. Denoting the spin matrix

$$\vec{\Sigma} = \gamma^5\gamma^0\vec{\gamma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} ,$$

equation (3.14) can be rewritten

$$\varepsilon L\psi = (\vec{\Sigma}.\vec{\pi})\psi \qquad (3.15)$$

where $\vec{\pi}$ is the generalized local momentum operator

$$\vec{\pi} = i\vec{e}^{\,\alpha}\,\partial_\alpha + i\,\frac{\omega}{2}\,\vec{n}_1 + \frac{\sqrt{2}}{4}\,\vec{\Omega}\gamma^5 , \qquad (3.16)$$

Where $\vec{n}_1 = (1,0,0)$ and

$$\vec{\Omega} = (0,0,\omega) \qquad (3.17)$$

is the vorticity of matter in the local frame (3.11). $\vec{\Sigma}.\vec{\pi}$ is the operator which acts on the space of neutrino energy modes (3.13). From (3.15) we have that the operator $L\vec{\Sigma}\cdot\vec{\pi}$ is the Hamiltonian of the system (expressed in terms of objects defined in the local frame determined by (3.11)), in the sense that the time development of any operator acting on the space of neutrino functions is proportional to the commutator of the operator and $L\vec{\Sigma}.\vec{\pi}$. With respect to this Hamiltonian $\vec{\Sigma}.\vec{\pi}$ is conserved, that is, the projection of the spin $\vec{\Sigma}$ on the direction of the local momentum $\vec{\pi}$ is conserved. In this sense $L = \vec{\Sigma}.\vec{\pi}/\varepsilon$ has a precise meaning as the helicity of neutrino, in the local Lorentz frames determined by (3.11). The wave functions (3.13) are energy and helicity eigenstates for neutrinos. Later we shall characterize neutrino amplitudes by $L = -1$, $\varepsilon>0$ and antineutrino amplitudes by $L = +1$, $\varepsilon>0$.

We now determine the motion of the local momentum $\vec{\pi}$. We have

$$\dot{\pi}^k = i \left[\pi^k, L\vec{\Sigma} \, . \, \vec{\pi}\right] \tag{3.18}$$

and since $\vec{\Sigma}$ comutes with γ^5, (3.18) reduces to

$$\dot{\pi}^k = iL\vec{\Sigma} \, . \left[\pi^k, \vec{\pi}\right]$$

After a straightforward calculation, using Ricci coefficients (3.9) calculated from (3.11), we obtain

$$\dot{\pi}_k = i \sqrt{2} \; L \; \varepsilon^k{}_{\ell m} \Sigma^\ell \Omega^m e^\lambda{}_{(0)} \partial_\lambda$$

which acting on the space of eigenmodes of energy (3.13) reduces to

$$\dot{\pi}^k = \sqrt{2} \; \varepsilon \; L \; \varepsilon^k{}_{\ell m} \Sigma^\ell \Omega^m$$

or

$$\dot{\vec{\pi}} = \sqrt{2} \; \varepsilon \, L \, \vec{\Sigma} \times \vec{\Omega} \tag{3.19}$$

Since the projection $\vec{\Sigma}.\vec{\pi}$ is conserved, that is, the helicity L of neutrinos is conserved, we have from (3.19) that, for a given sign of ε, the spin $\vec{\Sigma}$ precesses locally about the direction determined by $\vec{\Omega}$, with angular velocity proportional $\frac{\sqrt{2}\varepsilon}{\omega}$ $\frac{\vec{\Omega}}{|\vec{\Omega}|}$ and independent of the sign of L, that is, independent of being neutrino or antineutrino. The local angular velocity of precession will appear in the analysis of the exact solutions of Dirac's equation.

From the expression (3.16) of the local momentum operator $\vec{\pi}$, we observe that L behaves like a (leptonic) charge in the coupling of the vorticity field to the neutrino spinor structure. This fact also shall appear more clearly when we describe the Fourier space of neutrinos associated to the complete basis of energy-momentum-helicity modes $(\varepsilon, k_2, k_3, L)$, in next section.

To separate neutrino equations, we consider neutrino wave functions which belong to the complete set of modes $(\varepsilon, k_2, k_3, L)$, described by[15]

$$\psi = \begin{vmatrix} \phi(x^1) \\ L\phi(x^1) \end{vmatrix} e^{ik_2x^2+ik_3x^3-i\varepsilon t} \tag{3.20}$$

which are invariantly and globally defined, as we have discussed in section 2. Using (3.18) and the explicit expression of $e^{\alpha}_{(A)}$ from (3.11), Dirac's equation (3.14) reduces to

$$\phi_1 + \sqrt{2}\,\varepsilon\sigma^3\phi - k_3\sigma^2\phi + \sqrt{2}\,k_2\,e^{-\omega x^1}\sigma^3\phi + \frac{\omega}{2}\phi =$$
$$= L\,(i\varepsilon\sigma^1\phi + \frac{\sqrt{2}}{4}\,\omega\sigma^2\phi)$$

where $\phi_1 = \partial\phi/\partial x^1$, and expressing

$$\phi = \begin{pmatrix}\alpha(x^1)\\ \beta(x^1)\end{pmatrix}$$

we obtain (16)

$$\alpha_1 + \sqrt{2}\,\varepsilon\alpha + \sqrt{2}\,k_2\,e^{-\omega x^1}\alpha + \frac{\omega}{2}\alpha = i(k_3 + \frac{\sqrt{2}}{4}L\omega + \varepsilon L)\beta \qquad (3.21a)$$

$$\beta_1 - \sqrt{2}\,\varepsilon\beta - \sqrt{2}\,k_2 e^{-\omega x^1}\beta + \frac{\omega}{2}\beta = i\,(-k_3 - \frac{\sqrt{2}}{4}L\omega + \varepsilon L)\alpha \qquad (3.21b)$$

We examine first the case $k_2 \neq 0$. Introducing the variable $x = (u_0^2/\omega)e^{-\omega x^1}$, $0 < x < \infty$, equations (3.21) result

$$\alpha' - (\frac{\sqrt{2}\varepsilon}{\omega} + \frac{1}{2})\frac{\alpha}{x} - \frac{\sqrt{2}k_2}{u_0^2}\alpha = -i(\frac{\varepsilon L + \pi^3}{\omega})\frac{\beta}{x} \qquad (3.22a)$$

$$\beta' + (\frac{\sqrt{2}\varepsilon}{\omega} - \frac{1}{2})\frac{\beta}{x} + \frac{\sqrt{2}k_2}{u_0^2}\beta = -i(\frac{\varepsilon L - \pi^3}{\omega})\frac{\alpha}{x} \qquad (3.22b)$$

where $\pi^3 = k_3 + \frac{\sqrt{2}}{4}L\omega$ (cf. ref. (15)), and a prime denotes x-derivative. Introducing the notation

$$A_1 = -(\frac{\sqrt{2}\varepsilon}{\omega} + \frac{1}{2}) \qquad E_1 = \frac{\varepsilon L + \pi^3}{\omega}$$
$$A_2 = -(\frac{\sqrt{2}\varepsilon}{\omega} - \frac{1}{2}) \qquad E_2 = \frac{\varepsilon L - \pi^3}{\omega} \qquad (3.23)$$
$$B_1 = \frac{\sqrt{2}k_2}{u_0^2}$$

The second order equations resulting from (3.22) are

$$\beta'' + B_1(1 + A_1 + A_2)\frac{\beta}{x} - B_1^2\beta = -(E_1E_2 - A_1A_2)\frac{\beta}{x^2} \qquad (3.24a)$$

$$\alpha'' - B_1(1 - A_1 - A_2)\frac{\alpha}{x} - B_1^2\alpha = -(E_1E_2 - A_1A_2)\frac{\alpha}{x^2} \qquad (3.24b)$$

