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by

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0.1 Introduction

A new monography or review on an old and established subject as QFT should justify and measure itself relative to the many existing review articles and textbooks. The main motivation underlying these notes consists in the desire to unify two presently largely disconnected branches of QFT: (1) the standard (canonical, functional) approach which is mainly perturbative in the sense of an infinitesimal "deformation" of free fields[1] and (2) nonperturbative constructions of low-dimensional models as the formfactor-bootstrap approach (which for the time being is limited to factorizable models in d=1+1 spacetime dimensions) and the non-Lagrangian construction of conformal chiral QFT's..

The synthesis requires a significant step byond the concepts which were used in order to formulate the two mentioned separate branches. On the physical side the S-matrix regains some of its early prominence, however unlike in the proposals of Heisenberg (and later in the S-matrix bootstrap approach of Chew

[1]Unfortunately only in the sense of an infinitesimal deformation around free fields, and not in that of an approximation of an existing QFT.

et al.), here it remains subservient to the locality and causality principles encapsulated in the theory of local observables (the heartpiece of algebraic QFT). In the new context of the TCP- and the Tomita-reflection operators explained in these notes, the S-matrix takes the role of a powerful constructive tool in QFT. In particular local fields turn out to be just "coordinatizations" of local algebras, and different fields in the same local class describe the same physics. In addition to the charge superselection rules (including statistics and spin), the S-matrix reveals itself as the most valuable "net" invariant of the field net which incorporates all charge sectors.

On the mathematical side one needs the Tomita-Takesaki modular theory which is closely related to the concept of KMS states in Quantum Statistical Mechanics. In our present context they will be mainly used to implement localization and geometrical properties in Local Quantum Physics. It is the proximity of these concepts to TCP symmetry and other fundamental notions in QFT which gives confidence in our approach. The change of paradigm which accompanies the new approach is that those structures of QFT which were always thought of as fundamental but "kinematical" properties as TCP and Spin & Statistics, now move into the center of the "dynamical" stage. Of course such statements should be taken with the full awareness that the cut between "kinematics" and what was hitherto considered as "dynamics" was never as rigid and a priori as the textbooks make us believe.

A new approach is also expected to cast some new light on past successes. For this reason we will attempt a rather systematic presentation which includes a substantial part of the standard material of QFT (especially chapter 2), but occasionally somewhat different from the standard textbook treatments by emphasis and interpretation. Since the new method (build on old physical principles of local quantum theory) is still in its infancy, it is to early to expect an exhaustive treatment of the algebraic approach and the modular construction method.

In fact QFT, despite its age, is still in its conceptual infancy, especially from the new viewpoint. This goes contrary to the widespread (but unfounded) belief that the basic equations of the world below the Planck scale are already known, and that if we would be more clever in our computations (bigger computers etc.) we would have a description of all phenomena below that scale.

It is interesting to understand how this overly optimistic (and somewhat dull) picture about QFT developed in time. The greatest success of QFT and the raison d'etre for its continuing existence was certainly the renormalization of QED and its more recent extension via the standard model [2]. But it is helpful to remind oneself that in physics (and also outside) success and disaster often are close together. A big step forward naturally lends itself to an extension of the formalism through which it was obtained. In the case at hand, the perturbation theory was streamlined by the functional formulation without which one would

[2]However the remarkable aspect of the electro-weak theory beyond QED is not so much its precision, but rather its power to combine a overwhelming body of experimental facts into a reasonable quantum field theoretical shell which appears to be a good starting point for future conceptual advances.

be unable to understand most contemporary publications. To press ahead with a successful formalism and to see what it leads to, if extended beyond laboratory energies, is of course in the best tradition of theoretical physics, even if at the end it should turn out to be the wrong path.

Since formal extensions are easy and take less time than conceptional investigations, their rapid pursuit is very reasonable indeed. It only may lead to desaster if this develops into the main research topic for several generations and the if broad conceptual basis which led to the first success gets narrowed in its potential *conceptional* richness and at the end becomes lost with the younger generation. Whether we already reached this state of physical stagnation and crisis is left up to the judgement of the reader.

In any case this report is not written for those who still (after a quarter century of post standard model stagnation) are convinced that the extension of the standard formalism is all one needs in order to make progress. It rather tries to attract those physicists who, on the one hand still have not lost their conceptual curiosity, and on the other hand feel uneasy about the present predominance of formalism over conceptional insight. The present author has tried to analyse the contemporary situation in terms of the lost balance between the conceptual "Bohr-Heisenberg"- and the esthetical mathematical "Dirac"-approach[3].

Hopefully the reader will be able to appreciate the use of the unifying physical point of view which is encapsulated in the "local nets" and which gives e.g. all the low-dimensional richness without invoking special structures (as e.g. Virasoro- and Kac-Moody- algebras) which are restricted to low space-time dimensions only. After all, the main physical *value of low dimensional models is their use for a better conceptual and structural understanding of general QFT* (and perhaps also for basing universality explanations in condensed matter physics on firm grounds) and only in second place the increase of our mathematical culture. For this reason the presentation of low dimensional models (in particular chiral conformal QFT) may appear different from the way the reader may have met this material in other reviews.

Some ideas which enjoyed or still enjoy great popularity as e.g. supersymmetry and the Seiberg-Witten duality will not be presented here. The reason is not that they lack an experimental basis nor that I am against trends and fashions, but rather that I know no compelling *theoretical* principle which leads to such ideas. To the contrary, as recent investigations of the thermal behaviour of supersymmetry showed (D. Buchholz and I. Ojima: "Spontaneous Collapse of Supersymmetry", March 1997 University of Kyoto preprint), this symmetry is completely unstable in thermal KMS states[4].

Whereas the Lorentz-symmetry is only spontaneously broken as a result of the presence of a heat-bath rest frame, the space-time symmetries beyond ("super") are wiped out without leaving any intrinsic physical trace behind. This seems to be a general property, not just of spinorial currents, but of other non-

[3] B.Schroer "Motivations and Physical Aims of Algebraic QFT" hep-th 9608083, to appear in Ann.of Phys. March 1997

[4] The usual (broken) symmetry picture, which holds for locally generates internal and external symmetries, does not apply here.

causal currents as e.g. the nonlocal conserved axial current of the Schwinger model which is responsible for the chiral symmetry and its breaking through the "seized" θ−vacua. Certainly temperature physics has higher physical priority than supersymmetry and perhaps nonlocal currents reveal their unphysical nature mainly via temperature instabilities. This there seems to be a strong theoretical veto in addition to the experimental situation.

The reason for the absence of the Seiberg-Witten Duality in these notes is not only its proximity to supersymmetry, but also the fact that it is build on global concepts as "effective potentials" whose relation to the structural properties of real time QFT I do not understand (nor was anybody able to explain this to me). It is not reasonable for an autor to comment on observations which he only understands formally mathematically, but not in terms of physical concepts.

On the other hand the reader will find a detailed presentation of "Haag Duality" which is a pivotal property of nonperturbative local QFT and is known to lead to such fundamental issues as (braid group) statistics and an intrinsic understanding of spontaneous symmetry breaking. In d=1+1 we also study its relation to the Kadanoff order-disorder Duality which is the local version of the global Kramers-Wannier Duality in lattice systems. In these notes we will not sacrifice physical notions as equivalence classes of real time fields and TCP- and Tomita J-reflections in favour of (imaginary time) cohomology or S- and T-reflections.

I elaborated this material in the conviction that the best strategy in a time of stagnation and crisis is to return to the roots of QFT and re-analyse the underlying principles in the light of new concepts. For this reason we put heavy emphasis on Wigner's approach to particles which, long before algebraic QFT, was the first successful attempt to do relativistic quantum physics in an intrinsic fashion i.e. without recourse to quantization. Another cross road which merits careful scrutiny, is the concept of local gauge invariance.

As a result of lack of time, I was not able to write a planned mathematical appendix on nets on von Neumann algebras and their representation theory. In section 3 of the first chapter the reader finds some material on finite dimensional C^*-algebras. For the general case I refer the reader to a review article by Roberts and some sections of the book of Bratteli and Robinson. A survey of important mathematical structures of local quantum physics including more references can also be found in the book of Haag. the reader will find references at the end of each chapter. Lack of time also prevented me from elaborating certain other matters, but at least they will be briefly mentioned in the text at their logical position.

I am grateful to Prof. José Helayel for arranging this visit of the CNBPF at rather short notice.

Anybody who knows me, understands that my attachment to Brazil is not only due to its natural beauty, but there are also deep scientific roots. With feelings of nostalgia I remember those wonderful years (1968-1980) of collaboration with J.A.Swieca. These were times of free-roaming scientific endevour, long before the globalization of physics, which nowadays forces young physicist to build their carrier around some trendy formal ideas, started to do uncalculable harm

to QFT.

Never before in the history of QFT it took so many researchers in QFT to achieve so little of physical weight as in the post "electro-weak period". A small consolation is that at least some mathematicians seem to be able to build their beautiful mathematical castles on top of our physical ruins.

This presentation should not be misread as a moral judgement against one or the other approach to QFT. It rather is an attempt to revive some of the conceptually based Bohr-Heisenberg-Wigner spirit in the present times of "everything goes" (as long as it lives up to that entertaining high caliber scientific journalism which characterizes parts of QFT in the post standard model globalized physics[5]).

[5] I strongly advice any young person still in search of a permanent position not to say such things in public.

Chapter 1

The Basics of Quantum Theory

1.1 Multiparticle Wave Functions, Particle Statistics and Superselection Sectors

It is well-known that quantum mechanical multiparticle systems can be obtained by the tensor product construction from one-particle spaces. Let :

$$\psi(x,s) \in \mathcal{H}_1 = \mathcal{L}^2\left\{\mathcal{R}^3, \{-s,\ldots,+s\}\right\} \tag{1.1}$$

be a wave function of a nonrelativistic particle of spin s where, as usual, s_3 is the component of spin with respect to an arbitrarily chosen z-quantization axis. The L^2−space has the standard inner product

$$(\psi_2,\psi_1) := \sum_{s_3=-s}^{s} \int d^3x \psi_2^*(x,s_3)\psi_1(x,s_3) \tag{1.2}$$

and norm $||\psi|| = (\psi,\psi)^{\frac{1}{2}}$

The two-particle space:

$$\mathcal{H}_2 = \mathcal{H}_1^a \overline{\otimes_C} \mathcal{H}_1^b \qquad \text{a,b..type of particle} \tag{1.3}$$

is simply the closure of the algebraic tensor product over the complex numbers generated by vectors: $\sum_{ik} \psi_i^a \otimes_C \psi_k^b$ equipped with the scalar product (in the following we omit the subscript C) induced by the formula:

$$(\psi_{i'}^a \otimes \psi_{k'}^b, \psi_i^a \otimes \psi_k^b) := (\psi_{i'}^a, \psi_i^a)\cdot(\psi_{k'}^b, \psi_k^b) \tag{1.4}$$

continued by linearity in the right and antilinearity in the left factor. The generalization to N factors:

$$\mathcal{H}_N = \mathcal{H}_1 \overline{\otimes} \mathcal{H}_1 ..... \overline{\otimes} \mathcal{H}_1 \tag{1.5}$$

is straightforward . The tensorproduct structure is the mathematical formulation of kinematical "statistical independence" in the sense of quantum theory. The dynamical variables of the subsystems (say particles with spin) at a given time factorize:

$$\vec{X}_i = 1 \otimes ... \otimes \vec{x}_i \otimes 1 ... \otimes 1 \quad \text{and similarly for } \vec{P}_i \text{ and } \vec{S}_i \tag{1.6}$$

They act on N-particle wave functions $\psi(x_1 s_1, ...x_N s_N)$ with s_i the z-component of the i^{th}spin. Global symmetry transformations, as translations and rotations, act multiplicatively :

$$(U(\vec{a})\psi)(x_1 s_1, ...x_N s_N) = (U_1(\vec{a}) \otimes ... \otimes U_1(\vec{a})\psi)(x_1 s_1, ...x_N s_N) \tag{1.7}$$

$$(U(\vec{n},\theta)\psi)(x_1 s_1, ...x_N s_N) = (U_1(\vec{n},\theta) \otimes ... \otimes U_1(\vec{n},\theta))\,\psi(x_1 s_1, ...x_N s_N) \tag{1.8}$$

$$U(a) = e^{i\vec{P}\vec{x}}, \qquad \vec{P} = \sum_i 1 \otimes ... \otimes \vec{P}_i \otimes ... \otimes 1 \tag{1.9}$$

$U(\vec{n},\theta) = e^{i\vec{J}\vec{n}\theta}$, $\quad \vec{J} = \sum_i 1 \otimes ... \otimes \vec{J}_i \otimes ... \otimes 1$, $\quad \vec{J}_i = \vec{x}_i \times \vec{p}_i + \frac{1}{2}\vec{\sigma}_i$

Operators which implement an interaction as the hamiltonian violate this factorization:

$$U(t) = e^{-iHt}, \quad H = H_0 + H_{int}, \quad H_{int} = \sum_{i<k} 1 \otimes ... V_{ik}(x_i - x_k) ... \otimes 1 \tag{1.10}$$

