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On the connection between nonextensivity and financial markets, A Langevin and Fokker-Planck analysis

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Abstract

In this manuscript one determines the first and second Kramers-Moyal moments of the Dow Jones daily index return time series and compare them with the coefficients related to a generalised Langevin equation, proposed *a priori*, whose associated Fokker-Planck equation has as solution the distribution which optimises nonextensive entropy, known as *Tsallis distribution*. The results obtained show that coefficients, from data and from dynamics, are similar confirming, thus, that the application of the current nonextensive statistical mechanics formalism in the treatment of financial systems has, in fact, a dynamical foundation.

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The intricate character of financial systems has been one of the main motives for the physicists interest in the study of their statistical and dynamical properties [1–3]. In fact, the enormous number of degrees of freedom, the intricacy of the interactions and the set of power-law behaviour empirically observed, turned financial systems, and particularly, financial markets, in a standard case of complexity [4]. Besides their own importance, those empirical observations have been the source of several new and better performing models (developed at various system scales) [5–8] and inspired the application of classical physical treatments, such as phase transitions and stochastic processes in the study of complexity [9, 10].

Though, it is now completely established that financial markets, have an anomalous dynamics, corroborated by, *e.g.*, the non-Gaussianity in the return probability density function (PDF) [11, 12] or the multi-fractal character of return time series [13, 14], therefore incompatible with the celebrated Boltzmann-Gibbs statistical mechanics, the definition of an appropriate statistical framework remained as an open question. If many authors sustain that financial markets belong to some strange dynamical class out of Lévy regime [15], another trend [16, 17] defends that, due to their main features, financial markets can be analysed within non-extensive statistical mechanics (NESM) framework based on *Tsallis statistics* [18] which already produced a collection of important results in the area [6, 19–23].

In this Rapid Note one will compare the first and second Kramers-Moyal (KM) moments of the Dow Jones (DJ) daily return time series from 1900 until 2003 with the corresponding moments of a generalised Langevin equation introduced in a NESM. We show that, instead of a *"simple fitting parameter"* [24], the entropic index q, is intimately related with the anomalous dynamics of the problem, in such a way that, for the limit $q = 1$, an ordinary Brownian dynamics, like it was proposed by Bachelier/Einstein [25, 26], is recovered.

Although considered as the ideal description, purely deterministic microscopic models, where one knows the equations of evolution for every microscopic degree of freedom and therefore one could, in principle, rigorously determine system's state are in a large majority of the cases an inappropriate way of describing the evolution of systems macroscopic observables, particularly of complex systems. One way out is to consider the description of such macroscopic variables as a combination of deterministic terms and some stochastic dependence which aims to heuristically reproduce microscopic effects on the general state of the system [27]. In this kind of description one can include stochastic differencial equations like the Langevin equation (LE), composed by drift and diffusion coefficients. From these coefficients, closely related with the microscopic dynamics of the system, one can write the corresponding Fokker-Planck equation (FPE) which describes the evolution of the macroscopic observables PDFs.

On first place one will define the macroscopic observable under analysis, the daily (log-)return of a stock market index, as

$$\tilde{r}\left(t\right)=\log\left[S\left(t\right)\right]-\log\left[S\left(t-1\right)\right],$$

where S represents the index value. For simplicity reasons, one will deal with the *normalised daily return*, $r\left(t\right)$, by subtracting the average time series return and express it in standard deviation time series units, i.e,

$$r\left(t\right)=\frac{\tilde{r}\left(t\right)-\left\langle\tilde{r}\left(t\right)\right\rangle}{\sqrt{\left\langle\left(\tilde{r}\left(t\right)-\left\langle\tilde{r}\left(t\right)\right\rangle\right)^{2}\right\rangle}}.$$