We choose the constant u_0^2 such that $B_1^2 = \frac{1}{4}$, which implies

$$u_0^2 = \pm 2\sqrt{2}\ k_2 \qquad (3.25)$$

and, in this case, both equations (3.24) assume the form of Whittaker's equation[17], $A'' + \left[-\frac{1}{4} + \frac{\kappa}{x} + (\frac{1}{4} - \mu^2)/x^2\right]A = 0$,
with the linearly independent solutions

$$M_{\kappa,\mu}(x) = e^{-\frac{x}{2}}\, x^{\frac{1}{2}+\mu}\, M(\tfrac{1}{2} + \mu - \kappa,\ 1 + 2\mu,\ x)$$

$$(3.26)$$

$$W_{\kappa,\mu}(x) = e^{-\frac{x}{2}}\ {}^{\frac{1}{2}+\mu}\, U(\tfrac{1}{2} + \mu - \kappa,\ 1 + 2\mu,\ x)$$

$M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ are called Whittaker functions where M and U are confluent hypergeometric functions. For (3.24) we have

$$\frac{1}{4} - \mu^2 = E_1E_2 - A_1A_2\ , \qquad (3.27)$$

and $\kappa = B_1(1 + A_1 + A_2)$ for solutions β of (3.24a) ,

$\kappa = -B_1(1 - A_1 - A_2)$ for solutions α of (3.24b). We distinguish two types of neutrino solutions:

(I) Taking

$$\alpha = \Lambda_1 M_{\kappa,\mu} + \Lambda_2 W_{\kappa,\mu}$$

where Λ_1 and Λ_2 are constants, we must have for consistency

$$\beta = \frac{i}{E_1}\left\{\Lambda_1 (\tfrac{1}{2} + \mu + \kappa) M_{\kappa+1,\mu} - \Lambda_2 W_{\kappa+1,\mu}\right\} \quad \text{for } B_1 = + \frac{1}{2}$$

$$\beta = \frac{i}{E_1}\left\{\Lambda_1 (\tfrac{1}{2} + \mu - \kappa) M_{\kappa-1,\mu} - \Lambda_2 (\mu-\kappa+\tfrac{1}{2})(\mu+\kappa-\tfrac{1}{2}) W_{\kappa-1,\mu}\right\}$$

$$\text{for } B_1 = - \frac{1}{2}$$

where $\kappa = \pm A_1$ for $B_1 = \pm \frac{1}{2}$;

(II) Taking

$$\beta = \Lambda_1 M_{\kappa,\mu} + \Lambda_2 W_{\kappa,\mu}$$

we must have for consistency

$$\alpha = \frac{i}{E_2}\left\{\Lambda_1 (\tfrac{1}{2} + \mu - \kappa) M_{\kappa-1,\mu} - \Lambda_2 (\tfrac{1}{2} + \mu - \kappa)(\mu + \kappa - \tfrac{1}{2}) W_{\kappa-1,\mu}\right\}$$

$$\text{for } B_1 = \frac{1}{2}$$

$$\alpha = \frac{i}{E_2}\left\{\Lambda_1 (\tfrac{1}{2} + \mu + \kappa) M_{\kappa+1,\mu} - \Lambda_2 W_{\kappa+1,\mu}\right\} \quad \text{for } B_1 = - \frac{1}{2}$$

where $\kappa = \pm A_2$ for $B_1 = \pm \frac{1}{2}$. For both types (3.27) holds.

On these sets of solutions we now impose boundary conditions, namely, we impose that neutrinos - which are test fields in the sense that they do not contribute to the curvature of the cosmological background - are finite perturbations at any spacetime point. This is equivalent to impose that the above Whittaker functions must be bounded for all values of x. This condition is realized if and only if [17]

$$\frac{1}{2} + \mu - \kappa = \text{negative integer or zero} \qquad (3.28)$$

Using (3.23) and imposing conditions (3.28) on (α,β) of the above types, we have

<u>type</u> (I)

$$-\left(\frac{1}{2} + \mu - \kappa\right) = \begin{pmatrix} n \\ n+1 \end{pmatrix} , \quad \kappa = \begin{pmatrix} -\frac{\sqrt{2}\varepsilon}{\omega} - \frac{1}{2} \\ -\frac{\sqrt{2}\varepsilon}{\omega} + \frac{1}{2} \end{pmatrix} \quad \text{for } B_1 = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \qquad (3.29)$$

<u>type</u> (II)

$$-\left(\frac{1}{2} + \mu - \kappa\right) = \begin{pmatrix} n+1 \\ n \end{pmatrix} , \quad \kappa = \begin{pmatrix} -\frac{\sqrt{2}\varepsilon}{\omega} + \frac{1}{2} \\ \frac{\sqrt{2}\varepsilon}{\omega} - \frac{1}{2} \end{pmatrix} \quad \text{for } B_1 = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \qquad (3.30)$$

$n = 0,1,2,\ldots$ From (3.29), (3.30) and (3.27), we have for both types the dispersion relations

$$\varepsilon = \begin{pmatrix} -\sqrt{2}(n+1)\omega \pm \sqrt{(n+1)^2\omega^2 + (\pi^3)^2} \\ \sqrt{2}(n+1)\omega \pm \sqrt{(n+1)^2\omega^2 + (\pi^3)^2} \end{pmatrix} \quad \text{for } B_1 = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \qquad (3.31)$$

We remark that the only relevant momentum to the dispersion relation (3.31) is k_3; k_2 does not contribute to (3.31), except for determining signs (cf. (3.23))[18].

In the next section we will see that a necessary condition of normalization of the above we functions is $\mu > 0$, and this implies (by a straightforward examination) that the frequencies (3.31) must have a definite sign,

$$\varepsilon = \mp\left[\sqrt{2}(n+1)\omega + \sqrt{(n+1)^2\omega^2 + (\pi 3)^2}\right] \quad \text{for } B_1 = \pm\frac{1}{2}\,, \qquad (3.32)$$

and for all cases we have

$$\mu = -(n+1) + \frac{\sqrt{2}\,|\varepsilon|}{\omega}\,, \qquad (3.33)$$

which is always greater than zero.

Observing that $\pi^3 = k_3 + \frac{\sqrt{2}}{4} L\omega$ we see from (3.32) that the minimum absolute value of the frequency ε, for <u>normalizable modes</u>, occurs for $n = 0$, $k_3 = -L\frac{\sqrt{2}}{4}\omega$, and

$$\varepsilon = \mp(\sqrt{2}+1)\omega \qquad \text{for } B_1 = \pm\frac{1}{2}$$

In the following we consider for simplicity solutions of the above types with $\Lambda_2 = 0$, and which are normalizable, that is, $\mu > 0$. Thus general solutions for neutrinos in the helicity and energy-momentum modes (3.20) are expressed

1) $B_1 = -\frac{1}{2}\ ,\ \varepsilon > 0$

$$\psi^{(L)}_{(+)} = \begin{pmatrix} M_{\frac{\sqrt{2}|\varepsilon|}{\omega}+\frac{1}{2},\ \mu}(x) \\[2ex] \frac{-iL\omega(n+1)}{|\varepsilon| + L\pi^3}\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}-\frac{1}{2},\ \mu}(x) \\[2ex] L\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}+\frac{1}{2},\ \mu}(x) \\[2ex] \frac{-i\omega(n+1)}{|\varepsilon| + L\pi^3}\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}-\frac{1}{2},\ \mu}(x) \end{pmatrix} e^{-i|k_2|x^2+ik_3x^3-i|\varepsilon|t} \tag{3.34}$$