In the last expression the identity operator at the i^{th} and k^{th} place have been replaced by a conventional local pair interaction which, in the exponentiated form U(t), looses the pairing property of the infinitesimal generator. Therefore at first sight it appears, that the localized pair-interaction leaves no mark at all on U(t). Fortunately this is not quite correct, rather its marks are the important "cluster properties". In our context they are :

$$H_N \to H_{N_1} \oplus H_{N_2}, \quad \Omega_N \to \Omega_{N_1} \otimes \Omega_{N_2}, \quad S_N \to S_{N_1} \otimes S_{N_2}$$
$$\text{on clustering wave functions:}$$
$$lim_{a\to\infty}\psi(x_1,....x_{n_1}, x_{n_1+1} + a, ...x_{N_1+N_2} + a) \tag{1.11}$$

Here $\Omega^{\pm} = \lim_{t\to\infty} e^{iH_0 t}e^{-iHt}$ are the Møller operators and S is the S-matrix $S = \Omega^{+*}\Omega^{-}$. Equivalently one may introduce a partial translation $U_{C_{N_2}}(a)$ which translates the particles in the N_2-cluster infinitely far away from the rest:

$$\lim_{a\to\infty} U_{C_{N_2}}(a) H_N U^*_{C_{N_2}}(a) = H_{N_1} \oplus H_{N_2},$$
$$\text{and similarly } for\ e^{iHt}\ ,\ \Omega^{\pm} \text{ and S with } \otimes \text{ instead of } \oplus \tag{1.12}$$

The cluster property is therefore a kind of asymptotic factorization expressing statistical independence for long distances. As we will see, it follows from more fundamental locality structures in QFT. Although it is trivially satisfied for short- range quantum mechanical interactions, it cannot always be taken for

granted if the interactions become long-range. In that case the cluster decomposition requirement is expected to affect the boundary conditions of scattering theory. For example for the relative Aharonov-Bohm interaction between "dyons" (electrically and magnetically charged particles), the cluster decomposition property is expected to become a nontrivial imposition on the a priori unknown scattering boundary conditions ($\neq$ plane waves) on multiparticle scattering states. These quantum theoretical subtleties are of course irrelevant for the calculation of the Aharonov-Bohm phase shift, which can be understood in entirely quasiclassical terms. In the applications to braid group statistics of dyons however, that scattering boundary condition is the correct one, which yields the cluster decomposition property of the scattering amplitude, thus assuring that the N-particle S-matrix passes to the previously determined (N-1)-particle S-matrix upon removal of one particle to infinity. In this way the N-particle problem becomes related to the N-1-particle problem via the cluster property, even though there is no actual creation of particles. Hence what appeared at the beginning as a standard quantum mechanical problem, turns out to be a problem in which particle states for all N enter, i.e. a problem with a field theoretic aspect. In local QFT, cluster properties are a consequence of the general framework. In fact it is appropriate to think of the cluster decomposition property as the relic of "local quantum physics"(Haag) in the nonrelativistic limit. The statistics of particles must have this inclusive structure and the fact that the identity of particles is expressed by the braid group B_∞ which is the inductive limit of .. $\subset B_i \subset B_{i+1} \subset$ ... (the same holds for the permutation group S_∞ which is a special case of the braid group) reflects precisely the inclusive "russian matrushka" structure of the clustering on an algebraic level.

It is well-known that the dynamical variables obtained as tensor products form an irreducible system (~validity of Schur's lemma) if the single particle variables have this property in the one-particle Hilbertspace (true in our example of $\vec{p}$, $\vec{x}$, $\vec{s}$). In that case the requirement of identity of particles permits only one-dimensional representations of the permutation group i.e. Fermions and Bosons. In order to understand this, we have to incorporate the notion of indistiguishability into our tensor products. Writing simply $\psi_n(1,2,...n)$ as a short hand for our x- and spin-dependent wave functions, a permuted wave function (the P act on the indices)

$$\begin{aligned}(U(P)\psi)_n(1,2,...n) &:= \psi_n^P(1,2,...,n) := \\ &= \psi_n(P^{-1}(1),P^{-1}(2),...;P^{-1}(n))\end{aligned} \tag{1.13}$$

must describe the same physical state, even though the permutation leads to a different vector in Hilbertspace. This distinction between vectors and physical states (in the sense of defining expectation values on the algebra of observables) is crucial for the understanding of the (later presented) superselection rules. In terms of observables, this simply means that the $U(P)'s$ commute with the observables. If, as in standard Schrödinger theory, the observables form an irreducible set of operators, the representation $U(P)$ of the permutation group

S_n must be abelian and hence:

$$\begin{aligned} \psi_n^P(1,2,...n) &= \omega(P)\psi_n(1,2,...n) \\ \omega(P) &= \begin{cases} 1 & Bose \\ sign(P) & Fermi \end{cases} \end{aligned} \tag{1.14}$$

Mathematically this statement is a tautology and physically there is no reason to assume irreducibility. In fact the occurrence of nonabelian representations of S_n is equivalent to the appearance of reducible quantum mechanical observable algebras. In order to construct such a nonabelian example, a rudimentary knowledge of the representation theory of the permutation group is helpful. The equivalence classes of irreducible representations of S_N are characterized by partitions:

$$N_1 \geq N_2 \geq .... \geq N_n\,, \quad \sum_i N_i = N \tag{1.15}$$

They are pictured by so called Young tableaus, an array of N boxes in rows of nonincreasing size (the admissibility condition for tableaus) see Fig.1.

The representation of S_N corresponding to each tableau of depth d is most conveniently described by decomposing the natural representation on the N-fold tensor product of a d-dimensional complex vector space $V^{\otimes N}$ (of dimension d·N) into irreducibles. Up to equivalencies one obtains all irreducible components by applying S_N to the vector:

$$e_1^{\otimes(N_1-N_2)} \otimes (e_1 \wedge e_2)^{\otimes(N_2-N_3)} \otimes .... \otimes (e_1 \wedge e_2 ... \wedge e_d)^{\otimes N_d} \qquad e_1....e_d \text{ basis in V} \tag{1.16}$$

with the tensor product action:

$$U(P)\xi_1 \otimes \xi_2 \otimes .... \otimes \xi_N = \xi_{P(1)} \otimes \xi_{P(2)} .... \otimes \xi_{P(N)}\,, \qquad \xi_i \in \{e_1....e_d\} \tag{1.17}$$

The cyclically generated subspace is the irreducible representation space corresponding to 1.16. Here the $\wedge$ designates the wedge product leading to completely antisymmetric tensors. An "admissible" numbering of a given tableau is any numbering which is increasing in each row and column. The different admissible numbering correspond to the multiplicity with which the irreducible representation occurs in the regular representation (which is higher dimensional than the above natural representation) of the group algebra CS_N (see next section).

Another method to construct the representation theory is the inductive construction of tableaus according to Schur's rules of adding a small box to a previously constructed tableaux:

$$\pi_T \times box = \oplus_{adm.} \pi_{T+box} \tag{1.18}$$

The admissible ways to add a box are such that the resulting tableaus are admissible in the aforementioned sense. But now there is no maximal height d and

instead of the natural representation on tensor products on d-dim. vectorspaces we are inducing the so called regular representation. The iteration starting from n=1 gives a reducible representation of S_N with $m(T_N)$=multiplicity of occurrance of the tableaux T_N:

$$\oplus_{T_N} m(T_N)\pi_{T_N}, \tag{1.19}$$

The multiplicity is the same as that of T_N in the group algebra CS_N (next section). It is obviously equal to the number of possibilities to furnish admissible numbering (the box numbering inherited from the inductive Schur construction). It has a group-theoretic significance because it agrees with the size of a certain conjugacy class.

Now we are able to scetch a (rather trivial) counterexample against the irreducibility of the observable algebra. Imagine that we have Bose particle which carry besides spin a "hidden" quantum number ("flavor"or "color") i.e. an internal degree of freedom which can take d values. Assume that the measurability is restricted to flavor neutral operators:

$$\vec{x} = \frac{1}{d}\sum_{a=1}^{d} \vec{x}^{(a)}, \quad \vec{p} = ... \ \vec{s} = ... \tag{1.20}$$

Clearly the flavor averaged multiparticle observables act cyclically on a smaller "neutral" Hilbertspace which may be described in the following way:

$$\mathcal{H}_N^{red.} = \mathcal{H}_1 \otimes ....\mathcal{H}_1 \otimes CS_N \tag{1.21}$$

Here the one-particle wave function spaces H_1have no flavor degree of freedoms and CS_N is the N! dim. representation space of the regular S_N representation. We connect the reduced inner product with the one in the flavor description :

$$\left(\Phi \otimes P^{-1}, \Psi \otimes Q^{-1}\right)_{red} =$$
$$\sum_{i_1..i_N} \frac{1}{d^N} \left(\varphi_1 \otimes e_{i_{P(1)}} \otimes ..\varphi_N \otimes e_{i_{P(N)}}, \psi_1 \otimes e_{i_{Q(1)}} \otimes .. \otimes \psi_N \otimes e_{i_{Q(N)}}\right)_{sym} \tag{1.22}$$

Here the $\varphi's$ and $\psi's$ are the spatial wave functions without flavor and the e's are from the basis in d-dim. flavor space V. The inner product is the natural scalar product in the symmetrized tensor space $(H \otimes V)^{\otimes_N}_{symm.}$. On the left hand side Φ and Ψ are the tensor products of the φ_i and ψ_i. The reduction is implemented through averaging over the flavor degrees of freedom. The orthonormality relations if the e's allow to simplify the result to:

$$\begin{aligned} &\sum_{i_1...i_N} d^{-N} \sum_S \prod_j \left(\varphi_j, \psi_{S(j)}\right) \left(e_{i_{P(j)}}, e_{i_{Q(j)}}\right) = \\ &= \sum_S \prod_j \left(\varphi_j, \psi_{S(j)}\right) \varphi^{(N)}(PSQ^{-1}) \end{aligned} \tag{1.23}$$

$\varphi^{(N)}$denotes a tracial linear functional on CS_N which on the basis elements P is given by :

$$\varphi^{(N)}(P) = \frac{1}{d^N} tr U_N(P) \tag{1.24}$$

where $U_N(P)$ is the previously introduced natural representation of S_N on $V^{\otimes N}$.

The memory on the averaged flavor is completely absorbed in the first multiplicity factor. Note that flavored fermions would have given a similar result with the only difference of signP factors.

The conceptually aware reader will note, that this "cooked up" parastatistics illustration is precisely what an experimentalizer confronts, if in a nonrelativistic atomic problem the electron spin would have no dynamical maifestation (negleglible spin-orbit coupling) or if for a neutron-proton system he would not be able to measure electric charge. Internal symmetry is a very clever theoretical invention which trades the unpleasant nonabelian Young-tableaux against the more physical (*more local!*) standard compact internal symmetry-group description. However it is not universally applicable, see the later discussion of attempts to encode nonabelian braid group statistics into a "quantum symmetry" concept.

It is now easy to see that the normalized trace of the natural representation has an intrinsic characterization in terms of a tracial state φ (a positive linear function) on the group algebra CS_∞. Here S_∞ is the inclusive limit of the S_N groups:

$$S_2 \subset \ldots\ldots\ldots \subset S_N \subset S_{N+1} \subset \ldots\ldots \vec{C} S_\infty \tag{1.25}$$

The normalized tracial functional $\varphi^{(N)}(P)$ on CS_N (the extension from S_N to the group algebra CS_N is by linearity) has a natural extension φ to the inductive limit CS_∞. It is characterized by the following three properties:

$$\varphi(x) \text{ is tracial state on } CS_N \tag{1.26}$$
$$i.e.\quad \varphi(x^*x) \geq 0,\quad \varphi(1)=1,\quad \varphi(xy)=\varphi(yx)$$

$$\varphi(P = \text{transposition}) = \pm\frac{1}{d}, \quad \text{for flavoured fermions} \tag{1.27}$$

$$\varphi \text{ fullfills the Markov-property: } \varphi(P_1P_2)=\varphi(P_1)\varphi(P_2) \tag{1.28}$$

where $P_1 = P_1(\tau_1,\tau_2 \ldots . \tau_{N_1-1})$ is a permutation involving the first $N_1 - 1$ generators and $P_2 = P_2(\tau_{N_1},\ldots . \tau_{N-1})$ is a permutation involving the remaining generators.

The generators τ_i (transpositions) are most conveniently pictured as crossings of the i^{th} strand with its neighbor $i+1$, whereas all the other strands are running parallel (say upward).

These generators are subject to two relations:

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \qquad \tau_i^2 = 1 \tag{1.29}$$

The first relation by its own is most appropriately pictured by allowing over- and under-crossings in the τ_i i.e.by introducing $\tau_i^{\pm} = \{\tau_i, \tau_i^{-1}\}$ see Fig 2.. If one adds to this Artin relation the second one $\tau^2 = 1$, the braid group (which even for finite number of strands is an infinite group) B_∞ passes to the permutation group S_∞. Both groups owe their physical relevance to the fact that they are crucial for the understanding of particle statistics and normal commutation relations between charge-carrying fields. Their natural inclusive structure should

be seen as an algebraic counterpart of the physical cluster property. As we already pointed out, the S_∞ representations described by Young tableaus occur in the centralizer algebras of tensor products of group representations. Take e.g. the algebra generated by the d-dimensional defining matrix representation π of SU(n):

$$\pi(g) \otimes \pi(g) ..... \otimes \pi(g) \quad \text{in } V^{\otimes N} \tag{1.30}$$

In this case the commutant (or centralizer of the group representation) of these operators does not only contain the algebra CS_N (this is evidently the case for all tensor representations of groups) but this algebra is even identical to the centralizer. The following two questions are relevant for the statistics classification:

(1) Can one argue that the indistinguishability requirement of particles (or other localizable objects) together with other physical principles leads to the above tracial representations of S_∞ ?