Now, let one try to describe heuristically the evolution of normalised daily returns by splitting it into the two referred parts, deterministic and stochastic. The first one represents the internal mechanisms which intend to keep the market in the average returnand that can be written as a restoring force, with a constant k, similar to the viscous force in the Langevin equation [28–30]. This term is compatible with the (fast) exponencial-like decay in the auto-correlation function for returns observed in liquid (and stable) markets [1, 15]. The stochastic term will represent the microscopic response of the system to fluctuations of the return. Since, low probability (most unexpected) return values, are those that will certainly produce more instability in the market, *i.e.*, will make it more *volatile*, it is prefectly plausible that the stochastic term may have an inverse PDF dependence and so,

$$dr=-k\,r\,dt+\sqrt{\theta\,\left[p\left(r,t\right)\right]^{\left(1-q\right)}}\,dW_{t},\qquad\left(q\geq1\right),\tag{1}$$

where W_t represents a regular Wiener process with null mean and unitary variance (using Itô convention) [31](see typical run in Fig 1). From Eq. (1) one can determine the n^{th} order KM coefficients defined as

$$D^{\left(n\right)}\left(x,t\right)=\frac{1}{n!}\lim_{\tau\rightarrow0}\frac{1}{\tau}\left.M_{n}\left(x,t,\tau\right)\right|_{r\left(t\right)=x},\tag{2}$$

(where $M_n\left(x,t,\tau\right)=\left\langle\left(r\left(t+\tau\right)-x\right)^{n}\right\rangle$ is n^{th} order KM moment) namely the first, $D^{(1)}\left(r,t\right)$ and the second

$D^{(2)}(r,t)$ yielding [27],

$$D^{(1)}(r,t) = -k\,r, \quad D^{(2)}(r,t) = \frac{1}{2}\theta\,[p(r,t)]^{(1-q)}\,. \tag{3}$$

The respective FPE [33] is

$$\frac{\partial p(r,t)}{\partial t} = \frac{\partial}{\partial r}[k\,r\,p(r,t)] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left\{\theta\,[p(r,t)]^{(2-q)}\right\}. \tag{4}$$

If one assumes, as it is generally done, that financial markets are in a stationary state, one can restrict oneself to the evaluation of Eq. (4) stationary solution [34, 35] which is

$$p(r) = \frac{1}{Z}\left[1-(1-q)\,\frac{k\,r^2}{(2-q)\;Z^{q-1}\,\theta}\right]^{\frac{1}{1-q}}, \tag{5}$$

with

$$\theta = \frac{k\,(5-3\,q)\,\sigma^2\left\{\frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]}\sqrt{\frac{(q-1)}{(5-3\,q)\pi\,\sigma^2}}\right\}^{q-1}}{2-q} \tag{6}$$

Z is the normalisation constant and σ the PDF width. PDF 5 optimises Tsallis entropy [18],

$$S_q = \frac{1-\int[p(r)]^q\;dr}{q-1},$$

under appropriate constrains [36] and that has already been applied to adjust return PDF of several markets [37].

If the dynamics proposed in Eq. (1) is approriate to reproduce return time series, then the KM moments obtained from Eq. (3) and those obtained from the time series by

$$M_n(r,t,\tau) = \int (r'-r)^n\;P(r',t+\tau|r,t)\;dr' \tag{7}$$

should be, at least similar (See PDFs in Fig. 2 and Fig. 3). In other words, at the steady state,

$$M_1 \approx -\tau\,k\,r = -\tilde{k}\,r, \tag{8}$$

i.e., a straight line with slope $-\tilde{k}$ and

$$M_2 \approx \tau\,\theta\,[p(r)]^{(1-q)} = \frac{\tilde{k}}{2-q}\left[(5-3\,q)\,\sigma^2+(q-1)\,r^2\right], \tag{9}$$

which is a second order polynomial. As can be observed from symbols in Fig. 4 and Fig. 5 the KM moments, actually, *do present such linear and parabolic dependence!* Defining a normalised 2^{nd} KM moment as $\mathcal{M} = |M_2/M_2^{min} - 1|$ one verifies some discrepancy from theoretical curve $\mathcal{M} = r^2$ for a small interval between -0.1 and 0.1 which represents the slight skewness of the PDF. Besides DJ, such behaviour in the KM moments were also verified in other indices [28, 29, 38, 39]. It is important to highlight that the *substancial* difference between this and other works is that the KM moments (functionals) are defined *a priori* from the dynamics instead of being the outcome of some fitting procedure.