2) $B_1 = \frac{1}{2},\ \varepsilon < 0$

$$\psi^{(L)}_{(-)} = \begin{pmatrix} \frac{iL\omega(n+1)}{|\varepsilon| + L\pi^3}\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}-\frac{1}{2},\ \mu}(x) \\[2ex] M_{\frac{\sqrt{2}|\varepsilon|}{\omega}+\frac{1}{2},\ \mu}(x) \\[2ex] \frac{i\omega(n+1)}{|\varepsilon| + L\pi^3}\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}-\frac{1}{2},\ \mu}(x) \\[2ex] L\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega}+\frac{1}{2},\ \mu}(x) \end{pmatrix} e^{i|k_2|x^2+ik_3x^3+i|\varepsilon|t} \tag{3.35}$$

3) $B_1 = -\frac{1}{2}, \ \varepsilon > 0$

$$\psi^{(L)}_{(+)} = \begin{pmatrix} \dfrac{iL\omega\left(\frac{2\sqrt{2}|\varepsilon|}{\omega} - (n+1)\right)}{|\varepsilon| - L\pi^3} M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\ \mu} \\ M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\ \mu} \\ \dfrac{i\omega\left(\frac{2\sqrt{2}|\varepsilon|}{\omega} - (n+1)\right)}{|\varepsilon| - L\pi^3} M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\ \mu} \\ L\, M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\ \mu} \end{pmatrix} e^{-i|k_2|x^2 + ik_3x^3 - i|\varepsilon|t} \tag{3.36}$$

4) $B_1 = \frac{1}{2}, \ \varepsilon < 0$

$$\psi^{(L)}_{(-)} = \begin{pmatrix} M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\ \mu} \\ \dfrac{iL\omega\left(2\frac{\sqrt{2}|\varepsilon|}{\omega} - (n+1)\right)}{-|\varepsilon| + L\pi^3} M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\ \mu} \\ L\, M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\ \mu} \\ \dfrac{i\omega\left(2\frac{\sqrt{2}|\varepsilon|}{\omega} - (n+1)\right)}{-|\varepsilon| + L\pi^3} M^{(x)}_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\ \mu} \end{pmatrix} e^{i|k_2|x^2 + ik_3x^3 + i|\varepsilon|t} \tag{3.37}$$

Solutions (1) and (3), or (2) and (4), that is, with the same value of B_1, are linearly dependent for the same value of L, because

$$\left(\frac{-iL\omega(n+1)}{|\varepsilon| + L\pi^3}\right)\left(\frac{iL\omega\left[2\frac{\sqrt{2}|\varepsilon|}{\omega} - (n+1)\right]}{|\varepsilon| - L\pi^3}\right) = 1$$

We then select the two independent set of solutions (1) and (2), respectively for positive and negative energy, which are related by

$$\psi^{(L)}_{(+)}(k_3) = -i\gamma^5\gamma^2\psi^{(-L)*}_{(-)}(-k_3) \tag{3.38}$$

where * denotes complex conjugation.

We now examine the case $k_2 = 0$. Equations (3.21) assume the form

$$\alpha_1 + \sqrt{2}\,\varepsilon\alpha + \frac{\omega}{2}\alpha = i(k_3 + \varepsilon L + \frac{\sqrt{2}}{4}L\omega)\beta \tag{3.39a}$$

$$\beta_1 - \sqrt{2}\,\varepsilon\beta + \frac{\omega}{2}\beta = i(-k_3 + \varepsilon L - \frac{\sqrt{2}}{4}L\omega)\alpha \tag{3.39b}$$

The corresponding second-order equations are

$$\begin{pmatrix}\alpha_{11}\\ \beta_{11}\end{pmatrix} + \omega\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} + \left\{\frac{\omega^2}{4} - \varepsilon^2 - (\pi^3)^2\right\}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \tag{3.40}$$

whose unique bounded solutions are constants, related to a zero root of the characteristic equation of (3.40), $\lambda^2+\omega\lambda+(\omega^2/4-\varepsilon^2-(\pi^3)^2) = 0$. This equation has a zero root only for

$$\frac{\omega^2}{4} = \varepsilon^2 + (\pi^3)^2 \tag{3.41}$$

Condition (3.41) implies that $k_2 = 0$ modes have bounded energy and momentum modes, the absolute maximum of energy being $\varepsilon^2 = \frac{\omega^2}{4}$, for moment $k_3 = -\frac{\sqrt{2}}{4} L\omega$. They have the form of free plane waves propagating along the x^3 direction. While these modes $k_2 = 0$ exist for neutrinos, they are forbidden for scalar[23] and vector fields[24]. We also distinguish two types of solutions $\psi = \binom{\phi}{L\phi}$, namely

$$\phi^{(L)}_{(I)} = \begin{pmatrix} 1 \\ \\ \dfrac{-iL(\sqrt{2}\varepsilon + \omega/2)}{\varepsilon + L\pi^3} \end{pmatrix} e^{ik_3x^3 - i\varepsilon t} \tag{3.42}$$

$$\phi^{(L)}_{(II)} = \begin{pmatrix} \dfrac{iL(\sqrt{2}\varepsilon - \omega/2)}{\varepsilon - L\pi^3} \\ \\ 1 \end{pmatrix} e^{ik_3x^3 - i\varepsilon t} \tag{3.43}$$

Solutions (3.42) and (3.43) are linearly dependent for the same L. Denoting $\psi_{(\pm)}$ positive or negative energy solutions, we have for (3.42) and (3.43) the property

$$\psi^{(L)}_{(I)\,(+)}(k_3) = i\gamma^5\gamma^2\psi^{(-L)\,*}_{(II)\,(-)}(-k_3) \tag{3.44}$$

analogous to (3.38). Neutrinos (3.42), (3.43) poses an interesting question on the stability of the stationary background universe because in principle they could be created by a purely time-dependent vorticity perturbation $\delta\omega^{\alpha}(t)$, with the consequent perturbative Lagrangian $L_I \sim \delta\omega_A(t)\bar{\psi}\gamma^5\gamma^A\psi$ which could in principle create neutrino pairs from the vacuum of the neutrino field.

4. NORMALIZATION AND GENERALIZED FOURIER SPACE OF NEUTRINO AMPLITUDES

We consider first the case $k_2 \neq 0$. Since $B_{1/k_2} > 0$ always, we have for any mode $(\varepsilon, k_2, k_3, L)$ that k_2 has the opposite sign of ε and hence, for a given sign of the frequency ε, propagation along x^2 is allowed in only one direction. Because of this, although we can say that the modes $(\varepsilon, k_2, k_3, L)$ are energy stationary states in the usual sense, it is not possible - in a finite space-time volume - to have normalized waves along x^2, and this has a strong physical implication in the definition of normalization of neutrino functions. To see this, let us consider in Minkowski space-time a closed cubic box with stationary plane-waves $e^{i\vec{k}.\vec{x}-i\varepsilon t}$ inside, for a given value of the frequency ε; suppose now that along a given axis the waves can only propagate in one direction, of positive y axis, for instance. These stationary states cannot nor malized in the volume of the box, because in one of the boundary walls (transverse to y axis) of the box we should then have waves being <u>absorbed</u> in the same rate in which they are <u>emitted</u> in the opposite wall. In a finite space-time volume of Gödel universe the same problem would be posed because the stationary solutions (ε, k_2, k_3, L) have this property, and waves which propagate along the x^2 direction should in some way be reinjected into the volume[19]. We are led to normalize these stationary modes using a normalization integral which must then be taken over the whole manifold of the model.