(2) Is it natural to interpret the appearance of a tracial state with the Markov property in terms an inner symmetry,l and does one gain anything by introducing the symmetry multiplicities (Heisenberg's isospin and its "flavor" generalization) explicitly into the formalism?

Both questions have an affirmative answer, i.e.statistics and internal symmetry are inexorably linked. The relevant theorem is the following:

Theorem 1 *(Doplicher, Roberts) The most general statistics allowed in d=3+1 dimensions is abelian (Bose-Fermi) together with a compact internal symmetry group. The Boson Fermion alternative is related to that of integer versus halfinteger spin (the spin-statistics relation).*

Although the tracial nature and the Markov property can be derived from a properly adapted (to the indistinguishability principle) cluster decomposition property, the natural place for its proof is QFT (as is the case with other structural properties as spin and statistics). The QFT locality and positive energy requirements naturally imply the cluster property and the inclusive picture relating N particles with N+1. We will give a derivation in a later QFT section in connection with the theory of superselection sectors. It is this picture and not the N-particle quantum mechanics for fixed n (the quantum mechanical "proof" for the F-B alternative in the books is a tautology) which is responsible for the results on statistics (and in particular for new quantizations of statistical dimensions for braid group statistics which one finds in $d \leq 2+1$ dim. QFT.)

Superselection rules appeared first in the 1952 work of Wick, Wightman and Wigner. These authors pointed out that the unrestricted superposition principle of quantum mechanics or equivalently the unrestricted identification of self adjoint operators with observables (as formulated by von Neumann) suffers a restriction through the appearance of superselection rules. Their main example was a quantum theory which describes integer as well as halfinteger spin. Its Hilbertspace is a sum of $\mathcal{H}^{\pm}$ where minus corresponds to halfinteger spin. A linear combination of vectors from both spaces changes its relative sign under

2π -rotation:

$$\psi = \alpha\psi_- + \beta\psi_+ \rightarrow \psi^{2\pi} := U(2\pi)\psi = -\alpha\psi_- + \beta\psi_+ \qquad (1.31)$$

Whereas the projective nature allows state vectors to suffer phase changes (the quantum mechanical origin of halfinteger spin), observables and states (in the sense of expectation values) are unchanged under such a 2π rotation (they transform like classical quantities). The following calculation shows that this is only possible iff the observables have vanishing matrix elements between $\mathcal{H}^-$ and $\mathcal{H}^+$

$$(\psi, A\psi) = (\psi^{2\pi}, A\psi^{2\pi}) \Longleftrightarrow (\psi_-, A\psi_+) = 0 \qquad (1.32)$$
$$\text{for all observables } A \in \mathcal{A}$$

The proof just follows by inserting the above linear combinations. This selection rule is called the "univalence rule". In contradistinction to e.g. the $\Delta l = \pm 1$ angular momentum selection rules of atomic physics which suffer changes in higher order radiative corrections, *super*selection rules are universally valid. The vector state ψ above cannot be distinguished from a density matrix ρ:

$$(\psi, A\psi) = tr\rho A \quad with\ \rho = |\alpha|^2 |\psi_-\rangle\langle\psi_-| + |\beta|^2 |\psi_+\rangle\langle\psi_+| \qquad (1.33)$$

The formal generalization for the Hilbertspaces and observables is obviously:

$$\mathcal{H} = \oplus_i \mathcal{H}_i\,, \quad A = \oplus_i A_i\,, \quad A_i = A\,|_{\mathcal{H}_i} \qquad (1.34)$$

Such observable algebras in block form have a nontrivial center. In the following we will illustrate this decomposition theory by a simple but rich mathematical example, the superselection rules of the group algebra.

1.2 The Superselection Sectors of CG

As a mathematical illustration of superselection rules we are going to explain the representation theory of the group algebras.

Let G be a (not necessarily commutative) finite group.We affiliate a natural C*-algebra, the group-algebra CG with G in the following way:

(i) The group elements $g \in G$ including the unit e form the basis of a linear vectorspace over C:

$$x \in CG, \quad x = \sum_g x(g)g\,, \quad with\ x(g) \in \mathbf{C} \qquad (1.35)$$

(ii) This finite dimensional vector space CG inherits a natural convolution product structure from G:

$$\left(\sum_{g\in G} x(g)g\right) \cdot \left(\sum_{h\in G} y(h)h\right) = \sum_{g,h\in G} x(g)y(h)g \cdot h = \sum_{k\in G} z(k)k \qquad (1.36)$$

$$with\ z(k) = \sum_{h\in G} x(kh^{-1})y(h) = \sum_{g\in G} x(g)y(k^{-1}g)$$

(iii) A *-structure, i.e. an antilinear involution:

$$x \to x^* = \sum_{g\in G} x(g)^* g^{-1}\ ,\quad i.e. x^*(g) = x(g^{-1})^* \tag{1.37}$$

Since :

$$(x^*x)(e) = \sum_{g\in G} |x(g)|^2 \geq 0,\quad (= iff\quad x = 0) \tag{1.38}$$

this *- structure is nondegenerate.

(iiii) The last formula also suggests to introduce a norm which is compatible with the *-structure:

$$||x|| := \sqrt{(x^*x)(e)},\quad C^* - condition : ||x^*x|| = ||x^*||\,||x|| \tag{1.39}$$

A C^*−norm on a *-algebra is necessarily unique (if it exists at all). It can be introduced through the notion of spectrum (mathematical appendix).

It is worthwhile to note that (iii) also serves to introduce a tracial state on CG i.e.a positive linear functional φ with the trace property:

$$\varphi(x) := x(e),\quad \varphi(x^*x) \geq 0,\quad \varphi(xy) = \varphi(yx) \tag{1.40}$$

This state (again as a result of (iii)) is even faithful, i.e. the scalar product defined by:

$$(\hat{x}, \hat{y}) := \varphi(x^*y) \tag{1.41}$$

is nondegenerate. On the left hand side the elements of CG are considered as members of a vector space. The nondegeneracy and the completeness of the algebra with respect to this inner product (a result of the finite dimensionality of CG) give a natural representation (the regular representation of CG) on this Hilbert space:

$$x\hat{y} := \widehat{xy} \tag{1.42}$$

The norm of these operators is identical to the previous one.

This construction of this "regular" representation from the tracial state on the C*-group-algebra is a special case of the general Gelfand-Neumark-Segal (GNS-)construction presented in a later section.

Returning to the group theoretical structure, we define the conjugacy classes K_g and study their composition properties.

$$K_g := \{hgh^{-1}, h \in G\} \tag{1.43}$$

In particular we have K_e=$\{e\}$. These sets form disjoint classes and hence:

$$G = \cup_i K_i,\ |G| = \sum_{i=0}^{r-1} |K_i|\ ,\quad K_e = K_0,\ K_1,....K_{r-1},\ r = \#classes \tag{1.44}$$

We now define central "charges":

$$Q_i := \sum_{g \in K_i} g \in \mathcal{Z}(CG) := \{z, [z,x] = 0 \quad \forall x \in CG\} \tag{1.45}$$

It is easy to see that the center $\mathcal{Z}(CG)$ consists precisely of those elements whose coefficient functions z(g) are constant on conjugacy classes i.e. $z(g) = z(hgh^{-1})$ for all h. The coefficient functions of Q_i:

$$Q_i(g) = \begin{cases} 1 & if \ g \in K_i \\ 0 & otherwise \end{cases} \tag{1.46}$$

evidently form a complete set of central functions. The composition of two such charges is therefore a linear combination of the r independent $Q_i's$ with positive integer valued coefficients (as a result of the previous formula.):

$$Q_iQ_j = \sum_l N_{ij}^l Q_l \tag{1.47}$$

The fusion coefficients N can be arranged in terms of r commuting matrices

$$\mathbf{N}_j, \ with \ (\mathbf{N}_j)_i^l = N_{ij}^l \tag{1.48}$$

The associativity of the 3-fold product QQQ is the reason for this commutativity, whereas the abelianess of the central algebra (only valid for abelian groups!) results in the i-j symmetry of the fusion matrices.

Functions on conjugacy classes also arise naturally from characters χ of representations π:

$$\chi^\pi(g) = Tr\pi(g), \quad \chi^\pi(g) = \chi^\pi(hgh^{-1}) \tag{1.49}$$

This applies in particular to the previously defined left regular representation λ with $(\lambda_g x)(h) = x(g^{-1}h)$.Its decomposition in terms of irreducible characters goes hand in hand with the central decomposition of CG:

$$CG = \sum_l P_l CG, \quad Q_i = \sum_l Q_i^{\pi^l} P_l \tag{1.50}$$

The central projectors P_l are obtained from the algebraic spectral decomposition theory of the $Q_i's$ by inverting the above formula. The "physical" interpretation of the coefficients is: $Q_i^{\pi^l} = \pi_l(Q_i)$ i.e. the value of the i^{th} charge in the l^{th} irreducible representation. The central projectors P_l are simply the projectors on the irreducible components contained in the left regular representation. Since any representation of G is also a representation of the group algebra, every irreducible representation must occur in $\lambda(CG)$.One therefore is supplied with a complete set of irreducible representations, or in more intrinsic terms with a complete set of r equivalence classes of irreducible representations. As we had the intrinsic (independent of any basis choices) fusion rules of the charges,

we now encounter the intrinsic fusion laws for equivalence classes of irreducible representations.

$$\pi_k \otimes \pi_l \simeq \sum_m \tilde{N}_{kl}^m \pi_m \tag{1.51}$$

Here the fusion matrices $\tilde{N}$ are "dual" to those of the charges namely $\tilde{N}_{kl} = N_{lk}$ The unsymmetry of these two fusions is typical for nonabelian groups and corresponds to the unsymmetry of the character table: although the number of irreducible representations equals the number of central charges (= # conjugacy classes), the two indices in $\pi_l(Q_i)$ have a different meaning. This *character matrix* S_{li} appears via diagonalization of the commuting system of N's and the surprise is that it shows up in two guises, once as the unitary which diagonalizes this N-system and then also as the system of eigenvalues which can be arranged in matrix form. We will not elaborate on this point.

In passing we mention that closely related to the group algebra CG is the so-called "double" of the group (Drinfeld):

$$D(G) = C(G) \bowtie_{ad} G \tag{1.52}$$

In this crossed product designated by $\bowtie$, the group acts on the functions on the group $C(G)$ via the adjoint action:

$$\alpha_h(f)(g) = f(h^{-1}gh) \tag{1.53}$$

The dimension of this algebra is $|G|^2$ as compared to $\dim CG = |G|$. Its irreducible representations are labeled by pairs $([\pi_{irr}], K)$ of irreducible representation and conjugacy class and therefore their matrices N and S are selfdual. In this sence group duals are "more symmetric" than groups.

Finally we may notice that the equivalence classes of irr.representations appear with the natural multiplicity :

$$mult.(\pi_l \; in \; \lambda_{reg}) = |K_l| \tag{1.54}$$

The results may easily be generalized to compact groups where they are known under the name of Peter-Weyl theory.

Since group algebras are very special, some remarks on general finite dimensional algebras are in order.

Any finite dimensional C*-algebra R may be decomposed into irreducible components.and any finite dimensional irr.C*-algebra is isomorphic to a matrix algebra $Mat_n(\mathbf{C})$. If the irreducible component $Mat_{n_i}(\mathbf{C})$ occurs with the multiplicity m_i , the algebra R has the form (is isomorphic to) of the following matrix algebra:

$$R = \bigoplus_i Mat_{n_i}(\mathbf{C}) \otimes 1_{m_i} \quad in \; \mathcal{H} = \oplus \mathcal{H}_{n_i} \otimes \mathcal{H}_{m_i} \tag{1.55}$$

and the multiplicities are unrelated to the dimensionalities of the components. The commutant of R in H is:

$$R' = \oplus_i 1_{n_i} \otimes Mat_{m_i}(\mathbf{C}), \quad Z := R \cap R' = \oplus_i \lambda_i 1_{n_i} \otimes 1_{m_i} \tag{1.56}$$

The last formula defines the center with $\lambda_i \in \mathbf{C}$.