The determination of θ and q is made by computing the second-order moment and the kurtosis, $\kappa = \langle r^4 \rangle / \langle r^2 \rangle^2$ which *only depends* on the value of q. For the DJ daily return PDF one obtained $\sigma = 1.14$ and $q = 1.54$ (Fig. 2). From the first KM moment one can determine the restoring constant $k = 1.06$ and then the *volatility constant* $\theta = 0.883$ from Eq. (6). For $q = 1$, from Eq. (9) we get $D^{(2)} = \theta/2$, *i.e.*, a constant degree of volatility and hence a Gaussian PDF which only appears when one analyses returns for large aggregated times. This aggregation is related to the convolution of several returns associated to a Tsallis distribution with $q \leq 5/3$, leading hence to the Gaussian distribution [39]. An on-going analysis inspired in Drost and Nijman work on the convolution of GARCH processes [40], points that new r' variables generated by aggregating time series elements given by Eq. (1) can be mapped onto another Eq. (1) with a smaller q value which approaches unit as the lag increases like it seems to happen for data.

To summarise, one has analysed a possible connection between dynamical and statistical properties of the DJ daily return time series and an *a priori* proposed Langevin-like stochastic differential equation which has emerged within NESM framework. This model provides a fairly good agreement with both time series PDF and Kramers-Moyal moments explaining why models based in akin equations, like Borland's option-pricing theory [6], give so remarkable results. Despite other dynamics are capable to afford the same PDF [41, 42], the one presented herein, seems to be the only which has such KM moments and it is naturally dependent on the *volatility degree*. In other words, large returns, which correspond to low values in the PDF, are those that introduce large degrees of unstability and make the market more volatile leading to subsequent large positive/negative return values. Further improvements of this model should take into account the well-known slight asymmetry in the PDF (usually related to risk-aversion) as well as the subtle feedback effects that allow the implementation of artitrage on markets. Nevertheless, it is worthy to emphasise the simplicity of the model and its aptitude to provide a quite good portrait of financial markets.

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[1] J.P. Bouchaud and M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management* (Cambridge University Press, Cambridge, 2000).

[2] R.N. Mantegna and H.E. Stanley: *An introduction to Econopysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambrigde, 1999).

[3] M. Gell-Mann and C. Tsallis, *Nonextensive Entropy - Interdisciplinary Applications* (Oxford University Press, New York, 2004).

[4] J. Doyne Farmer, Int. J. Appl. Fin. **3**, 311 (2000).

[5] M. Potters, R. Cont and J.-Ph. Bouchaud, Europhys. Lett. **41**, 239 (1998).

[6] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002).

[7] V. Plerou, P. Gopikrishnan, X. Gabaix and H.E. Stanley, Phys. Rev. E **66**, 027104 (2002).

[8] J. Doyne Farmer, L. Gillemot, F. Lillo, S. Mike and A. Sen, Quantit. Finance **4**, 383 (2004).

[9] J. Voit, *The Statistical Mechanics of Financial Markets* (Springer-Verlag, Berlin, 2003).

[10] H.E. Stanley, L.A.N. Amaral, S.V. Buldyrev, P. Gopikrihnan, V. Plerou and M.A. Salinger, Proc. Nat. Acad. Sci. USA **99**, 2561 (2002).

[11] R.N. Mantegna and H.E. Stanley, Nature **376**, 46 (1995).

[12] S.M. Duarte Queirós, Physica A **344**, 279 (2004).