To proceed, let us consider the local classical current of neutrinos

$$j^{(A)} = \bar{\psi}\gamma^{A}\psi = e_{\alpha}^{\ (A)}(x)\,\bar{\psi}\gamma^{\alpha}(x)\,\psi \tag{4.1}$$

which for $\psi = \binom{\phi}{L\phi}$ assumes the form

$$j^{(A)} = 2\,(\phi^{+}\phi,\ L\phi^{+}\vec{\sigma}\phi) \tag{4.2}$$

The component $j^{(0)} = 2\phi^{+}\phi$ of (4.2) is the <u>local number density</u> of neutrinos. As expected, $j^{(0)}$ transforms like the zeroth component of a Lorentz vector with respect to Lorentz transformations (3.3), and it is a scalar function with respect to coordinate transformations (and/or point transformation) of the space-time. The local number $j^{(0)}(x)\sqrt{-g}\, d^4x$ is thus a scalar[20], and integrated over a given volume of the manifold,

$$\int \sqrt{-g}\, j^{(0)} d^4x \tag{4.3}$$

yields a number which is coordinate invariant. Furthermore the integral is positive definite since $\sqrt{-g}\, j^{(0)}$ is easily seen to be positive definite for the complete solutions (3.34), (3.35).

In view of this, neutrino wave functions are normalized according to the integral (4.3), taken over the whole Gödel manifold for reasons discussed above, and for the complete set (3.34), (3.35) we then have the δ normalization

$$\langle\psi_{(i)}(\varepsilon',k_2',k_3',L)|\psi_{(j)}(\varepsilon,k_2,k_3,L)\rangle = (2\pi)^3 N^2\, \delta_{ij}\cdot$$
$$\delta(|\varepsilon|-|\varepsilon'|)\,\delta(|k_2|-|k_2'|)\,\delta(k_3-k_3'), \tag{4.4}$$

where $i,j = +,-$, corresponding respectively to positive (3.34) and negative (3.35) frequency solutions, and

$$N^2 = \frac{\sqrt{2}\,u_0^2}{\omega^2}\int_0^\infty \left\{ \left| M_{\frac{\sqrt{2}|\varepsilon|}{\omega}+\frac{1}{2},\,\mu} \right|^2 + \frac{\omega^2(n+1)^2}{(|\varepsilon|+L\pi^3)^2}\left| M_{\frac{\sqrt{2}|\varepsilon|}{\omega}-\frac{1}{2},\,\mu}\right|^2 \right\} \frac{dx}{x^2} \tag{4.5}$$

Using expressions (3.26) of Whittaker functions in terms of confluent hippergeometric functions[(17,21)], (4.5) exists provided

$$\mu > 0\,, \tag{4.6}$$

and can be calculated

$$N^2 = \frac{4|k_2|}{\omega^2}\,\Gamma(2\mu)\Gamma(2\mu+1)(n+1)!\left\{\frac{1}{\Gamma(2\mu n+2)} + \frac{(n+1)\omega^2}{(|\varepsilon|+L\pi^3)^2}\,\frac{1}{\Gamma(2\mu+n+1)}\right\} \tag{4.7}$$

As we have discussed, condition (4.6) imposes that the frequency ε has a definite sign for a given sign of k_2 (cf. (3.32)), and that for all cases of types (I), (II) neutrino solutions, $\mu = -(n+1) + \frac{\sqrt{2}|\varepsilon|}{\omega}$ (of. (3.33)). The factor $(2\pi)^3 N^2$ in the right-hand-side of (4.4) can be interpreted as proportional to the local number density of neutrino states (ε, k_2, k_3, L), that is, the local number density in the Fourier space associated to the complete basis of solutions (3.34), (3.35).

Since we have used the local number density $j^{(0)}$ to normalize the wave functions, the normalization depends on the orientation of the field of tetrad frames $e^{\alpha}{}_{(A)}(x)$, with an arbitrariness due to local Lorentz transformations (3.3). The present orientation of the tetrad frame in which (4.4) and (4.7) were calculated, is nevertheless a <u>preferred orientation</u> in the sense that (3.11) is based on the matter flow of the model - actually the zeroth vector of the tetrad frame $e^{\alpha}{}_{(0)}$ is defined by the four-velocity field of matter, $e^{\alpha}_{(0)} = \delta^{\alpha}_{0}$, and (4.4) and (4.7) are invariant under Lorentz transformations which preserve this condition, that is, $L^0_A = \delta^0_A$. The matter flow of the model singles out (4.4), (4.7).

The Fourier space associated to the complete basis (3.34), (3.35) can be constructed as follows. The kernel of the transformation is defined

$$K(k_2,k_3,\varepsilon,x) = K_{(+)}(|k_2|,k_3|\varepsilon|,x) + K_{(-)}(|k_2|,k_3,|\varepsilon|,x) \tag{4.8}$$

where

$$K_{(+)} = \mathrm{diag}\left(\frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\mu}}{\langle M_+\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\mu}}{\langle M_-\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\mu}}{\langle M_+\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\mu}}{\langle M_+\rangle^{1/2}} \right) \cdot \exp\left[i|k_2|x^2 - ik_3x^3 + i|\varepsilon|t \right] \tag{4.9}$$

and

$$K_{(-)} = \mathrm{diag}\left(\frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\mu}}{\langle M_-\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\mu}}{\langle M_+\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\mu}}{\langle M_-\rangle^{1/2}}, \frac{M_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\mu}}{\langle M_+\rangle^{1/2}} \right) \cdot \exp\left[-i|k_2|x^2 - ik_3x^3 - i|\varepsilon|t \right], \tag{4.10}$$

and we have denoted

$$<M_{\pm}> = \int_0^{\infty} \frac{dx}{x^2} \left| M_{\frac{\sqrt{2}|\varepsilon|}{\omega} \pm \frac{1}{2}, \mu} \right|^2$$

The Fourier transform of a neutrino solution ψ has the form

$$F[\psi] = \psi_F(|\varepsilon'|, |k_2'|, k_3') = \int \sqrt{-g}\, d^4x\; k(k_2', k_3', \varepsilon', x)\psi(x) \qquad (4.11)$$

where the integration is taken over the whole manifold.

For (4.8) we have the unitarity property

$$\int \sqrt{-g}\, d^4x\; K^+(k_2', k_3', \varepsilon', x) K(k_2, k_3, \varepsilon, x) =$$

$$= 4\pi^3 \cdot \mathbb{1} \cdot \delta(|\varepsilon| - |\varepsilon'|) \cdot \delta(|k_2| - |k_2'|) \cdot \delta(k_3 - k_3') \qquad (4.12)$$