Let us conclude with some remarks on states over (finite dim.) C*-algebras. Since finite dimensional C*-algebras decompose into irreducible components (this decomposition agrees with the central decomposition), it suffices to look at states on irreducibles i.e.on the matrix algebra $Mat_n(\mathbf{C})$. The linear functionals can be identified with these matrix space since one can use the unique normalized tracial state $\varphi(A) = \frac{1}{n} TrA$ to define a nondegenerate inner product which does this identification. By restricting these linear functionals to be positive and normalized one obtains the well known representation of states in terms of density matrices:

$$\varphi_\rho(A) = tr\rho A, \quad \rho \geq 0, \quad tr\rho = 1 \tag{1.57}$$

In the simplest case of $Mat_2(C)$, which corresponds to the spin algebra generated by the Pauli matrices together with the identity matrix, the convex space of states (the space of density matrices ρ) is a 3-dim.ball:

$$\rho = \frac{1}{2}(1 + \vec{r}\vec{\sigma}), \quad \vec{r}^2 \leq 1 \tag{1.58}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Besides the normalization we used the positivity of ρ (which requires positive eigenvalues and hence a positive determinant) to derive the inequality. The pure states correspond to one-dimensional projectors and therefore cover the surface of the ball. Note that pure states can be defined intrinsically (i.e.without referring to one-dimensional projectors or state vectors) by the property of indecomposability of φ :

$$\nexists \ \varphi_1, \varphi_2 \ s.t. \quad \varphi = \alpha\varphi_1 + \beta\varphi_2, \ \alpha, \beta \geq 0, \ \alpha + \beta = 1 \tag{1.59}$$

Starting from $Mat_3(\mathbf{C})$, one meets the new phenomenon of a stratification of the surface into convex subregions called "faces". Density matrices with 3 different eigenvalues correspond to faithfull states inside the convex state space. If one of the eigenvalues vanishes, one looses faithfulness and although these states are "purer", they are not pure in the sense of indecomposability. They form a "face" which looks precisely like the previous ball. In higher dimensions one finds lower dimensional faces inside higher dimensional ones. Once one is on a face, any further purification takes place inside this face. Returning to the situation of a general finite dim.C*algebra, we now see that a state is described by a collection of positive traces or $\rho's$ (one for each central component) with normalized total trace.

In order to appreciate the structural differences to classical observables and states, one should remember that the classical observable algebra consists simply of (continuous) functions on phase space and pure states are represented by Dirac δ–functions whereas the Liouville measure and the Radon Nikodym derivatives with respect to it represent the mixtures and may be viewed as

continuously smeared out δ−functions. Since the notion of coherent superposition is absent, a classical analogon of the states for the $Mat_n(\mathbf{C})$ toy models would be a higher dimensional simplex . The pure states are the vertices and every mixed state is obtained by a unique convex combination of the vectors corresponding to the vertices. This is quite different from the the ball-shaped region of quantum physics where the representation of a point in the interior in terms of pure states on the surface is highly nonunique. In fact this structure of states is, more than anything else, the most characteristic property of quantum physics. The presence of superselection rules tells us that at least partially there exists a classical structure within quantum physics: the central decomposition of a C*algebra is unique, and the unrestricted superposition principle holds only within each component (similar to the classical case).

The concepts needed in the infinite dimensional case are more subtle and will be presented in a mathematical appendix.

The reader may ask the question of why superselection rules despite their importance are rarely mentioned in quantum mechanics. The reason is the validity of the Stone-von Neumann unicity-theorem on irreducible (regular)representations of the Schrödinger theory . In QM of finite degrees of freedom it is only only through topological nontriviality of configuration- or phase-space that inequivalent representations may enter the quantization procedure. Let us look at a typical example, the quantum mechanics of a particle in a circle. The geometric argument in favour of many representations is as follows. Diagonalyzing first the algebra of the position operator we represent the state vectors by (periodic) wave functions $\psi(\varphi)$ on S^1. In order to fulfill the Heisenberg-Weyl commutation relations,the most general form for the momentum is $p = i\frac{\partial}{\partial\varphi} + f(\varphi)$. The p only commutes with itself iff the real function f is a constant (in higher dimensions the p would be like a gauge covariant derivative, and the constancy of f like the flatness of a connection or the vanishing of the field strength associated with a vector potential). Its exponential function representing the translation shows that this constant is only determined mod 2π :

$$p = i\frac{\partial}{\partial\varphi} + \theta, \;\; \theta \; mod\, 2\pi \tag{1.60}$$

Hence there are many Schrödinger theories parametrized by a theta-angle which have different physical content (e.g. the spectrum of $H_0^{(S_1)}$). An equivalent form is obtained by keeping the standard Schrödinger-form of p but accepting quasiperiodic wave functions. The "θ-obstruction" is intrinsic, it can be shifted from the algebra to the states, but only in a simply connected space it can be removed by a nonsingular operator transformation;this is the case in ordinary Schrödinger theory.

This geometric viewpoint has one disadvantage: there is no natural way to consider the different θ−theories as just different manifestations of one system, rather they are different geometrical objects (generally inequivalent vectorial fibre bundles). Here the algebraic view of superselection rules is physicall superior: the different θ−theories are simply different representation of one more

abstract C*algebra (in our case the "rotational algebra"). We will return to this issue in a later section.

The mathematical example of a particle on a circle is closely related to the Aharonov-Bohm effect. As long as the solenoid has not passed to the infinite thin limit, the A.-B.system falls into the ordinary Schrödinger description. It is only through the limiting overidealization that the $\theta-$dependent circular mechanics with its nonsimple C*-algebra enters. There is a general message in this example: all topologically nontrivial quantum mechanics result from an overidealization of Schrödinger theory. Only for infinite degrees of freedom in QFT it becomes possible to encounter superselection rules which have a fundamental origin (e.g. a phase transitions).

One lesson to be drawn for QFT from these illustrations is that one is not limited by "quantization" methods. Rather one may use the superselection idea and try to classify and construct QFT's by studying representations of observable algebras instead of quantizing classical physics. It was realized by Haag, Haag and Kastler as well as Borchers and Araki (see Haag: "Local Quantum Physics") already at the beginning of the 60's that the principle of locality makes such a formulation very consistent and structurally rich. But the path from those early studies to the more recent advances in e.g. properties of low dimensional QFT with surprising nonperturbative insights was very thorny indeed. Our approach to QFT is strongly influenced by this "algebraic QFT". In particular the problem of particle statistics will be presented as part of the understanding of superselection charges.

1.3 Illustration of Important Quantum Concepts

In this section some additional quantum-physical concepts will be introduced in a finite dimensional setting (for simplicity).

We start with the GNS-construction. It associates in a canonical way with a given C*-algebra A and a state ω on it a so called GNS-tripel $(\mathcal{H}_\omega,\pi_\omega(\mathcal{A}),\Omega_\omega)$ which consists of a representation space H_ω with a distinguished vector Ω_ω on which A through its representation $\pi_\omega(\mathcal{A})$ acts cyclically. The construction is analogous to that of the regular representation of CG. Again one uses the state ω in order to construct a positive semidefinite sesquilinear form on the linear space of A:

$$(\psi_A,\psi_B) := \omega(A^*B) = \overline{\omega(B^*A)} \tag{1.61}$$

Here we use a notation which distinguishes the vectors from the elements of the algebra. The reason is that whereas the tracial state φ on the CG was faithfull i.e. the sesquilinear form was strictly positive, a general state ω on A leads to a nontrivial nullspace N_ω

$$\mathcal{N}_\omega = \{A \in A \mid \omega(A^*A) = 0\} \tag{1.62}$$

Fortunately this Nullspace N_ω is also a left ideal (the Gelfand ideal) of A i.e. with $A \in \mathcal{N}_\omega$ also $BA \in \mathcal{N}_\omega$ for any $B \in \mathcal{A}$. This follows from the Cauchy-Schwartz inequality for states:

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B) \tag{1.63}$$

if we write the latter in the adapted form:

$$\omega\left((BA)^*BA\right) \leq \omega(C^*C)\omega(A^*A), \quad C := ((BA)^*B)^* \tag{1.64}$$

In this form it is obvious that the left hand side must vanish. The proof of the Cauchy-Schwartz inequality for states is identical to that for scalar products of vectors in Hilbert space.

If one now defines the state vectors ψ as elements of A mod N_ω and the action of A on these vectors as:

$$\pi_\omega(B)\psi_A := \psi_{BA}, \quad B \in \mathcal{A} \tag{1.65}$$

then one obtains the desired relation between the state and the representation:

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = \omega(A) \tag{1.66}$$

Here Ω_ω is the distinguished vector in $\mathcal{H}_\omega$ which corresponds to the Gelfand ideal. The only additional step for infinite dimensional algebras is to form the closure of the linear space and to continue the definition of $\pi_\omega(A)$ to this Hilbert space closure: $\mathcal{A}$ mod $\mathcal{N}_\omega$. Since a dense set of vectors is obtained by applying A to Ω_ω, the proof is finished. It remains to be added that every other cyclic representation $\pi_\omega(A)$ with the same $\omega(A)$ turns out to be unitary equivalent to the canonical GNS representation.

As a preparation for the next topic let us use a tracial state as a reference state. On a factor $Mat_n(C) \otimes \underline{1}_m$ in the central decomposition of a semisimple algebra a tracial state is unique and has the standard form:

$$\tau(A) = \frac{1}{n} TrA \equiv trA \quad TrA = \sum_i A_{ii} \tag{1.67}$$

On a semisimple algebra there is a family of tracial states parametrized by $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$:

$$\tau_{(\lambda_1\lambda_2\ldots\lambda_k)}(A) = \sum_{i=1}^{k} \lambda_i \tau(A_i) \tag{1.68}$$

$$\tau_{(\frac{1}{k},\frac{1}{k}\ldots\frac{1}{k})}(A) =: \tau(A) = \frac{1}{N} TrA, \quad N = \Sigma_i n_i$$

where $\mathcal{A}_i$ is the i^{th} central component of $\mathcal{A}$ and τ is the standard faithfull tracial state. We have seen before that the most general state φ may be described in terms of a density matrix ρ :

$$\varphi(A) = \tau(\rho A) \tag{1.69}$$

This formula is just a concrete realization of the GNS construction via the trace formalism (in which case the Nullspace vanishes and the the correspondence between matrices and vectors in the representation space is one to one):

$$\begin{aligned} \tau(A^*B) &= (\Omega_\tau, \pi_\tau(A^*)\pi_t(B)\Omega_t) \\ \varphi(A) &= (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi) = \tau(\rho^{\frac{1}{2}}A\rho^{\frac{1}{2}}) \\ \Omega_\omega &= \rho^{\frac{1}{2}}\Omega_\tau, \quad \Omega_t = \underline{1} \quad \text{in space of matrices} \mathcal{H}_\tau \end{aligned} \tag{1.70}$$

Note the analogy of this construction with the regular representation of CG: in both cases the algebra in its role as a space together with the trace served as the the arena for GNS faithful representations. Such a Hilbertspace $\mathcal{H}_\tau$ may be written in a more suggestive notation:

$$\mathcal{H}_\tau = L^2(Mat_N, \tau), \quad \Omega_\tau = \underline{1} \tag{1.71}$$

In addition to the "left" GNS representation $\pi_l = \pi_\tau = \pi$, we introduce a right representation:

$$\pi_r(B)\psi_A = \psi_{AB^*} = \pi_t(A)\underline{1}\pi_t(B^*) \tag{1.72}$$

In the group case, the right action unravels the multiplicity structure of the irreducible representations (the irreducible representations occur according to the Peter-Weyl theory with a multiplicity identical to the dimension of the representation as a result of the complete symmetry between left and right in the case CG). In our present more general setting, the existence of the two commuting left-right actions furnish the germ of a deep and general theory: the Tomita-Takesaki modular theory. One first defines the antiunitary "flip" operator J:

$$J\psi_A = \psi_{A^*}, \curvearrowright j(\pi(A)) := J\pi(A)J = \pi_r(A) \tag{1.73}$$

In this case the J not only implements the flip, but it also transforms the vector $A\Omega_\tau$ into $A^*\Omega_\tau$. In the more general φ-representation one has two different involutive operators:

$$\begin{aligned} S\pi(A)\Omega_\varphi := \pi(A^*)\Omega_\varphi \quad \Omega_\varphi = \rho^{\frac{1}{2}}\underline{1} \\ J\pi(A)J = \pi_r(A) \end{aligned} \tag{1.74}$$

Fom this one reads off the new operator S:

$$S = J\Delta^{\frac{1}{2}}, \quad \Delta = \pi(\rho)\pi_r(\rho^{-1}) \tag{1.75}$$

since the identity: $S\pi(A)\rho^{\frac{1}{2}}\underline{1} = J\rho^{\frac{1}{2}}\pi(A)\rho^{\frac{1}{2}}\underline{1}\rho^{-\frac{1}{2}} = \pi(A^*)\rho^{\frac{1}{2}}\underline{1}$ follows from the definitions. The formula for S agrees with the polar decomposition formula into an "angular" part J and positive radial part $\Delta^{\frac{1}{2}}$. The above formulas require ρ to be invertible i.e. the φ—representation to be faithful (or Ω_φ to be a separating vector in $\mathcal{H}_\tau$) Remembering the notation of the von Neumann commutant $\mathcal{A}'$, the algebraic role of these operators, the following important properties are an easy consequence:

$$j(\mathcal{A}) = \mathcal{A}', \quad \sigma_t(A) := \Delta^{it}A\Delta^{-it} \in \mathcal{A} \tag{1.76}$$

j(·) is the modular conjugation and $\sigma_t(\cdot)$ is called the modular group. Since every ρ may be written as a Gibbs formula in terms of a (ad hoc) hamiltonian H:

$$\rho = \frac{1}{Z} e^{-\beta H}, \quad Z = Tre^{-\beta H} \tag{1.77}$$

In this case $\Delta^{it} = \pi_l(e^{-i\beta tH})\pi_r(e^{i\beta tH})$ and the modular automorphism on $\mathcal{A}$ is apart from a stretching factor $-\beta$ equal to the hamiltonian automorphism. Note that the infinitesimal generator H of the time translations is not simply H but rather: $\mathbf{H} = \pi_l(H) \otimes \underline{1} \oplus \underline{1} \otimes \pi_r(-H)$. This fact becomes important in the realistic (infinite dimensional) case, since the heat bath fluctuations of the hamiltonian H become infinitly large in the thermodynamical limit.