[13] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner and Y. Dodge, Nature **381**, 767 (1996).

[14] A. Arnéodo, J.-F. Muzy and D. Sornette, Eur. Phys. J. B **2**, 277 (1998).

[15] H.E. Stanley, L.A.N. Amaral, X. Gabaix, P. Gopikrishnan and V. Plerou, Physica A **299**, 1 (2001).

[16] C. Tsallis, C. Anteneodo, L. Borland and R. Osorio, Physica A **324**, 89 (2003).

[17] S.M. Duarte Queirós, C. Anteneodo and C. Tsallis in Noise and Fluctuations in Econophysics and Finance, D. Abbot, J.-P. Bouchaud, X. Gabaix and J.L. McCauley (eds.), Proc. of SPIE **5**848, 151 (SPIE, Bellingham, WA, 2005).

[18] C. Tsallis, J. Stat. Phys. **52** , 479 (1988). Bibliograhy

http://tsallis.cat.cbpf.br/biblio.htm.

[19] L. Borland, Quantit. Finance **2**, 415 (2002).

[20] R. Osorio, L. Borland and C. Tsallis in *Nonextensive Entropy - Interdisciplinary Applications*, M. Gell-Mann and C. Tsallis (eds.) (Oxford University Press, New York, 2004), 321.

[21] C. Anteneodo, C. Tsallis and A.S. Martinez, Europhys. Lett. **59**, 635 (2002).

[22] S.M. Duarte Queirós and C. Tsallis, Europhys. Lett. **69**, 893 (2005).

[23] S.M. Duarte Queirós, Europhys. Lett. **7**1, 339 (2005).

[24] A. Cho , Science **297**, 1268 (2002) 1268; S. Abe and A.K. Rajagopal; A. Plastino; V. Latora, A. Rapisarda and A. Robledo, *idem* **300**, 249-251 (2003).

[25] L. Bachelier, *Théorie de la spéculation,* Ann. Sci. École Norm. Sup. III-17, 21 (1900).

[26] A. Einstein, Ann. der Phys. **17**, 549 (1905).

[27] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, 2^{nd} edition (Springer-Verlag, Berlin, 1989).

[28] R. Friedrich, J. Peinke and Ch. Renner, Phys. Rev. Lett **84**, 5224 (2000).

[29] K. Ivanova, M. Ausloos and H. Takayasu in *The Applications of Econophysics*, H. Takayasu (ed.) (Springer Verlag, Berlin, 2004), 161.

[30] J.P. Bouchaud and R. Cont, Eur. Phys. J. B **6**, 543 (1998).

[31] Albeit this equation can be transformed in another with a additive-multiplicative noise structure (Ref. [27] sec. 3.4 or [32]), one will keep this form because it is quite more physical.

[32] C. Anteneodo and C. Tsallis, J. Math. Phys. **44**, 5149 (2003).

[33] L. Borland, Phys. Rev. E **57**, 6634 (1998).

[34] A.R. Plastino and A. Plastino, Physica A **222**, 347 (1995).

[35] C. Tsallis and D.J. Bukman, Phys. Rev. E **54**, (R)2197 (1996).

[36] C. Tsallis, Braz. J. Phys **29**, 1 (1999).

[37] See the list of works at the URL of [18].

[38] M. Ausloos and K. Ivanova, Phys. Rev. E **68**, 046122 (2003).

[39] S. M. Duarte Queirós, Quantit. Finance (in press), 2005.

[40] F.C. Drost and T.E. Nijman, Econometrica **61**, 909 (1993).

[41] L. Borland, Phys. Lett. A **245**, 67 (1998).

[42] C. Beck, Phys. Rev. Lett **87**, 180601 (2001).