We remark that the first term $K_{(+)}$ of the kernel (4.8) can be considered as a projector - in the sense of (4.11) - into positive energy states, since its action on negative energy states (3.35) gives zero; analogously the second term $K_{(-)}$ in (4.8) is a projector into negative energy states since its action on positive energy states (3.34) gives zero. Because the inverse of a projector is not a one-one map, the inverse Fourier transform is then defined separately for positive and negative energy amplitudes, with kernels $K_{(+)}$ and $K_{(-)}$ respectively, that is,

$$F^{-1}\left[\psi_F(|k_2|, k_3, |\varepsilon|, +)\right] = \int_{\varepsilon>0} \frac{dk_2\, dk_3\, d\varepsilon}{(2\pi)^3} K^+_{(+)}(|k_2|, k_3, |\varepsilon|, x)\psi_F \qquad (4.13)$$

for positive energy states, and

$$\overset{-1}{F}\left[\psi_F(|k_2|,k_3,|\varepsilon|,-)\right] = \int_{\varepsilon>0} \frac{dk_2\,dk_3\,d\varepsilon}{(2\pi)^3} K^{+}_{(-)}(|k_2|,k_3,|\varepsilon|,x)\psi_F \qquad (4.14)$$

for negative energy states. (4.13) and (4.14) are consequence of the unitary properties

$$\int_{\varepsilon'>0} \frac{dk_2'dk_3'd\varepsilon'}{(2\pi)^3} K^{+}_{(+)}(|k_2'|,k_3',|\varepsilon'|,x')K_{(+)}(|k_2'|,k_3',|\varepsilon'|,x) = \mathbb{1}\,\frac{\delta^4(x-x')}{\sqrt{-g}} \qquad (4.15)$$

$$\int_{\varepsilon'>0} \frac{dk_2'dk_3d\varepsilon'}{(2\pi)^3} k^{+}_{(-)}(|k_2'|,k_3',|\varepsilon'|,x')K_{(-)}(|k_2'|,k_3',|\varepsilon'|,x) = \mathbb{1}\,\frac{\delta^4(x-x')}{\sqrt{-g}} \qquad (4.16)$$

which actually imply $F\overset{-1}{F} = \overset{-1}{F}F = 1$, as expected. We remark that the inverse unitarity properties (4.15), (4.16) do not hold for the total kernel (4.8); only separately for positive and negative energy kernels does the complete unitarity hold.

The Fourier transform of a normalized positive energy state (3.34), for instance, is the four-spinor

$$\psi_F(|k_2|,k_3,|\varepsilon|,L,+) = \frac{(2\pi)^3}{N} \cdot \begin{pmatrix} \langle M_+\rangle^{1/2} \\ \dfrac{-iL\omega(n+1)}{|\varepsilon| + L\pi^3} \langle M_-\rangle^{1/2} \\ L\langle M_+\rangle^{1/2} \\ \dfrac{-i\omega(n+1)}{\varepsilon + L\pi^3} \langle M_-\rangle^{1/2} \end{pmatrix} \cdot$$

$$\cdot\delta(|k_2'|-|k_2|)\,\delta(k_3'-k_3)\,\delta(|\varepsilon'|-|\varepsilon|) \tag{4.17}$$

which under the local Lorentz group (3.3), (3.4) transforms as

$$\tilde{\psi}_F(|k_2|,k_3,|\varepsilon|,L,+) = \int \frac{dk_2'dk_3'd\varepsilon}{(2\pi)^3} S(k_2,k_3,\varepsilon;k_2',k_3',\varepsilon').$$

$$\cdot\psi_F(|k_2'|,k_3',|\varepsilon'|,L,+) \tag{4.18}$$

where

$$S(k_2,k_3,\varepsilon;k_2',k_3',\varepsilon') = \int\sqrt{-g}\; d^4x\; K_{(+)}(|k_2|,k_3,|\varepsilon|,x).$$

$$S(x)K^+_{(+)}(|k_2'|,k_3',|\varepsilon'|,x) \tag{4.19}$$

The group of transformations (4.18), (4.19) is induced on the Fourier space by the local Lorentz rotations (3.3) of the field of Lorentz frames, with respect to which the spinor structure is defined.

The Fourier space described above is actually a <u>momentum space</u> for neutrinos. In fact, expressing a positive energy state (3.34) as

$$\psi^{(L)}_{(+)} = \int \frac{dk_2 dk_3 d\varepsilon}{(2\pi)^3} \underset{\sim}{K}^{+}_{(+)}(|k_2|, k_3, |\varepsilon|, x)\psi_F(|k_2|, k_3, |\varepsilon|, L, +)$$

and using Dirac's equation $\gamma^A \nabla_A \psi = 0$, we obtain after a long calculation the transformed Dirac's equation

$$-i\pi_A \gamma^A \psi_F = 0 \qquad (4.20)$$

where π_A is given by[22]

$$\pi_A = \left\{ |\varepsilon|, \frac{i}{2}\left[\left(\frac{\langle M_-\rangle}{\langle M_+\rangle}\right)^{1/2} \omega(n+1) - \left(\frac{\langle M_+\rangle}{\langle M_-\rangle}\right)^{1/2} \frac{\varepsilon^2 - (\pi^3)^2}{\omega(n+1)}\right], \right.$$

$$\left. -\frac{1}{2}\left[\left(\frac{\langle M_+\rangle}{\langle M_-\rangle}\right)^{1/2} \frac{\varepsilon^2 - (\pi^3)^2}{\omega(n+1)} + \left(\frac{\langle M_-\rangle}{\langle M_+\rangle}\right)^{1/2} \omega(n+1)\right], -\left(k_3 + \frac{\sqrt{2}}{4} L\omega\right) \right\}$$

$$(4.21)$$

We have

$$\pi_A \pi^A = 0 \qquad (4.22)$$

as expected for a massless particle, where $\pi^A = \eta^{AB}\pi_B$. The spatial components of the local four-vector π_A are the eigenvalues of the momentum operator (3.16), corresponding to positive energy, and momentum eigenstates (3.34). The imaginary character of π^1 is connected to the fact that $j^{(1)} = 0$ (cf. (4.2)). The form of the component π_3 (along the direction of the vorticity vector) shows that the "leptonic charge" L behaves like the coupling constant in the coupling of the spinor structure of neutrino to the vorticity field. For a negative energy solution

$$\psi^{(L)}_{(-)} = \int \frac{dk_2 dk_3 d\varepsilon}{(2\pi)^3} K^{+}_{(-)}(|k_2|,k_3,|\varepsilon|,x)\psi_F(|k_2|,k_3,|\varepsilon|,L,-) \tag{4.23}$$

we analogously obtain (4.20), where π_A is now

$$\pi_A = \left\{-|\varepsilon|,\ \frac{i}{2}\left[\left(\frac{\langle M_-\rangle}{\langle M_+\rangle}\right)^{1/2}\omega(n+1) - \left(\frac{\langle M_+\rangle}{\langle M_-\rangle}\right)^{1/2}\frac{\varepsilon^2-(\pi^3)^2}{\omega(n+1)}\right],\right.$$
$$\left.\frac{1}{2}\left[\left(\frac{\langle M_-\rangle}{\langle M_+\rangle}\right)^{1/2}\omega(n+1) + \left(\frac{\langle M_+\rangle}{\langle M_-\rangle}\right)^{1/2}\frac{\varepsilon^2-(\pi^3)^2}{\omega(n+1)}\right],\ -\left(k_3+\frac{\sqrt{2}}{4}L\omega\right)\right\} \tag{4.24}$$

with $\pi_A\pi^A = 0$. Comparing (4.21) with (4.23), we see that π_A (positive energy) and π_A (negative energy) have opposite signs in the 0 - and 2 - components. We remark that the component

π_3 is the same for (4.21) and (4.24) due to our definition of (4.10); indeed if in (4.10), (4.23) we change $k_3 \to -k_3$, $L \to -L$ we have in (4.24) that $\pi_3 \to -\pi_3$ without altering other components, analogous to the fact that positive and negative energy solutions related through property (3.38) shall have π_3 with opposite signs. This fact is important when we consider symmetry transformations between particle and antiparticle amplitudes, in next section.