It is therefore not surprising that the modular theory was discovered independently (and at the same time) by physicists in the study of temperature states on bosonic or fermionic algebras. In our toy case, instead of writing down Gibbs formula, one may also characterize the faithful state φ by the so-called KMS condition:

$$\varphi(A\sigma_t(B)) = \varphi(\sigma_{t+i\beta}(B)A) \tag{1.78}$$

In the finite dimensional case the analytic dependence of σ on t is automatic whereas for the general case the analyticity of a function $F_z(A,B) := \varphi(A, \sigma_z(B))$ (for A, B from a dense subalgebra of $\mathcal{A}$) and the formulas:

$$\begin{gathered} F_z(A;B) \text{ analytic in strip: } 0 \leq \operatorname{Im} z \leq \beta \\ \text{with } F_z = \begin{cases} \varphi(A\sigma_t(B)) \text{ for } \operatorname{Im} z = 0 \text{ i.e. } z = t + i0 \\ \varphi(\sigma_{t+i\beta}(B)A) \text{ for } \operatorname{Im} z = t + i\beta \end{cases} \end{gathered} \tag{1.79}$$

constitute part of the *definition* of the KMS property of φ. This characterization of thermal equilibrium states is more general than the Gibbs formula, since the latter looses its meaning as a result of the volum divergencies in the thermodynamic limit $V \to \infty$ with particle densities kept fix. In addition, even in case of finite volume, the calculation of φ by KMS boundary condition is much easier than by calculating traces (this practical advantage was the reason why Kubo, Martin and Schwinger introduced this condition, whereas the mathematical physics connection was made much later by Haag,Hugenholtz and Winnink).

Besides the notion of states and representation, the concepts of inclusions of von Neumann algebras will play an important role in later sections. Here we will only present a "cartoon" version. Suppose that $Mat_2(\mathbf{C})$ acts not on its natural irreducible space $\mathbf{C}^2$ but by left action on the 4-dim Hilbertspace $\mathcal{H}(Mat_2(\mathbf{C}), \frac{1}{4}Tr)$. In that space the commutant is of equal size and consists of $Mat_2(\mathbf{C})$ acting in the opposite order from the right which will be shortly denoted as $Mat_2(\mathbf{C})^{opp}$. Explicitely the realization of $\mathcal{H}$ as $\mathbf{C}^4$ may be defined as

$$\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \to \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{12} \\ \xi_{22} \end{pmatrix} \tag{1.80}$$

and the action of $\mathcal{A} = Mat_2(\mathbf{C})$ takes the following form:

$$a = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \simeq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \underline{1} \tag{1.81}$$

The most general matrix in the commutant $a' \in \mathcal{A}'$ has evidently the form:

$$a' = \begin{pmatrix} a'_{11} & 0 & a'_{12} & 0 \\ 0 & a'_{11} & 0 & a'_{12} \\ a'_{21} & 0 & a'_{22} & 0 \\ 0 & a'_{21} & 0 & a'_{22} \end{pmatrix} \simeq \underline{1} \otimes \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$$

The norm $\|\xi\| = \left(\frac{1}{4} Tr \xi^* \xi\right)^{\frac{1}{2}}$ is invariant under the involution $\xi \to \xi^*$ which in the $\mathbf{C}^4$ representation is given by the isometry:

$$J = \begin{pmatrix} K & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & K \end{pmatrix}, \quad K : \text{natural conjugation in } \mathbf{C} \tag{1.82}$$

We have:

$$j(\mathcal{A}) := J\mathcal{A}J = \mathcal{A}', \quad \text{antilin. map } \mathcal{A} \to \mathcal{A}' \tag{1.83}$$

which may be rewritten in terms of a linear anti-isomorphism:

$$a \to Ja^*J, \quad \mathcal{A} \to \mathcal{A}' \tag{1.84}$$

Consider now the trivial algebra $\mathcal{B} = \mathbf{C} \cdot \mathbf{1}_2$ as a subalgebra of $\mathcal{A} = Mat_2(\mathbf{C})$. In the $\mathbf{C}^4$ representation the B-algebra corresponds to the subspace:

$$\mathcal{H}_B = \left\{ \begin{pmatrix} \xi \\ 0 \\ 0 \\ \xi \end{pmatrix}, \xi \in \mathbf{C} \right\}, \quad \mathcal{H}_B = e_B \mathcal{H}, \quad e_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{1.85}$$

The projector e_B commutes clearly with $\mathcal{B}$ i.e.$e_B \in \mathcal{B}'$. We now define a measure for the relative size of $\mathcal{B} \subset \mathcal{A}$ the Jones index:

$$[A : B] = \tau_{B'}(e_B)^{-1}, \quad \tau : \text{ normalized trace in } \mathcal{B}'$$

In our example $\tau(e_B) = \frac{1}{4}(\frac{1}{2} + 0 + 0 + \frac{1}{2}) = \frac{1}{4}$ i.e. the satisfying result that the Jones index is 4. The same method applied to the inclusion:

$$Mat_4(\mathbf{C}) \supset Mat_2(\mathbf{C}) \otimes \mathbf{1}_2 = \left\{ \begin{matrix} X & 0 \\ 0 & X \end{matrix}, X \in Mat_2(\mathbf{C}) \right\} \tag{1.86}$$

also gives the expected result:

$$[A : B] = \frac{\dim Mat_4(\mathbf{C})}{\dim Mat_2(\mathbf{C})} = 4 \tag{1.87}$$

If, as in the previous cases B is a finite dimensional subfactor (i.e.a full matrix algebra) of A, the Jones index is the square of a natural number. For inclusions of finite dimensional semisimple algebras the index takes on more general values ≥ 4. For example:

$$Mat_2(\mathbf{C}) \oplus \mathbf{C} = \begin{pmatrix} X & & \\ & X & \\ & & x \end{pmatrix} \subset Mat_2(\mathbf{C}) \oplus Mat_3(\mathbf{C})$$
$$X \in Mat_2(\mathbf{C}), \quad x \in \mathbf{C}1$$

Here the index is 3. It is easy to see that instead of the projector formula one may also use the incidence matrix formula:

$$[A : B] = \|\Lambda_n^m\|^2$$

The incidence matrix Λ is describable in terms if a bipartite graph. From a sequence of ascending graphs one obtains important infinite graphs (Bratelli diagrams) which are very useful in the "subfactor theory" which will appear in the later chapter on algebraic QFT.

Some more remarks on inclusions should be helpful.

1.4 Measurement and Superselection Rules

The interpretation of quantum theory requires an observer which also may be a registration apparatus outside the observed system. Therefore notions like the"state or wave function of the universe" have to be handled with great care and are mostly meaningless, at least within the standard physical interpretation of QFT. Somewhere between the (generally microscopic) observed system and the observer a "cut" is needed. As already pointed out by Heisenberg, this cut may be somewhat shifted, but it must be there somewhere.

According to von Neumann the observed system is described by a selfadjoint operator and the measured values are the eigenvalues α of the "observable" A (with suitable mathematical adaptation in case of continuous spectrum). The state immediately after the measurement is obtained by a "quantum jump" i.e. cannot be computed via the Schrödinger time evolution of the observed system. Taking for A a projector P whose eigenvalues are just 0 and 1, the new state created by the measurement can be described as follows:

$$\omega_{after}(\mathcal{O}) = \omega_{before}(POP) + \omega_{before}((1-P)\mathcal{O}(1-P)), \quad \mathcal{O} \in \mathcal{A}, \text{ algebra of observables} \tag{1.88}$$

If $\omega_{before}$was a pure state described by a state vector ψ, ω_{after} corresponds to the density matrix:

$$\begin{aligned} \rho_{after} &= |P\psi\rangle\langle P\psi| + |(1-P)\psi\rangle\langle(1-P)\psi| = \\ &= p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| \end{aligned} \tag{1.89}$$

$$\psi_1 = \frac{P\psi}{||P\psi||},\ p_1 = ||P\psi||^2, \qquad (1.90)$$
$$\psi_2 = \frac{(1-P)\psi}{||(1-P)\psi||},\ p_2 = ||(1-P)\psi||^2$$

The last formula represents the mixed state associated with ρ_{after} as a sum of two orthogonal minimal projectors. However, as stressed before, an impure state permits myriads of decompositions into minimal projectors. If we could find a physical argument in favor of an orthogonal decomposition (as the one above in terms of ψ_1, ψ_2) then uniqueness follows. But no such principle is known. In addition, to have a change of states as the above reduction of wave packet(or quantum jump) which is outside the unitary time development of the Schrödinger equation is somewhat mysterious (the paradoxon of Schrödinger's cat). Any hamiltonian dynamics leading to unitary propagation in time necessarily preserves the purity of states.

A satisfactory way out of these mysterious aspects has been indicated by Hepp and strengthened by Bell. The essential idea is that the macroscopic measuring apparatus has superselection sectors which, as we have learned, is a generic phenomenon for systems with infinitely many degrees of freedom. Hepp's idea is that although the complete system including the apparatus is governed by a unitary time development :

$$\alpha_t(A) = e^{iHt} A e^{-iHt},\ A \in \mathcal{A} \qquad (1.91)$$

the limit for t$\rightarrow \infty$ may well be only *a positive map* instead of an automorphism of the observable algebra. In terms of Hilbertspace concepts the limit of e^{iHt} may only be an isometry mapping the total Hilbertspace on a subspace. Therefore the initial state which may be a pure state implemented by a vector in one superselection sector leaves the coherent subspace in the limit and acquires components to other superselection sectors. Although it remains formally a vector in the total space, it describes physically a mixture since it has projections to several coherent subspaces.

$$\omega_{before} \rightarrow \omega_{after} = \sum_i \lambda_i \omega^i_{after} \qquad (1.92)$$

Note that this central decomposition is completely intrinsic. It is a special case of a partial orthogonal decomposition. Whereas the latter is only unique within one superselection sector, the former is unique in general and in this sense behaves like a classical decomposition.

It is easy to provide mathematical illustrations of Hepp's ideas, but for realistic physical use one needs models of an apparatus with natural superselection sectors i.e.one with infinite degrees of freedom. Systems with phase transitions as infinite spin systems have been used in this context.

In Hepp's approach the "reduction of the wave packet" is achieved in the limit t$\rightarrow \infty$.There are also attempts (even experimentally) to investigate the loss of coherence in time by constructing "Schrödinger cat" states with the help of a few photons in a cavity.

Chapter 2

The Construction of Fock-Space.

2.1 The Bosonic Fock-Space

There are several reasons for combining N-particle spaces together with a one-dimensional "no-particle" space (vacuum) into a big "Master" space, the so called Fock-space:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ..... \tag{2.1}$$

One obvious reason is that relativistic local interactions do not conserve the particle number but only total charges (i.e. particle-antiparticle creation is allowed as long as it obeys the energy-momentum conservation). This is also valid in the infinite volume limit (the so called thermodynamic limit) in nonrelativistic systems for which the ground state (which has generally a finite density of particles) becomes the reference state for nonconserved "quasiparticle" excitations. The Fock-space is also the natural framework for the formulation of cluster-properties.

The bosonic Fock-space $\mathcal{H}^B$ is obtained by projecting the full n-particle spaces onto their symmetrized subspaces $\mathcal{H}_N^B$. In the following we will introduce the creation and annihilation operator formalism in x-space having in mind wave functions in Schrödinger theory. If we interprete the formulas in momentum space, there will be no difference between the relativistic and nonrelativistic formalism apart from possible changes in the normalization. A general vector in Fock-space is given by a finite norm sequence of symmetric wave functions:

$$\begin{aligned} &\mathcal{H}_F^B \ni \Psi = (\psi_0, \psi_1(\vec{x}), \psi_2(\vec{x}_1, \vec{x}_2), ...) \\ &(\Psi, \Psi) := |\psi_0|^2 + \textstyle\sum_{n=1}^{\infty} (\psi_n, \psi_n) < \infty \end{aligned} \tag{2.2}$$

The creation operator depends linearly on the wave function f and the n-particle

component after its application on Ψ is defined as:

$$(a^*(f)\Psi)_n(\vec{x}_1,.....,\vec{x}_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(\vec{x}_i)\psi_{n-1}(\vec{x}_1...\hat{\vec{x}}_i...\vec{x}_n), \quad n \geq 1,$$
$$(a^*(f)\Psi)_0 = 0 \tag{2.3}$$

Here the "roof" $\hat{}$ indicates deletion of the i-th coordinate. The formula for the hermitian adjoint annihilation operator a(f) follows from the defining property:

$$\left(\Psi, a^*(f)\Psi'\right) = \left(a(f)\Psi, \Psi'\right), \quad \text{namely}$$

$$(a(f)\Psi)_n(\vec{x}_1....\vec{x}_n) = \sqrt{n+1}\int d^3x \bar{f}(\vec{x})\psi_{n+1}(\vec{x},\vec{x}_1....\vec{x}_n) \tag{2.4}$$

The annihilation operator depends antilinear on the f. In particular for the vacuum $\Omega = (1,0,0...)$

$$(a(f)\Omega)_n = 0 \quad \forall n \tag{2.5}$$

i.e. a(f) annihilates Ω.

The multiple application of these operators leads to lengthy formulas, however the commutators are very simple:

$$[a(g), a^*(f)] = (g,f)\,1, \quad \text{with } (g,f) = \int \bar{g}(\vec{x})f(\vec{x})d^3x$$
$$[a(f), a(g)] = 0 = [a^*(g), a^*(f)] \tag{2.6}$$

This simplicity was the reason for the choice of normalization in the definition of a^*.