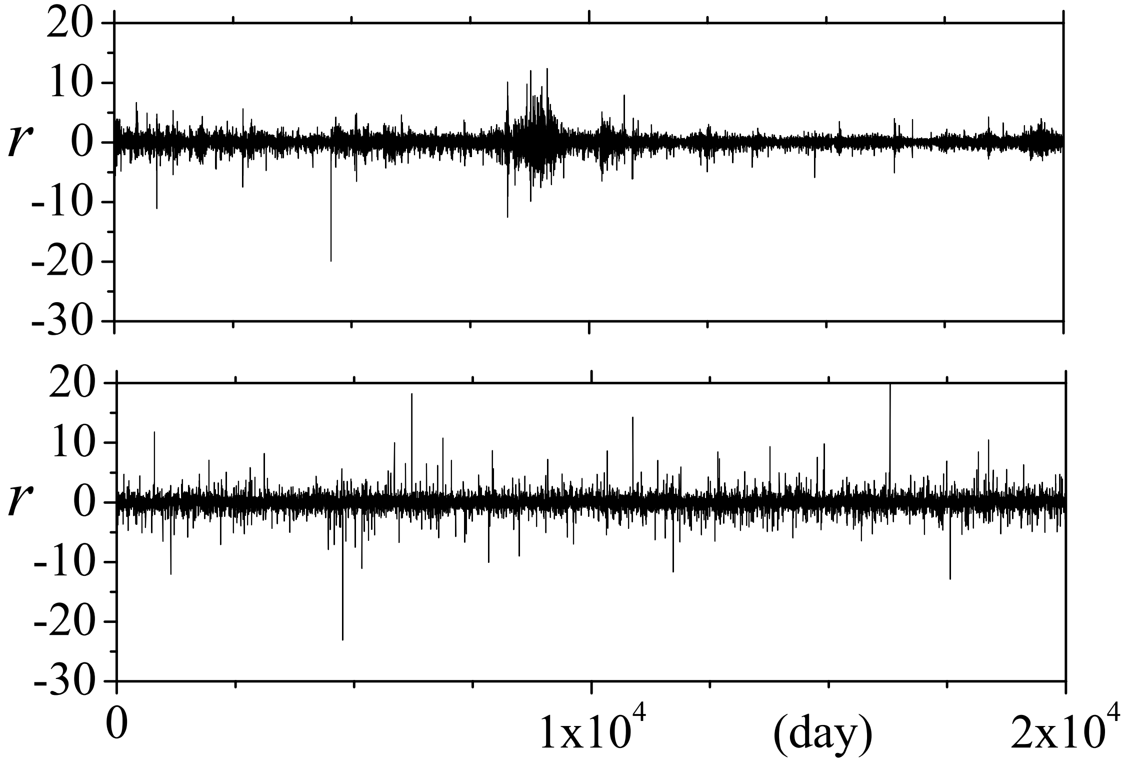

FIG. 1: Upper Plot: Time series of the 20k first days of the DJ daily return beginning in 1900. Lower Plot: A numerical realisation of Eq. (1) for typical values of $k = 1.06$, $\theta = 0.883$ and $q = 1.54$ ($t_0 \ll 0$).