We can now calculate the component $j_F^{(3)}$ (that is, along the local vorticity field $\vec{\Omega}$) of the local four current

$$j_F^{(A)} = \bar{\psi}_F \gamma^A \psi_F = (\psi_F^+ \psi_F \, , \, \bar{\psi}_F \vec{\gamma} \psi) \tag{4.25}$$

For the positive energy state (4.17) we have

$$j_F^{(3)} = 2L \, \frac{(2\pi)^6}{N^2} \left\{ \langle M_+ \rangle - \frac{\omega^2 (n+1)^2}{(|\varepsilon| + L\pi^3)^2} \langle M_- \rangle \right\} . \ \delta(|k_2''| - |k_2|) .$$

$$\cdot \delta(k_3'' - k_3) . \delta(|\varepsilon''| - |\varepsilon|)$$

By using that

$$\langle M_+ \rangle = \frac{\Gamma(2\mu)(n+1)!\,\Gamma(2\mu+1)}{\Gamma(2\mu+n+2)} \quad \text{and} \quad \langle M_- \rangle = \frac{\Gamma(2\mu)\, n!\, \Gamma(2\mu+1)}{\Gamma(2\mu+n+1)}$$

and after some simplifications we obtain

$$j_F^{(3)} = 2J^2 \left\{ L(\pi^3)^2 + |\varepsilon|\pi^3 \right\} \delta(|k_2''| - |k_2|) . \delta(k_3'' - k_3) . \delta(|\varepsilon''| - |\varepsilon|) \tag{4.26}$$

where J^2 is the positive definite quantity

$$J^2 = 2\,\frac{(2\pi)^6}{N^2}\;\frac{(n+1)!\;\;\Gamma(2\mu)\;\;\Gamma(2\mu+1)}{\Gamma(2\mu+n+1)\,.\;\;(|\varepsilon|+L\pi^3)^{2(2\mu+n+1)}}\;.$$

We use the expression (4.26) in next section, to discuss the microscopic asymmetry of neutrino emission in presence of a local vorticity field. The Fourier space associated to the plane wave modes $k_2 = 0$ has a trivial construction, as in the case of plane wave spinor solutions in Minkowski space-time. Since all conclusions for this case are analogous to the case $k_2 \neq 0$, we do not consider it here.

5. SYMMETRY TRANSFORMATIONS FOR NEUTRINO AMPLITUDES AND THE MICROSCOPIC ASYMMETRY OF NEUTRINO EMISSION

In order to examine questions connected to neutrino-antineutrino symmetry of some processes, we shall try to define amplitudes for particle and antiparticle states. To this end we obtain transformations which can be interpreted as leading from particle to antiparticle amplitudes, and which are actually symmetry transformations for the present neutrinos - in the sense that they preserve the Hilbert space of neutrino solutions generated by the basis (3.34), (3.35). These transformations can be reasonably understood as corresponding locally to known symmetries of particle physics.

As we have discussed already, the use of tetrads is practically unavoidable to describe the interaction of fermions with gravitation [12,25] and, in this context, the theory has two groups of transformations involved: the local Lorentz rotations (3.3) of the tetrads, and the isometry group of the manifold. Spinors are defined with respect to the local Lorentz structure, in the sense that they provide a basis space for a spinorial representation of the local Lorentz group. By other hand, these spinors provide a basis space for a scalar representation of the isometry group of the manifold. For the present case of neutrinos, we are restricted to a subspace of spinor functions which are eigenstates of γ^5, that is, the Hilbert space of neutrino solutions generated by (3.34), (3.35).

In the definition of neutrino and antineutrino amplitudes, both groups will be involved, for instance the energy eigenmodes are related to the Killing vector $\partial/\partial t$ of the isometry group, while the charge conjugation operation must take into account the local spinor structure. Our procedure will be to obtain consistent neutrino-antineutrino symmetry transformations of the Hilbert space of neutrino solutions [26] generated by (3.34), (3.35) and which then necessarily takes into account the two group structures present.

Starting from a negative energy solution (3.35).

$$\psi^{(L)}_{(-)}(k_3) = \begin{pmatrix} \phi^{(L)}_{(-)}(k_3) \\ L\phi^{(L)}_{(-)}(k_3) \end{pmatrix}$$

where

$$\phi^{(L)}_{(-)}(k_3) = \begin{bmatrix} \dfrac{iL\omega(n+1)}{|\omega|+Lk_3+\frac{\sqrt{2}}{4}\omega} \, M_{\frac{\sqrt{2}|\varepsilon|}{\omega} - \frac{1}{2},\mu} \\ \\ M_{\frac{\sqrt{2}|\varepsilon|}{\omega} + \frac{1}{2},\mu} \end{bmatrix} e^{i k_2 x^2 + ik_3 + i|\varepsilon|t}$$

we define the transformation

$$\psi^{(L)}_{(-)}(k_3) \to S^{-1}\bar{\psi}^{(L)T}_{(-)}(k_3) \tag{5.1}$$

where S is a matrix of the algebra of Dirac matrices, which satisfies

$$S\gamma^{\mu}S^{-1} = -\gamma^{\mu T} \tag{5.2}$$

In the present representation [14], (5.2) is satisfyed by

$$S \sim \gamma^2\gamma^0 \tag{5.3}$$

where $\sim$ denotes equality up to a constant phase factor. An explicit calculation of (5.1) gives

$$\gamma^2\gamma^0\bar{\psi}^{(L)T}_{(-)}(k_3) \sim \psi^{(-L)}_{(+)}(-k_3) \tag{5.4}$$

Transformation (5.1) has the following properties: (i) it is a symmetry transformation of the Hilbert space of neutrino solutions, since it takes a negative energy solution (3.35) to a positive energy solution (3.34), and vice-versa; (ii) the S matrix (5.2), (5.3) has the character of a charge-conjugation operator on the amplitudes (3.34), (3.35); (iii) neutrino amplitudes related by (5.1) have opposite helicity L and momentum k_3; this implies that the eigenvalues of the local momentum $\vec{\pi}$ change sign under (5.1), as may be seen from the real components of π_A (cf. (4.21), (4.24) and the remarks below (4.24)). We note that (5.4) is precisely the symmetry (3.38) between positive and negative energy solutions. From the above properties, we interpret (5.1) as a charge-conjugation-parity (CP) transformation for neutrino amplitudes, and hence we have the independent <u>positive energy</u> wave functions interpreted as

$$\psi^{(L)}_{(+)}(k_3) = \text{neutrino amplitude},$$

$$\psi^{(-L)}_{(+)}(-k_3) = \text{corresponding anti-neutrino amplitude} \tag{5.5}$$

The <u>positive energy</u> amplitudes (5.5) are said CP related in the sense that the corresponding negative energy amplitude

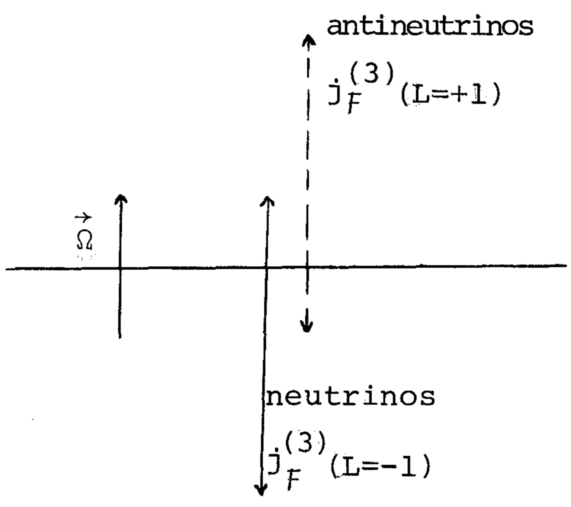
$\psi^{(L)}_{(-)}(k_3)$ $(\psi^{(-L)}_{(-)}(-k_3))$ one is transformed into the other

$\psi^{(-L)}_{(+)}(-k_3)$ $(\psi^{(L)}_{(+)}(k_3))$ under (5.1). From the local CP invariance of neutrinos physics (only negative helicity neutrinos exist) we take L = -1 for neutrinos, which implies L = +1 for antineutrinos (cf. (5.5)). Neutrino and antineutrino amplitudes (5.5) have their respective momentum $\vec{\pi}$ with opposite signs.