The number operator N is defined to be that positive semidefinite operator which multiplies each N-particle vector with N. Its commutation rules with $a^{\#}$ is:

$$[\mathrm{N}, a(f)] = -a(f), \quad [\mathrm{N}, a^*(f)] = a^*(f) \tag{2.7}$$

In terms of an orthonormal basis it looks as $\mathrm{N} = \sum_i a^*(f_i)a(f_i)$.

It is convenient to liberate the formalism from the wave functions by introducing operator-valued distributions $a^{\#}(\vec{x})$:

$$a^*(f) = \int a^*(\vec{x})f(\vec{x})d^3x, \quad a(f) = \int a(\vec{x})\bar{f}(\vec{x})d^3x$$
$$\text{with } [a(\vec{x}), a^*(\vec{y})] = \delta(\vec{x}-\vec{y}) \text{ etc.} \tag{2.8}$$

One can then introduce the improper basis (vector-valued distributions) in Fock-space:

$$|\vec{x}_1,....\vec{x}_N\rangle = \frac{1}{\sqrt{N!}}a^*(\vec{x}_1)....a^*(\vec{x}_N)\,|0\rangle, \quad |0\rangle := \Omega \tag{2.9}$$

$$|\Psi\rangle = \psi_0\,|0\rangle + \sum_N \int \psi_N(\vec{x}_1....\vec{x}_N)\,|\vec{x}_1....\vec{x}_N\rangle\, d^3x_1....d^3x_N \tag{2.10}$$

The action of the $a^{\#}$ on the basis vectors is (as always) contragredient to that on the wave functions. It is more common to use the former. Of frequent use (especially in the application of Fock-space in statistical mechanics) is the so

called occupation number representation. One chooses an orthonormal set of wave functions f_i i=1,2,...∞ and defines a basis in Fock-space by:

$$|n_{i_1}, n_{i_2}, ...n_{i_r}\rangle = \frac{1}{\sqrt{n_{i_1}!}}....\frac{1}{\sqrt{n_{i_r}!}} a^*(f_{i_1})^{n_{i_1}}....a^*(f_{i_r})\,|0\rangle\,, \quad \sum_{k=1}^{r} n_{i_k} = N \tag{2.11}$$

Often (in particular in particular in Stat. Mach.) one encloses the system in a box and uses the discrete set of plane waves as the orthonormal system for the occupation number representation.

The creation and annihilation operators in x-space are useful for rewriting the Schrödinger theory into the Fockspace formalism. One easily verifies the validity of the following formulas:

$$\begin{aligned} \mathbf{H} &= \int \mathbf{H}(x)d^3x \;\;,\;\; \mathbf{H}(x) = \mathbf{H}_0(x) \;+\; \mathbf{V}(x) \\ \mathbf{H}_0(x) &= \tfrac{1}{2m}\vec{\partial}a^*(x)\cdot\vec{\partial}a(x) \\ \mathbf{V}(x) &= \tfrac{1}{2}a^*(x)\int d^3y V(x-y)a^*(y)a(y)a(x) \end{aligned} \tag{2.12}$$

applied to the previously introduced N-particle state $|\Psi\rangle$ give the N-particle Schrödinger-operator:

$$\begin{gathered} \mathbf{H}\,|\Psi\rangle = |\Phi\rangle\,, \;\; \Leftrightarrow \\ \left(\textstyle\sum_{i=1}^{N} \frac{1}{2m}\Delta_i + \sum_{i<j} V(x_i - x_j)\right)\psi_N(\vec{x}_1....\vec{x}_N) = \varphi_N(\vec{x}_1....\vec{x}_N) \end{gathered} \tag{2.13}$$

The verification only uses the commutation relations of the $a^{\#}(x)$ and the annihilation property of $a(x)$ applied to the vacuum. The various terms in the N-body Schrödinger operator result from the following commutators which arise in the process of moving H through the N $a^*(x)'s$ onto the no-particle state:

$$[\mathbf{H}, a^*(\vec{x})] = -\frac{1}{2m}\Delta_x a^*(x), \quad [\mathbf{V}, a^*(\vec{x})a^*(\vec{y})] = V(\vec{x}-\vec{y})a^*(\vec{x})a^*(\vec{y}) \tag{2.14}$$

The hamiltonian H in Fockspace is used to define time-dependent operators:

$$a(\vec{x},t) = e^{i\mathbf{H}t}a(\vec{x})e^{-i\mathbf{H}t} \tag{2.15}$$

Only in the case $\mathbf{H} = \mathbf{H}_0$ and for "external interactions"

$$\mathbf{H} = \mathbf{H}_0 + \int V(x)a^*(x)a(x)d^3x \tag{2.16}$$

is the time dependent a(x,t) linear in a(x).

Before we consider an application some remarks on the mathematical status of the $a^{\#}$'s and related operators are helpful. Since the N-particle states for arbitrary N form a dense set of states, the $a^{\#}$ are densely defined. Using the number operator it is easy to compute:

$$\left\|a(f)\mathbf{N}^{-\frac{1}{2}}\right\|_{H_F^{\perp}} = (f,f)\,, \quad H_F^{\perp} = \text{subspace of } H_F \perp \Omega \tag{2.17}$$

We remind the reader that the norm of an operator A is related to the vector norm in the Hilbertspace:

$$||A|| = \sup_{\psi} \frac{||A\psi||}{||\psi||} \tag{2.18}$$

The technique for computing such norms is always the same: first one uses the defining formula for a(f) in order to prove the inequality and then one exhibits a particular vector for which the equality sign holds. In our case $\int f(\vec{x})a^*(\vec{x})\Omega$ is such a vector. The norm of the adjoint is the same $||A|| = ||A^*||$.

The relative boundedness with respect to N e.g. $||a^*(f)\psi|| \leq ||N\psi||^{\frac{1}{2}}$ may be used to show that these unbounded operators are closable and hence admit e.g. a polar decomposition.

Most physicist's calculations do not touch these fine points. They only check equations for densely defined bilinear forms . In case of the above formulas for H this means that one checks this formula for $\langle\Psi, |H|\Phi\rangle$ with the vectors running through e.g.the dense set of smooth states of finite particle number. The extension to a relation between densely defined closable or selfadjoint operators is in most physically relevant cases possible and follows a standard scheme (Reed-Simon). In those cases we will be satisfied with the check for matrix elements which is easily done with the commutation relations for $a^{\#}$and the above "pulling through onto the vacuum" rule. All perturbative calculation in Fock space are done with these rules and this applies also to the derivation of Feynman rules in relativistic QFT.

In order to illustrate the application of bosonic (symmetrized) Fockspace techniques to coherent states, we first convince ourself that the $a^{\#}$- formalism for an oscillator is a special case of the present formalism (specialization to one degree of freedom). For a one-dimensional space $\mathcal{H}_1 = \mathbf{C}$, all the tensor product spaces are also one dimensional and the "single degree of freedom" operator $a^{\#}$ does not require any additional label. An arbitrary vector may be written as:

$$|\Psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle + \psi_2 |2\rangle + ..., \quad |n\rangle = \frac{a^{*n}}{\sqrt{n!}} |0\rangle, \quad a|0\rangle = 0 \tag{2.19}$$

Writing instead of the $a^{\#'}$s standard dynamical variables of QM p and q (with natural units $\hbar = 1$) :

$$a^* = \sqrt{\frac{\omega}{2}}x + i\sqrt{\frac{1}{2\omega}}p \tag{2.20}$$

The $a^{\#}$- commutation relations go over into the Heisenberg c.r. and the standard oscillator hamiltonian takes the form:

$$H_{osc} = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 = \omega(a^*a - \frac{1}{2}) \tag{2.21}$$

The x-space wave functions $\langle x \mid n\rangle$ of the eigenstates $|n\rangle$ turn out to be the well-known Hermite functions. Coherent states are obtained by asking for the eigenstates of the perturbed hamiltonian:

$$H = H_{osc} + \lambda(a + a^*) = H_{osc} + \lambda\sqrt{2\omega}x \tag{2.22}$$

The linear perturbation can be obtained (modulo an uninteresting c-number term) by applying a spatial translation by $a = \frac{\lambda}{\sqrt{2\omega}}$ to H_{osc}. In terms of $a^{\#}$ this translation $U(a)$ is:

$$U(\frac{\lambda}{\sqrt{2\omega}}) = e^{-\frac{\lambda}{\omega}(a^* - a)} \tag{2.23}$$

Different from the previous use of Fockspace formalism, the number operator for the oscillator quanta $\mathbf{N} = a^*a$ does not commute with the perturbation. Hence the eigenstates of $\mathbf{H}$ do not have a well defined number such quanta. In order to obtain explicit formulas for $U\,|n\rangle$ we use the Campbell-Baker-Hausdorff formulas:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+....} \tag{2.24}$$

where the.... stands for higher commutator terms. This is easily established for matrices and (modulo domain problems) by perturbative arguments in the general case. Due to the absence of higher commutators we get:

$$e^{-\frac{\lambda}{\omega}a^*} e^{\frac{\lambda}{\omega}a} = e^{-\frac{\lambda}{\omega}(a^* - a)+\frac{1}{2}\frac{\lambda^2}{\omega^2}} \tag{2.25}$$

Therefore the ground state of H is an "eigenstate" of the annihilation operator:

$$|\Psi_0\rangle = U\,|0\rangle = e^{-\frac{1}{2}\frac{\lambda^2}{\omega^2}} e^{-\frac{\lambda}{\omega}a^*}\,|0\rangle\,, \quad a\,|\Psi_0\rangle = -\frac{\lambda}{\omega}\,|\Psi_0\rangle \tag{2.26}$$

Here in the first step we used the BCH formula to separate the annihilation part of U to the right (where an $e^{\alpha a}$ factor on $|0\rangle$ becomes the identity). For the eigenvalue equation use the translation property. On the higher eigenstates $U\,|n\rangle$ the application of a leads to an additive modification of the eigenvalue relation by $U\,|n-1\rangle$. The probability distribution of the oscillator quanta follows the Poisson distribution:

$$|\langle n \mid \Psi_0\rangle|^2 = \frac{e^{-\frac{\lambda^2}{\omega^2}}}{n!}\left(\frac{\lambda}{\omega}\right)^{2n} \tag{2.27}$$

Physically the perturbed oscillator may be thought of as resulting from a constant electric field:

$$H = H_{osc} - eEx \tag{2.28}$$

This field causes the expectation values of the $a^{\#}$'s to be nonvanishing:

$$\langle \Psi_n \mid a^{\#} \mid \Psi_n \rangle \sim E \tag{2.29}$$

The free time development on the the state vectors leads to the classical oscillatory behaviour of expectation values:

$$\begin{aligned} \langle \Psi_n(t)\,|x|\,\Psi_n(t)\rangle &\sim E\cos(\omega t - \varphi) \\ \langle \Psi_n(t)\,|p|\,\Psi_n(t)\rangle &\sim E\sin(\omega t - \varphi) \end{aligned} \tag{2.30}$$

This oscillatory behaviour would be the result of a sudden switching on of the field:

$$H(t) = \begin{cases} H & for\ t < 0 \\ H_{osc} & for\ t \geq 0 \end{cases} \tag{2.31}$$

by which the coherent states are created. The following classical behaviour of expectation values of functions in $a^{\#}$ is characteristic for coherent states:

$$\langle \Psi_0 | f(a^{\#}) | \Psi_0 \rangle = f\left(\langle \Psi_0 | a^{\#} | \Psi_0 \rangle \right) \tag{2.32}$$

In case of a time dependent source:

$$H(t) = H_{osc} + H_{int}(t), \quad H_{int}(t) = -eE(t)x \tag{2.33}$$

we are dealing with time dependent unitary transformations which implement the time dependent canonical transformations:

$$U(t)a^{\#}U^{*}(t) = a^{\#} - \sqrt{2\omega}E \tag{2.34}$$

which lead from H_{osc} to $H(t)$. In this simple case the U(t) has the same form as in the stationary case except that the constant in front of the a-a* term in the exponential is now time dependent. It is usefull to have a more systematic method which also works for cases for which the $U(t)$ is less simple. Such a method probably goes back to Dirac (and flourished in QFT thanks to Dyson) and treats the time-dependent problems in the "interaction picture" which is between the Heisenberg picture and the Schrödinger picture. All these pictures agree on the level of physical states i.e. in their expectation values, but they differ in how the total time development is distributed between operators and state vectors. Whereas in the Heisenberg - and Schrödinger-picture the full time development is on the operators respectively on the state vectors, the interaction picture is characterized by the property that the operators only suffer the free time development and the rest (the interaction picture time development V(t)) is dumped on the vectors. According to this definition the interaction operator $H_{int}(t)$ becomes:

$$H_w(t) = e^{iH_0 t} H_{int}(t) e^{-iH_0 t} \tag{2.35}$$

The time development operator $V(t_2, t_1)$ which propagates the vector state (or wave function) from one time to a later time is:

$$V(t_2, t_1) = Te^{-i \int_{t_1}^{t_2} H_W(t)dt} \tag{2.36}$$

It is a solution of the Schrödinger equation in the interaction picture:

$$i\frac{d}{dt}V(t,t') = H_W(t)V(t,t') \tag{2.37}$$

The time- (or path-) ordering is defined as:

$$\begin{aligned} TA_1(t_1)A_2(t_2)....A(t_n) &= A_{i_1}(t_{i_1})A_{i_2}(t_{i_2})....A_{i_n}(t_{i_n}) \\ for \quad t_{i_1} &\geq t_{i_2} \geq .... \geq t_{i_n} \end{aligned} \tag{2.38}$$

and the above time ordered exponential is defined by the power series with time ordered integrands or as the limit of subsequent products with decreasing length of the time intervalls as :

$$\begin{aligned} &Te^{-i \int_{t_1}^{t_2} H_W(t)dt} = \\ &\lim_{\Delta t \to 0} Te^{-i \int_{t_{n-1}}^{t_n} H_W(t)dt} ....Te^{-i \int_{t_2}^{t_3} H_W(t)dt} Te^{-i \int_{t_1}^{t_2} H_W(t)dt} \end{aligned} \tag{2.39}$$