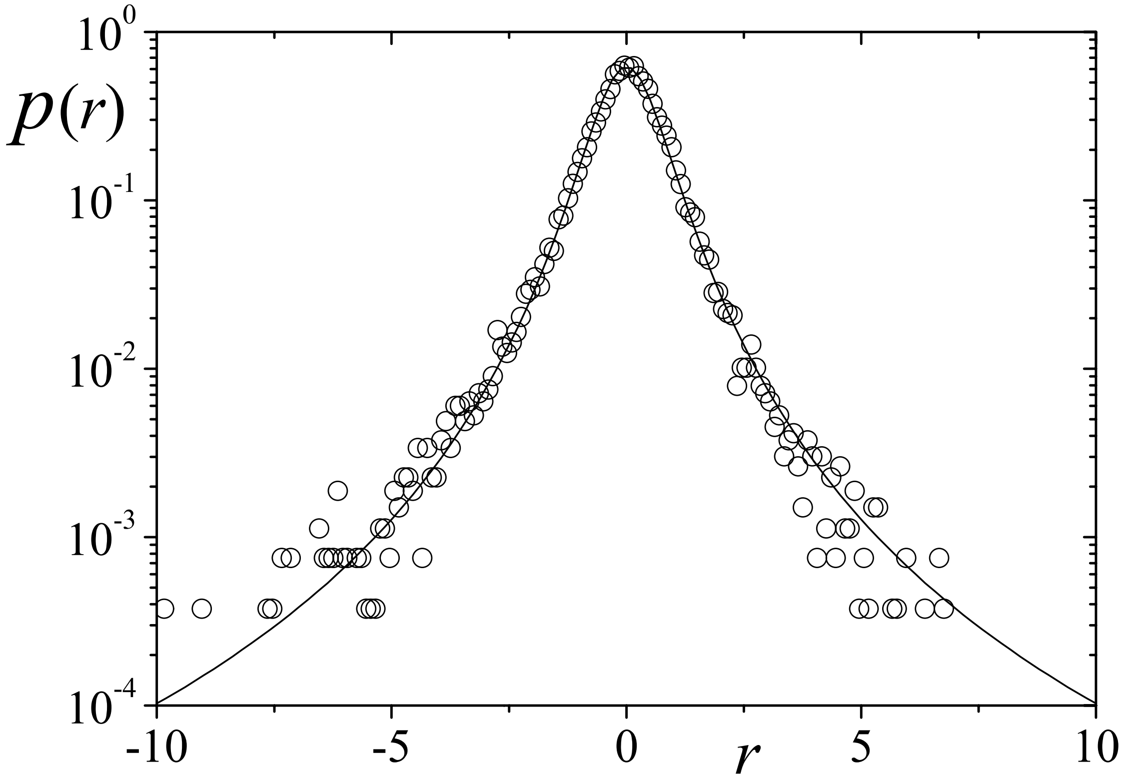

FIG. 2: Probability density function for the DJ daily return *versus* return from 1900 until 2003 (symbols). The line represents the best q-Gaussian (Eq. (5)) with $q = 1.54$ and $\sigma = 1.14$.