We can now discuss the microscopic asymmetry of neutrino emission along the direction determined by the vorticity vector field. From expression (4.26) for the component of $\vec{j}_F$ along $\vec{\Omega}$, we take the relevant factor

$$j_F^{(3)} \simeq |\varepsilon|\pi^3 + L(\pi^3)^2 \qquad (5.6)$$

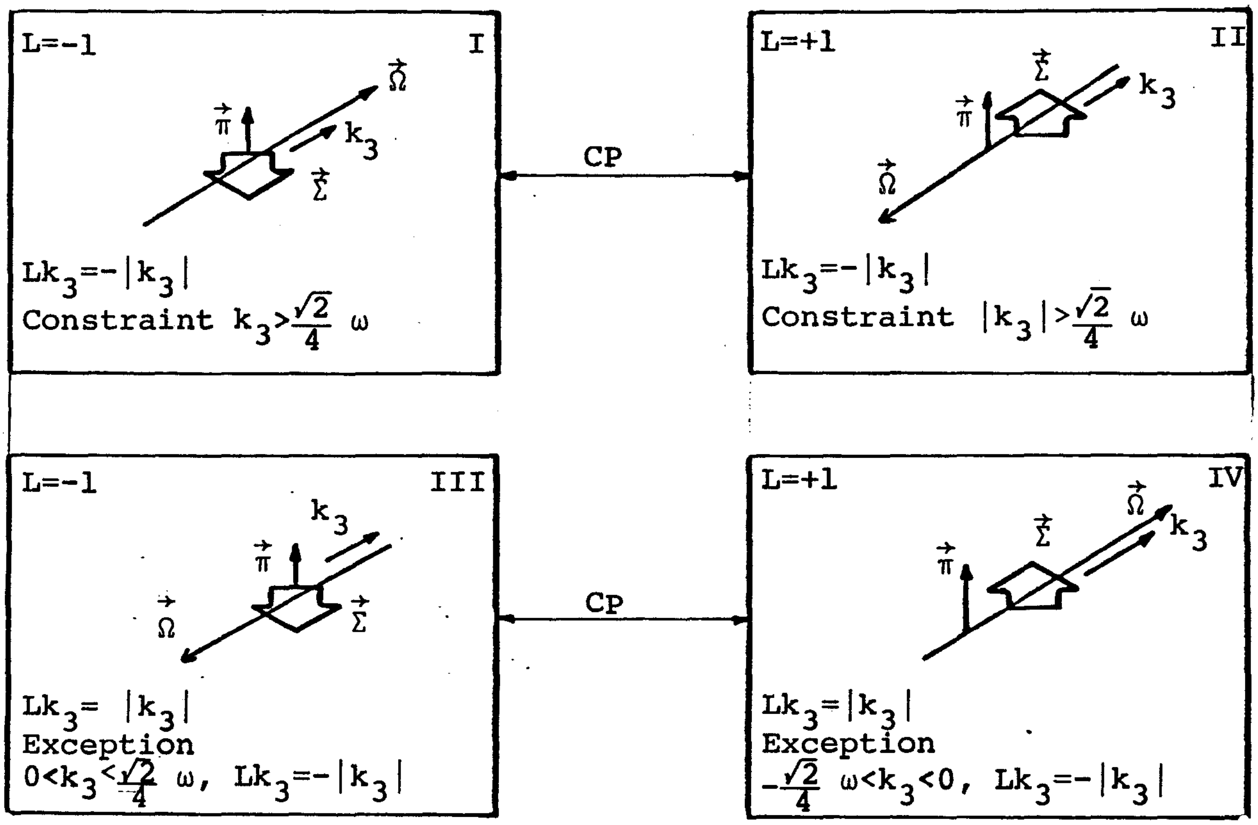
and considering that $|\varepsilon|$ is always greater than $|\pi^3|$ (cf. (3.32)) we have that $j_F^{(3)}$ has the same sign of π^3 and: for neutrinos (L= -1), (5.6) is larger for $\pi^3 < 0$ than for $\pi^3 > 0$; for antineutrinos (L= +1), (5.6) is larger for $\pi^3 > 0$ than for $\pi^3 < 0$. In the other words, for neutrinos (L= -1) we have that $\vec{j}_F$ is larger along the direction antiparallel to $\vec{\Omega}$ than along the parallel direction; for antineutrinos (L= +1) $\vec{j}_F$ is larger along the direction parallel to $\vec{\Omega}$. For both cases (neutrino and antineutrino) the asymmetry in $j_F^{(3)}$ is proportional to $2|\varepsilon\pi^3|$. The following diagram is illustrative (note that $\vec{\Omega} = (0,0,\omega)$).



The preferential emission of neutrinos (antineutrinos) along the direction antiparallel (parallel) to the local vorticity $\vec{\Omega}$ has a macroscopic analog in the case of neutrino evaporation by a rotating black-hole[27,28]. Besides the microscopic/macroscopic distinction in the calculated effects, a basic difference however lies in the local character of the vorticity field of matter flow as well in the local interpretation of L as the helicity fo neutrino spinor fields, for the present case, in contrast to the asymptotic meaning of rotation and other quantities in the space-time of a rotating black-hole.

Finally we draw some interesting conclusions concerning the number density of neutrino and antineutrino states, CP violation and lepton asymmetry, for the present problem. To this end we note that the number density of states - which is proportional to $\left[(|\varepsilon|-\sqrt{2}(n+1)\omega)/(k_3+\frac{\sqrt{2}}{4} L\omega)\right]N$ where N is given by (4.7), and which we denote by $n(Lk_3)$ - depends strongly on the sign of Lk_3 (through $|\varepsilon|$ and $L\pi^3$; cf. also (3.33)), for $|k_3|$ of the order of ω. Consequently for a given value of (k_2,k_3,n,ω), such that $|k_3|$ is of the order of ω), such that $|k_3|$ is of the order of ω, we could have a number density of states different for L= -1 and L= +1. This fact can be significative in the presence of CP violating interactions, as we shall discuss now for the case of creation of neutrino-antineutrino pairs in the presence of a CP violating perturbation, when a neutrino-antineutrino number asymmetry may possible occurs.

Having in mind the CP symmetry of (5.5) (cf.remarks below (5.5)) and that $\pi^3 = k_3 + \frac{\sqrt{2}}{4} L\omega$ we can draw the following diagram for the amplitudes (5.5) according to the sign of Lk_3 :



In the previous diagram of currents, the large components of neutrino and antineutrino currents correspond to amplitudes I and II, and are CP related. The small components correspond to CP related amplitudes III and IV, which shows clearly that the asymmetric emission of neutrinos is CP invariant.

In case of creation of neutrino-antineutrino pairs in the present universe, we can distinguish two possibilities: (i) neutrino-antineutrino pairs whose amplitudes are CP related, namely $(\nu_I \; \bar{\nu}_{II})$ or $(\nu_{III} \bar{\nu}_{IV})$ according to the above diagram; for each case the corresponding current diagram is CP invariant, and the number density of neutrino states is equal to the number density of antineutrino states.