The proof consists in rewriting the Heisenberg time development operator $U(t,s)$ (which in the case of time independent interactions just reduces to $e^{-iH(t-s)}$ as $e^{-iH_0(t-s)}$ times a remaining operator V (this is the split into the the free motion on the operators and the remainind interaction picture time development V on the state vector) :

$$U(t,s) = e^{-iH_0(t-s)}V(t,s) \quad i.e.\, V(t,s) = e^{iH_0(t-s)}U(t,s) \tag{2.40}$$

The Schrödinger equation for $U(t,s)$ is then equivalent to the following differential equation for $V(t,s)$:

$$\begin{aligned} i\frac{d}{dt}V(t,0) &= e^{iH_0t}(H-H_0)e^{iHt} = e^{iH_0t}(H_{int})e^{-iH_0t}e^{iH_0t}e^{-iHt} \\ &= H_w(t)V(t,0) \end{aligned} \tag{2.41}$$

The rest of the proof consists in deriving the time-ordered representation from the formal integration of this differential equation. On first convers this into an integral equation (using $V(0,0) = 1$ as an initial condition):

$$V(t,s) = 1 - i\int_s^t \frac{d}{dt'}H_w(t')V(t',s) \tag{2.42}$$

Clearly the perturbative solution is the geometric series:

$$V(t,s) = 1+(-i)\int_s^t H_w(t')dt'+(-i)^2\int_s^t dt_2\int_s^{t_2} dt_1 H_w(t_2)H_w(t_2)+.... \tag{2.43}$$

where the n^{th} term is integrated over the simplex $s \le t_1 \le t_2 \le .... \le t_n \le t$. The use of the the (nonlocal!) time-ordering prescription allows to convert the integration over a simplex into one over the n-dim. hypercube $s \le t_i \le t$, $i = 1...n$:

$$\begin{aligned} V(t,s) = 1 + (-i)\textstyle\int_s^t H_w(t')dt' + \frac{1}{2!}\int_s^t\int_s^t TH_w(t_2)H_w(t_1)dt_1dt_2 + ... \\ +\frac{1}{n!}\int ....\int .... + ... \end{aligned} \tag{2.44}$$

which has the desired exponential time-ordered form. These somewhat formal manipulations may be mathematically justified in two different ways. Either one finds a bound for the n^{th} term, or one shows the equivalence of the time-orderd expression with an exact unitary transformation which, like the one at the beginning of this section is a "dressing transformation" i.e. applied to the free hamiltonian it generates the interaction. Let us briefly explain this for the infinite degrees of freedom analog of the perturbed oscillator: a bosonic field system under the influence of an external source described by the hamiltonian:

$$H(t) = H_0 + a(j_t) + a^*(j_t), \quad H_0 = \int(\frac{\vec{p}^2}{2m} + \mu)a^*(\vec{p})a(\vec{p})d^3p, \tag{2.45}$$

where $j_t(\vec{x}) = j(\vec{x},t)$ and we have added a chemical potential term in order to avoid infrared divergencies of the p-integrals in subsequent calculations. The

dressing transformation is:

$$U(t) = e^{a(g_t)-a^*(g_t)}, \quad g(\vec{x},t) = \frac{1}{(2\pi)^3}\int \tilde{j}(\vec{p},t)(\frac{\vec{p}^2}{2m}+\mu)^{-1}e^{i\vec{p}\vec{x}}d^3p \qquad (2.46)$$

The connection with the time development U(t,s) is evidently (since it dresses the free operator):

$$U(t,s) = U(t)e^{-iH_0(t-s)}U^*(t) \quad or \quad V(t,s) = e^{iH_0(t-s)}U(t)e^{-iH_0(t-s)}U^*(t) \qquad (2.47)$$

The direct calculation of the time-ordered representation for V uses the previously mentioned infinite product representation:

$$V(t,s) = \lim_{\Delta t\to 0}\prod_{ord} e^{-i\int_{t_i}^{t_{i+1}} H_w(t')dt'} \qquad (2.48)$$

where the product is ordered with ascending times going to the left, but the time ordering within each factor is omitted. This formula is similar to the famous Trotter formula; in integrals over shrinking intervalls the difference between the time ordered and the unordered expression disappears. To this product form we may apply the BCH-formula in order to collect all operators within one unordered exponential:

$$\begin{aligned} V(t,s) = \lim_{\epsilon\to 0}\exp(-i\textstyle\sum_{j=0}^{n-1}\int_{s+j\epsilon}^{s+(j+1)\epsilon} H_w(t')dt' \\ -\tfrac{1}{2}\textstyle\sum_{j\neq k}\int_{s+j\epsilon}^{s+(j+1)\epsilon}dt'\int_{s+k\epsilon}^{s+(k+1)\epsilon}dt''\,[H_w(t'),H_w(t'')]) \end{aligned} \qquad (2.49)$$

Since the commutators are c-numbers, the result is of the form:

$$V(t,s) = \exp(-ia(j_{t,s}) - ia^*(j_{t,s}) - i\frac{\alpha}{2}) \qquad (2.50)$$

where α is a numerical phase (resulting from the commutator) and $j_{t,s}(\vec{x})$ is the result of time propagating the original source function in H_{int}. We obtain agreement between the two methods. Furthermore we learn that the time ordered exponential leads to a phase factor which is not present in dressing approach. Specializing now to the limit $t \to \infty, s \to -\infty$ (assuming that the interaction only extends over a finite time or that the integrals over time in V converge) we define the S-operator as the full interaction picture transition operator V which relates the free system before and after the interaction:

$$S = \lim_{t,s\to\pm\infty} V(t,s) \qquad (2.51)$$

Clearly the application of $S = exp(-ia(g) - ia^*(g) - i\frac{\alpha}{2})$ with $g = lim_{t,s}j_{t,s}$ onto the vacuum Ω gives a coherent state vector:

$$S(g)\Omega = e^{-i\frac{\alpha}{2}}\Omega(ig), \qquad (2.52)$$

The successive action of sources leads to the composition law:

$$S(f)\Omega(ig) = c(f,g)\Omega(ig+if), \quad |c| = 1 \qquad (2.53)$$

Therefore even if we eliminate the phase factor in the definition of $S\Omega$, it will reappear in form of a so called 2-cocycle in the composition law. as in the case of the oscillator, the source generates a coherent distribution of say photons with a poisson probability distribution. On a coherent state vector $\Omega(ig)$ the action of $S(f)$ will change the mean particle number to:

$$\begin{aligned}\Delta N &= (\Omega(if+ig), \mathrm{N}\Omega(if+ig)) - (\Omega(ig), \mathrm{N}\Omega(ig)) \\ &= ||f+g||^2 - ||g||^2 = ||f||^2 + 2Re(f,g)\end{aligned} \tag{2.54}$$

The interference term 2Re(f,g) describes induced absorption or emission depending on the sign. Many important results of laser physics may be developed in this formalism.

2.2 The Fermion Fockspace

The antisymmetric N-particle space was obtained by acting with the antisymmetric projector $\mathrm{P}_a$on the full N-fold tensor product H_N of one-particle spaces:

$$H_N^{(-)} = \pi(P_a)H_N, \qquad P_a = \frac{1}{N!}\sum_{P\in S_N} sign(P)P \tag{2.55}$$

Here $\pi(P)$ $P \in S_N$ stands for the natural representation of S_N on the full tensor space H_N. The Fermionic Fockspace is simply the direct sum of all antisymmetrized N-particle spaces augmented by the one dimensional no-particle state.

$$H^{(a)} = H_0 + H_1 + \sum_{N=2}^{\infty} H_N^{(a)} \tag{2.56}$$

The only difference to the Bosonic case (besides the antisymmetry of the wave functions) is the sign appearing in the formula for the creation operator:

$$(a^*(f)\psi)_n(\vec{x}_1, ...\vec{x}_n) = \frac{1}{\sqrt{n}}\sum_i (-1)^{i+1} f(\vec{x}_i)\psi_{n-1}(\vec{x}_1, ..\widehat{\vec{x}}_i, ..\vec{x}_n) \tag{2.57}$$

Here the roof on the $\vec{x}_i$ indicates omission of this variable. In a completely analogous fashion we obtain the anticommutation relations:

$$\{a(f), a^*(g)\} = (f,g), \quad \{a(f), a(g)\} = 0, \quad \{a^*(f), a^*(g)\} = 0 \tag{2.58}$$

and removing the wave packets:

$$\{a(\vec{x})a^*(\vec{y})\} = \delta(\vec{x}-\vec{y}), \quad \{.,.\} = 0 \text{ in all other cases} \tag{2.59}$$

There is no change in the formulas which carry the Schrödinger theory on antisymmetric N-particle wave functions to the Fockspace (at least if one writes H exactly in the same order in the $a^{\#}$'s). A significant difference to the bosonic case begins to show up, if one realizes that as a consequence of:

$$\{a(f)a(f)\} = 2a(f)^2 = 0 \tag{2.60}$$

and the hermitian adjoint relation, we obtain the Pauli exclusion principle: in an occupation number representation any quantum level can maximally be singly occupied:

$$|n_1, n_2, ...\rangle = \frac{a_1^{*n_1}}{\sqrt{n_1!}} \frac{a_2^{*n_2}}{\sqrt{n_2!}} .... |0\rangle , \quad n_i = 0, 1 \tag{2.61}$$

This principle holds only if all quantum numbers of a particle (including spin and possible internal charges) have been taken into account. Closely related is the ability of fermion-systems to form a new reference state by simply *occupying* a given set of levels (orthonormal one-particle vectors f_i i=1...N):

$$|\Psi_0\rangle = a_1^*....a_N^* |0\rangle , \quad a_i^* \equiv a^*(f_i) \tag{2.62}$$

This vector is annihilated by the new annihilation operators:

$$b(f) = \begin{cases} a^*(f) & if\ f \in H(f_1, ...f_N) \\ a(f) & if\ f \in H^\perp(f_1, ...f_N) \end{cases} \tag{2.63}$$

Here $H(f_1...f_N)$ is the subspace of the one-particle space spanned by the system of vectors f_i. Note that the $b^\# s$ obey the same commutation relations as the $a^\# s$. The annihilation property $b|\Psi_0\rangle = 0$ is an easy consequence. Note that that the hermitian adjoint $b^*(f_i)$ creates holes in $|\Psi_0\rangle$. This ability to create states which are annihilated by transformed Fermion variables $b^\#$ through occupying levels is typical for CAR. On the other hand for coherent states (relevant in e.g.laser physics) and Poisson-distributions one needs Bosons. Mathematically the CAR- structure (canonical anticommutation relations) is well behaved since the $a^\#(f)$ are bounded operators:

$$\begin{gathered} (\Phi, \{a(f), a^\#(f)\}\Phi) = (f, f)(\Phi, \Phi) \\ i.e. \quad ||a(f)\Phi||^2 \leq (f, f)(\Phi, \Phi) \end{gathered} \tag{2.64}$$

By taking $\Phi = a^*(f)\Omega$ we establish saturation (=) and therefore $||a(f)|| = (f, f)$ for the operator norm. The counterpart of the one-dimensional oscillator is:

$$\sigma_x = a + a^* \quad \sigma_y = i(-a + a^*) \quad \sigma_z = aa^* - a^*a \tag{2.65}$$

With σ being the Pauli-matrices i.e. the smallest irreducible representation of the Clifford algebra structure defined by $a^\#$ is in terms of Pauli matrices. This observation can be generalized:

Theorem 2 $Alg(a_i^\#, i = 1....N) \equiv Cliff(C^N) = \otimes^N Mat_2(C) = Mat_{2^N}(c)$

The proof consists in starting with a generating system of matrix units for the N-fold tensor product of Mat_2(C):

$$e_{ij}^{(k)} = \underline{1} \otimes ...\underline{1} \otimes e_{ij} \otimes \underline{1}...\underline{1} \tag{2.66}$$

Clearly the four 2×2 matrix units e_{ij} are linear combinations of the four Pauli-matrices and the system commutes for different k. The step towards anticommutation requires the introduction of the famous Jordan-Pauli transformormation. With the help of:

$$\mu_k = \prod_{i=1}^{k}(e_{11}^{(k)} - e_{22}^{(k)}) = \prod_{i=1}^{k}(1 - 2a^*a) \tag{2.67}$$

we define $a_i = \mu_{i-1}e_{12}^{(k)}$. and its hermitean adjoint. In commuting one of such objects through one of the e_{12} meets one of the $(e_{11} - e_{22})$ factors which leads to the -sign. The relation between the matrix units and the $a^{\#}s$ can be inverted and the generatel algebras are identical.