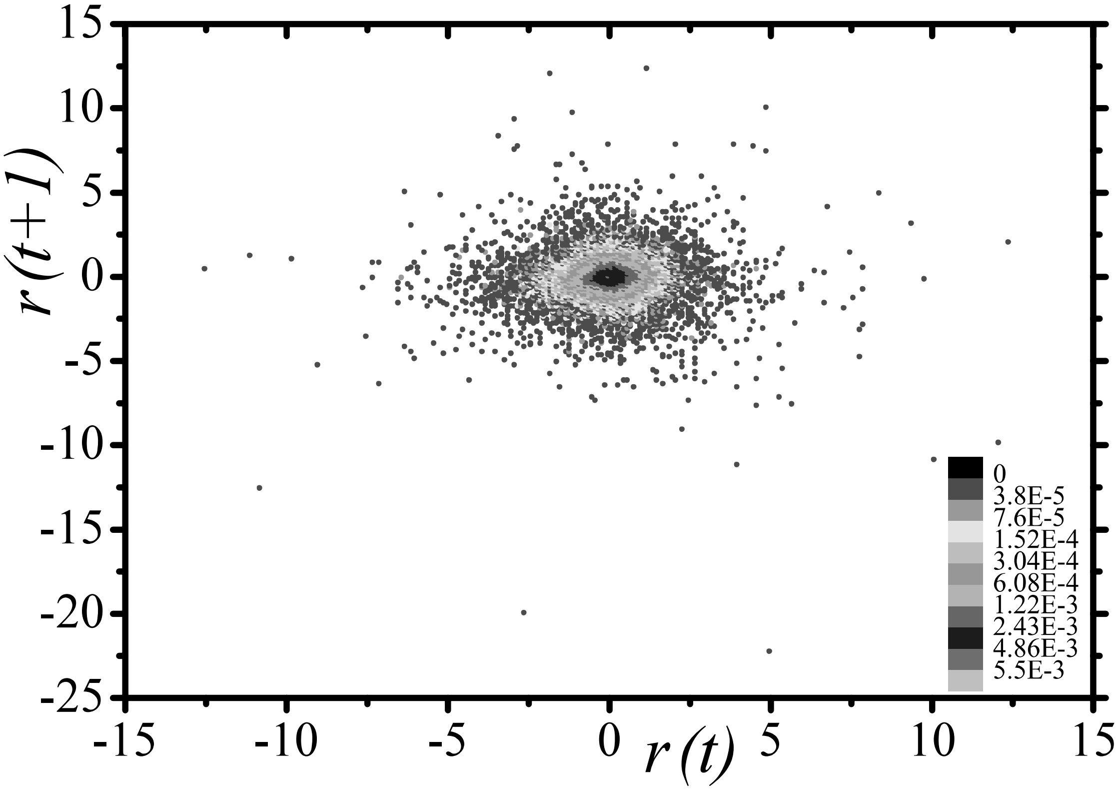

FIG. 3: A map representation of the immediate DJ daily return conditional probability density function $p\left(r\left(t\right)|r\left(t+1\right)\right)$.

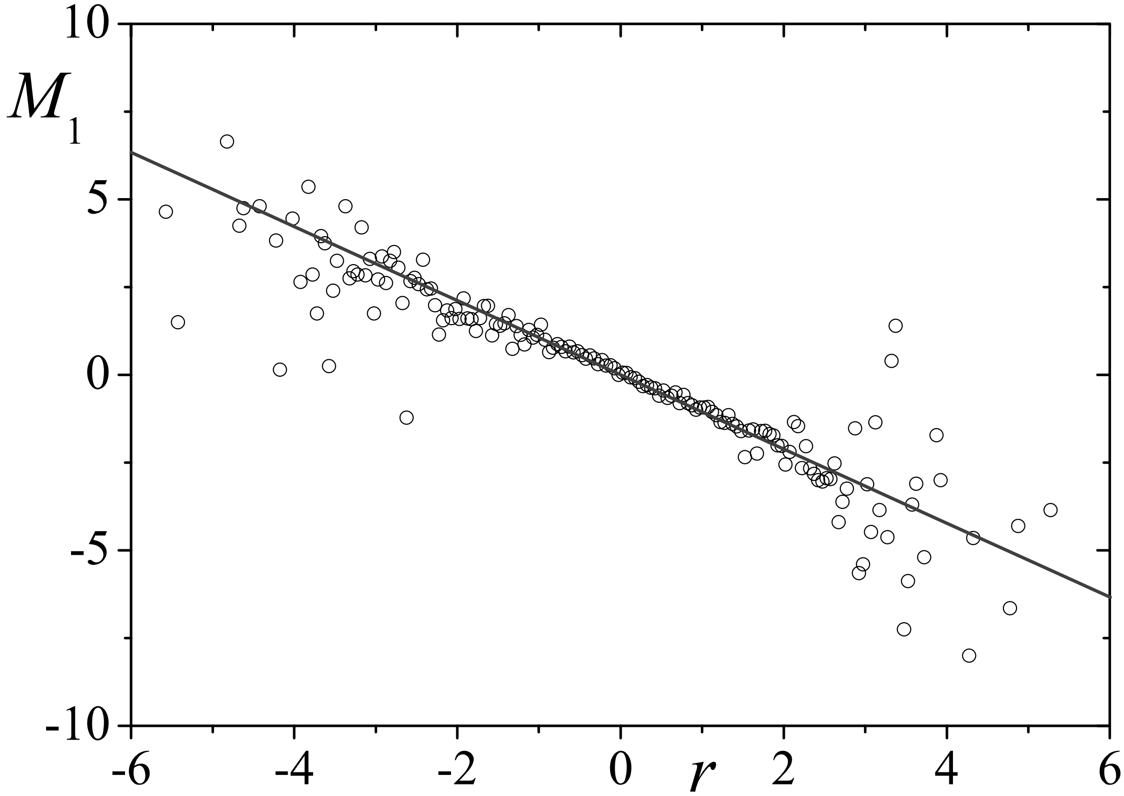

FIG. 4: First Kramers-Moyal moment for DJ daily return time series (symbols), M_1, *versus* return. The line represents Eq. (8) with $k = 1.06$.