(ii) neutrino-antineutrino pairs whose amplitudes are not CP related, namely $(\nu_I \bar{\nu}_{IV})$ or $(\nu_{III} \bar{\nu}_{II})$. In both cases we note that Lk_3 has opposite signs for neutrino and antineutrino amplitudes, which corresponds to a number density of states differents for neutrinos and anti-neutrinos. For $(\nu_I \bar{\nu}_{IV})$ or $(\nu_{III} \bar{\nu}_{II})$ we have respectively the number densities of states $(n(|k_3|), n(-|k_3|))$ and $(n(-|k_3|), n(|k_3|))$. Nevertheless if the creation of pairs is due to a CP invariant perturbation both cases will be equally probable since $(\nu_I \bar{\nu}_{IV}) \xleftrightarrow{CP} (\nu_{III} \bar{\nu}_{II})$ and no net asymmetry in neutrino-antineutrino number is possible. A net asymmetry (due to different number density of states avaiable for neutrinos and antineutrinos) will appear if the pair production perturbation violates CP. Indeed if pairs $(\nu_I \bar{\nu}_{IV})$ are produced, the pairs $(\nu_{III} \bar{\nu}_{II})$ are then forbidden and a net asymmetry between neutrino and antineutrino number will appear, proportional to the ratio

$$\delta_{k_3} = \frac{n(k_3) - n(-k_3)}{n(k_3) + n(-k_3)} \qquad (5.7)$$

<u>for positive values of</u> k_3 only. The ratio (5.7) is significantly non-zero <u>only for</u> k_3 <u>of the order of</u> ω. We also remark that

the above discussion is independent of the space-time point considered, since in our analysis we have dealt with scalar quantities only.

The same analysis and conclusions follow for the modes $k_2 = 0$.

6. CONCLUSIONS

The basic conclusion of our investigation is that the presence of a vorticity field of matter can generate, via gravitation, microscopic asymmetries in neutrino physics. We have shown this in the context of Einstein theory of gravitation, and for operational simplicity we have considered Gödel universe as the cosmological background, because it is the simplest known solution of Einstein field equations which is stationary and in which the matter content has a non-null vorticity. The results follow:

1) The local dynamics of neutrinos is obtained from Dirac equation in the given background. The spin of the neutrino precesses locally about the direction of the vorticity field. The direction of the angular velocity vector is parallel to the vorticity field, both for neutrino and anti-neutrino, and the absolute value of the angular velocity of precession depends on the energy of the neutrino/antineutrino. The Hamiltonian which determines the local dynamics of neutrinos is defined with respect to the global time-like killing vector $\partial/\partial t$, and we have that the helicity L of neutrino (defined with respect to the local Lorentz frames of the tetrads) is conserved.

2) By separation into invariant modes defined by the global Killing vector fields of the space-time, we obtain a complete set of solutions for neutrino amplitudes in the modes (ε, k_2, k_3, L). The modes $k_2 = 0$ exist for neutrinos, while they are forbidden (as test fields) for scalar and vector fields; they have the form of free plane waves propagating along the x^3 - direction, with bounded values of momentum k_3 and energy. Pairs $\nu\bar{\nu}$ of this type could in principle be created by purely time-dependent perturbations $\delta\omega^{\alpha}$ (t) of the vorticity vector field, but this process should be restricted by the stability of the cosmological background. We construct the Fourier space associated to these complete bases. In the case $k_2 \neq 0$, the complete unitary relations for the kernel of the transformation are separately defined for the positive energy part and negative energy part of the kernel (cf. (4.15), (4.16));

3) From the symmetry properties of the Hilbert space of neutrino solutions and its corresponding Fourier space we are able to define neutrino amplitudes and antineutrino amplitudes, which are CP related as expected from the laws of neutrino physics.

4) The Fourier current associated to neutrino amplitude is asymmetric along the direction determined by the vorticity field: the component of neutrino current along the direction antiparallel to the vorticity field is larger than the component along the opposite direction. Also the Fourier component associated to antineutrino amplitude is also asymmetric, since its component along the direction antiparallel to the vorticity vec

tor is smaller than the component along the direction parallel to the vorticity vector. Therefore at the microscopic level, neutrino are preferentially emitted antiparallel to the local vorticity field, as well antineutrinos are preferentially emitted parallel to the local vorticity field. This result is CP invariant. In case of production of pairs under CP violation, a net number asymmetry appears between neutrinos and antineutrinos, which is significantly non-zero for k_3 of the order of the vorticity value ω.

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4) This is the multiplication law of hyperbolic quaternions, cf. ref. (1) and I. Ozsvath, J. Math. Phys. 11, 2871 (1970).

5) A left unvariant vector field X on a group G is defined by vX(a) = X(va) for all v, a ε G.

6) The commutator of two vector fields e, f is defined $[e,f] = \mathcal{L}_e f$, where $\mathcal{L}$ denotes Lie derivative.

7) Correspondingly, by imposing the orthogonality conditions $g(X_a,X_b) = \frac{1}{\omega^2}$ diag $(2,-1,-1,-1)$, a,b = 0,1,2,3.

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10) Expression (2.17) was calculated using the form (3.11) for Gödel's metric.

11) $\partial/\partial x^2$ is the element $f^\mu_1 \frac{\partial}{\partial a^\mu}$ of the right invariant basis (2.6), in coordinates (t,x^1,x^2).

12) D.R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).

13) Capital Latin indices are tetrad indices and run from 0 to 3; they are raised and lowered with the Minkowski metric

η^{AB}, η_{AB} = diag (+1,-1,-1,-1). Greek indices run from 0 to 3 and are raised and lowered with $g^{\alpha\beta}$, $g_{\alpha\beta}$; throughout the paper we use units such that $\hbar = c = 1$.

14) γ^A are the constant Dirac matrices; we use a representation such that $(\gamma^A)^+ = \gamma^0\gamma^A\gamma^0$, with $(\gamma^0)^2 = -(\gamma^k)^2 = \mathbb{1}$, k= 1,2,3, and $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$.

15) The component along $\vec{\Omega}$ of the local momentum $\vec{\pi}$ has eigenvalue $\pi^3 = k_3 + \frac{\sqrt{2}}{4} L\omega$ with respect to the complete set (3.20). This shows the degeneracy of the linear momentum k_3 (associated to the globally defined operator $-i \frac{\partial}{\partial x^3}$) which is raised by the local vorticity field of matter.

16) We use Pauli matrices in the representation $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

17) M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Chapter 13, Dover, New York 1965.

18) This should be expected, since both t and x^2 are coordinates of time character, that is, both $\partial/\partial t$ and $\partial/\partial x^2$ are time-like vector fields; $\partial/\partial x^2$ becomes a null vector for $x^1 \to -\infty$.

19) This fact is related to the existence of closed time-like lines in Gödel model.

20) For (3.11), we have $\sqrt{-g} = \frac{\sqrt{2}}{2} e^{\omega x^1} = \frac{\sqrt{2}}{2} \frac{u_0^2}{\omega} \frac{1}{x}$ where the variable $x = \frac{u^2{}_0}{\omega} e^{-\omega x^1}$ has been introduced in section 3.

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26) This space is obviously non-invariant under the transformations induced by the local Lorentz inversions $\theta^0 \to \theta^0$, $\theta^i \to -\theta^i$, and $\theta^0 \to -\theta^0$, $\theta^i \to \theta^i$. Also we can easily verify that the inversion $t \to -t$ does not leave (as a scalar transformation) the space of neutrino solutions invariant. We remark that a transformation of one group can be a symmetry of the neutrino space, while the analogous transformation in the other group is not: for instance, the simultaneous inversion $x^2 \to -x^2$, $t \to -t$ leaves the neutrino space invariant (as a scalar transformation) while the simultaneous local inversion $\theta^0 \to -\theta^0, \theta^2 \to -\theta^2$ is not a symmetry of the neutrino space.

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