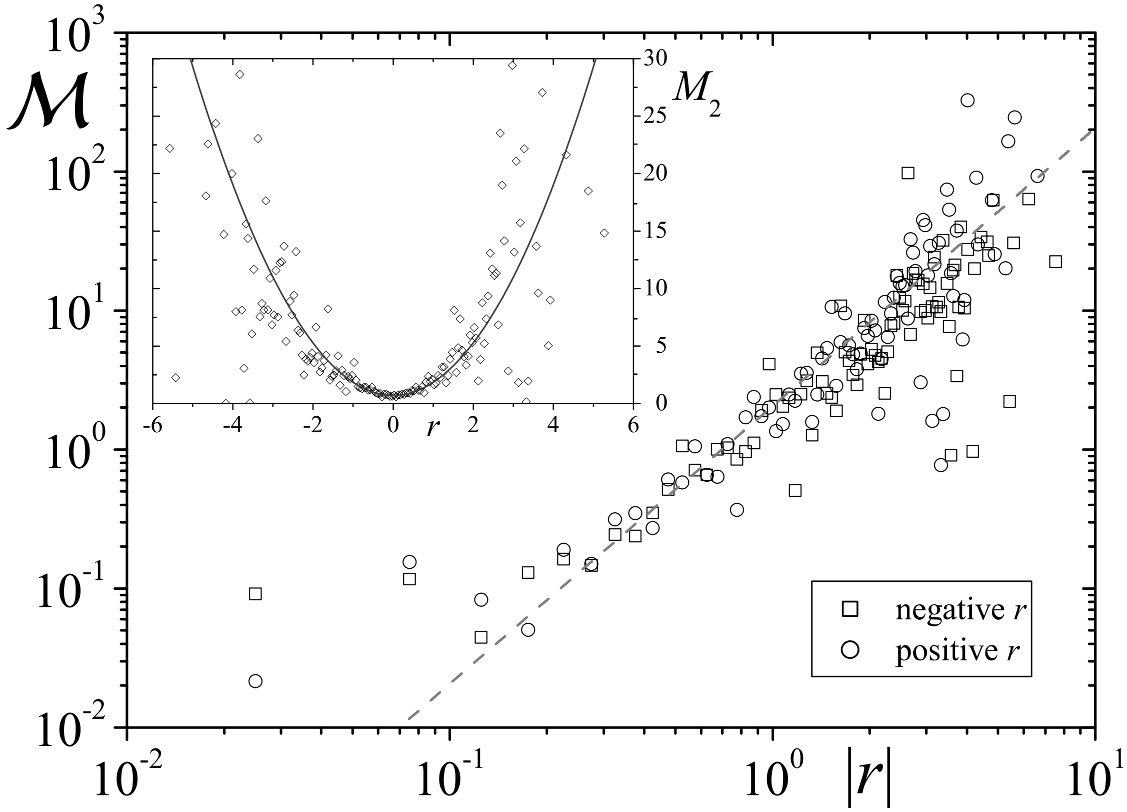

FIG. 5: Symbols $\mathcal{M}$ *versus* absolute return; dashed line r^2 curve in $\log - \log$ scale. The agreement between 0.1 and 10 is rather good. Inset: Second Kramers-Moyal moment for the DJ daily return time series (symbols), M_2, return. The line represents Eq. (9) with $\theta = 0.883$.