In the "Paulion" formalism, the filling operation is described by the unitary:

$$U = \sigma_1 \otimes \sigma_1 \otimes .... \otimes \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.68}$$

From this one reads off the filling operator in the $a^{\#}$ representation.

Clearly the filling mechanism is as typical for Fermions as the coherent state properties are for Bosons. Without the former there would be no periodic table (nor we) and without the latter no laser.

2.3 The CCR and CAR Algebras.

Wereas in the case of the Fermion Fock space an abstraction to a C*-algebra is straightforward (just take the C*-algebra generated by the $a^{\#}s$ subject to the anticommutation relations, uniqueness will be shown later), a construction of a C*-algebra from unbounded operators generally poses serious obstacles. Following Weyl, one formally converts the $a^{\#}$ into unitary operators:

$$W(f) = e^{i\Phi(f)}, \quad \Phi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f)) \tag{2.69}$$

The application of the BCH-formula leads to:

$$W(f)W(g) = e^{-i\frac{1}{2}\sigma(f,g)}W(f+g) = e^{-i\sigma(f,g)}W(g)W(f) \tag{2.70}$$

where $\sigma(f,g) = Im(f,g)$ is a non degenerate symplectic form. Formally the unbounded operators $a^{\#}(f)$ may be reobtained by multiplying f with a parameter t and it and differentiating the modified W with respect to t at t=0. We now take this Weyl relation or rather $Alg(W(f), f \in H)$ as our basic definition of the Boson algebra. This algebra is clearly an infinite degree of freedom generalization of the well-known Heisenberg-Weyl algebra which underlies standard QM:

$$\begin{aligned} U(\vec{\alpha}) &: = e^{i\vec{\alpha}\vec{p}}, \quad V(\vec{\beta}) := e^{i\vec{\beta}\vec{q}} \\ W(\vec{\gamma}) &: = e^{i\frac{1}{2}\vec{\alpha}\vec{\beta}}U(\vec{\alpha})V(\vec{\beta}), \quad \vec{\gamma} = \vec{\alpha} + i\vec{\beta} \end{aligned} \tag{2.71}$$

One easily checks with BCH that the W fullfill the above Weyl relation with $f = \gamma \in C^N$ and the symplectic form being the standard symplectic form being of the standard type known from 2N dim. phase space of classical mechanics. The following theorem collects the important structural properties of the CAR and CCR(Weyl) algebras.

Theorem 3 *The CAR and CCR algebras are simple (no ideals) C*-algebras generated by the CAR resp.CCR commutation relations.*

We only indicate the proof and refer to Bratteli-Robinson Vol 2 for details.

In the CAR case we know from the previous consideration that for finite degrees of freedom Fermions $a_i^\#$ i=1...N can be replaced by "Paulions". For infinite degrees of freedom we take a basis f_i i=1...∞ in the one particle space H. The uniqueness of the limiting algebra follows from the continuity resulting from $||a(f)|| = ||f||$. The full algebra is in fact an inductive limit of finite degree of freedom algebras. The separability is inherited from the Mat_N-algebras.

The proof in the CCR case is somewhat different. In this case one reduces the problem to a projective unitary representation of an (infinite) abelian group H with the multiplier $exp.-i\sigma(f,g)$ being a character. In this way the problem is reduced to that of uniqueness of C*-group algebras. The triviality of the ideal is established by showing that the kernel of every representation is trivial. Separability and the inductive uniform limit property does not hold. As in the CAR case one may ask about the uniqueness of irreducible representations (up to unitary equivalence). This indeed holds in the important class of "regular" representations i.e. representations π for which the unitaries $\pi(W(tf))$ are strongly continuous in t.

Theorem (Stone-von Neumann uniqueness theorem). Every regular irreducible representation of the Heisenberg-Weyl algebra for a finite number of degrees of freedom is unitarily equivalent to the Schrödinger representation, or alternativly the algebraic structure of standard Q.M. has no nontrivial superselection rules.

The proof uses the infinitesimal generators Φ, which thanks to the regularity property turn out to have a densely defined domain which allows to construct the $a(f)^\#$ and the number operator $\mathbf{N} = \sum a_i^* a_i$. The positivity of the latter requires the existence of of a "lowest" vector which is the required reference state for the annihilation operators. If on the other hand we are dealing with infinite degrees of freedom (i.e.dimH=∞), the sum in N need not to converge. In such representations the number operator does not exist. Examples are easily given.

Bosonic illustration If the shift function c(x) in a(x)$\rightarrow$b(x)=a(x)+c(x) is not square integrable (physically because of short distance [ultraviolet] or long distance [inf rared] divergencies) then N does not exist and the formal expression for the unitary implementer $U(c)$ cannot be given a meaning.

Fermionic illustration If the "occupied" Hibertspace is infinite dimensional no unitary implementer can exist. The reason is that such a vector

$\Phi = \prod_i a_i^* \Omega$ is orthogonal on each basis vector of the particle number representation:

$$(\Phi, \Psi(n_1, n_2 .... n_N)) = 0,$$
$$\Psi(n_1, n_2 .... n_N) = a^{*n_1} a^{*n_2} .... a^{*n_N} \Omega, \quad n_i = 0 \text{ or } 1 \tag{2.72}$$

This is because for any arbitrary large N the Φ contains infinitly many creation operators which remain uncompensated. The formal expression for Φ cannot be given meaning in Fockspace.

There is another way of looking at this illustration. The infinite sequence of 0 and 1 in $(n_1, n_2 .....)$ may be considered as a binary fraction. Whereas the Fock-basis consists of binary fractions with $n_i = 0$ for sufficiently large i (which may become arbitrarily large), the binary fraction for the above Φ is the constant sequence $(1, 1, 1, ..1....)$. This sequence is not in the vacuum class $(0, 0, ..0....)$ where "class" here means the set of sequences which deviate from each other only in an arbitrary large but finite number of places. Each class belongs to a basis in one Hilbertspace and the the different basis elements $\Psi(n_1, n_2, ...)$ are obtainable from one reference element in the class by the application of a finite (but arbitrarily large) number of $a^{\#'s}$. The various irreducible representation spaces obtained from the different classes are orthogonal subspaces of an inseperable unwieldy (and unphysical) Hilbertspace generated by all binary fractions (which form a continuous set). The same idea of classes of sequences works for bosons. In that case the n_i run through all natural numbers including zero. One obtains myriads of inequivalent irreducible representations and this construction is not even exhaustive. Most of them are physically uninteresting and one needs a physical selection principle. Many of the physically interesting ones are in the subset of "quasifree" states explained in the next section.

We close this section by commenting on automorphisms of the CCR and CAR C*-algebras which are linear maps of the algebra onto itself which preserve the algebraic structure i.e. in physical terms they are symmetry transformations which preserve the Weyl resp. CAR relation. Of particular interest are the Bogoliubov automorphisms. They are induced by (anti-)linear invertible transformations of the underlying linear wave function space H. In the CCR case they are required to leave the symplectic form σ on H invariant and map the Weyl generators as follows:

$$\sigma(Tf, Tg) = \sigma(f, g), \quad \alpha(W(f)) = W(Tf) \tag{2.73}$$

In the CAR case we must use (anti-)unitary operators in order to preserve the anti-commutator:

$$(Uf, Ug) = \begin{cases} (f, g) & \text{unitary} \\ \overline{(f, g)} & \text{antiunitary} \end{cases} \tag{2.74}$$

A slightly more general automorphism is obtained by combining these two possibilities:

$$\alpha(a(f)) = a(Uf) + a^*(Vf)$$
$$UU^* + VV^* = 1 = U^*U + V^*V, \quad V^*U + U^*V = 0 = UV^* + VU^* \tag{2.75}$$

where U is linear and V antilinear. The previous case is obtained by specialization V=0 resp. U=0. Clearly the earlier occupation transformation corresponds to the automorphism $\alpha(a(f)) = a^*(Vf)$. The crucial question is now whether the automorphism is really a bona fide symmetry i.e. implementable by a unitary transformation.Take as an example the shift $a^*(\vec{x}) \rightarrow a^*(\vec{x}) + c(\vec{x})$ which is formally implemented by the unitary:

$$Ua^*(\vec{x})U^* = a^*(\vec{x}) + c(\vec{x}), \quad U = e^{a(c)-a^*(c)} \tag{2.76}$$

But without the condition $\int |c(\vec{x})|^2 d^3x < \infty$ the operator U woud not be bona fide unitary in Fockspace. Physical intuition would tell us to expect that Bogoliubov tranformation in the one particle space H need to be sufficiently close to the identity in order to have an implementation in Fockspace. This is indeed the case, the deviation from $\underline{1}$ should be in the Hilbert-Schmidt class. Since Bogoliubov transformations leave the property of "quasi-freeness" invariant, the natural place for presenting the relevant implementation formulas is the next section.

2.4 Quasifree States

The most convenient way to obtain representations of the CCR and CAR C^*-algebras is through states ω on these algebras. We have seen (appendix A) that the GNS-construction gives a canonical association of a representation π_ω with a state ω. Since there are too many inequivalent states and associated representations which nobody has been able to classify, we need some limitation. It turns out that the class of quasifree states and their representations can be classified completely. They are defined by their two-point functions together with a combinatorial formula which expresses their n-point functions in terms of the given two-point functions. On the generators $a^\#$ we specify the state ω by giving first its two-point functions:

$$\begin{gathered}\omega(a(f)a(g)), \quad \omega(a(f)a^*(g)), \\ \text{or} \quad \omega(a(x)a(y)), \quad \omega(a(x)a^*(y))\end{gathered} \tag{2.77}$$

The remaining two-point functions $\omega(a^*(g)a^*(f))$ is (according to the reality properties of states following from their positivity) just the complex conjugate and $\omega(a^*(g)a(f))$ may be obtained by (anti-)commutation. The higher correlation functions of ω are given in terms of the two-point function by the following combinatorical formula:

$$\omega(a^\#(f_1)a^\#(f_2)....a^\#(f_{2n})) = \sum_{pairings\ P} signP \prod_{i_k<i_{k+1}} \omega(a^\#(f_{i_k})a^\#(f_{i_{k+1}}))$$

$$\omega(a^\#(f_1)a^\#(f_2)....a^\#(f_{2n+1})) = 0 \tag{2.78}$$

We have to prove that ω is positive on the polynomial algebra generatet by the $a^\#$'s:

$$\omega(A^*A) \geq 0 \quad A = polyn(a^\#) \tag{2.79}$$

For the CAR-algebra the bound from the anticommutation relations:

$$\omega(a(f)a^*(f)) \leq ||f||^2 \tag{2.80}$$

gives immediatly :

$$\omega(a(f)a^*(g)) = (f, Tg) \quad 0 \leq T \leq 1 \tag{2.81}$$

The positivity on monomials $A = a^\#(f_1)....a^\#(f_n)$ is a result of the basic two-point positivity:

$$\begin{aligned} ||(a(f) + a^*(g))\,\Omega||^2 &= \\ \omega\,(a^*(f)a(f) + a(g)a^*(g) + a^*(f)a^*(g) + a(g)a(f)) &\geq 0 \end{aligned} \tag{2.82}$$

(The latter holds as the result of the positivity of T and the Cauchy-Schwartz inequality) and the combinatorical definition of the n-point function.

A particular subclass of quasifree states are the gauge invariant quasifree states. By definition only those correlation functions are nonvanishing which contain the same number of a and a^*. Instead of working with unbounded operatorsone prefers to define the quasifree gauge invariant states directly in the Weyl algebra:

$$\omega(W(f)) = \exp(-\frac{1}{4}\,||f||^2 - \frac{1}{2}\left|\left|T^{\frac{1}{2}}f\right|\right|^2) \tag{2.83}$$

The standard Fock-representation reemerges as the special case T=0. Quasifree states are regular (by construction) so that we can return to unbounded $a^\#$ in a similar manner as we introduce Lie algebra generators in noncompact group representation theory. The corresponding GNS representation is most conveniently written in terms of an auxilary "doubled Fockspace", for CAR:

$$\begin{aligned} a_\omega(f) &= a(\sqrt{1-T}f) \otimes 1 + \gamma \otimes a^*(K\sqrt{T}f), \quad T \leq 1 \\ a^*(f) &= a^*(\sqrt{1-T}f) \otimes 1 + \gamma \otimes a(K\sqrt{T}f) \end{aligned} \tag{2.84}$$

For CCR we obtain the analogous formula:

$$\begin{aligned} a_\omega(f) &= a(\sqrt{1+T}f) \otimes 1 + 1 \otimes a^*(K\sqrt{T}f) \\ a^*_\omega(f) &= a^*_\omega(\sqrt{1+T}f) \otimes 1 + 1 \otimes a(K\sqrt{T}f) \end{aligned} \tag{2.85}$$

Here K is the standard conjugation (Kf,Kg)=(g,f) and γ (only defined in the CAR case) is the unitary operator which implements the Z_2 gauge transformation (distiguishes even from odd numbers of Fermions) in Fockspace. The proof consists in a simple calculation of the two-point function in the vector $\Omega_{double} = \Omega \otimes \Omega$.

The irreducibility condition for gauge invariant quasifree representations is that T is a projector T=P. The equivalence criterion for two gauge invariant quasifree representations is:

Theorem 4 *Two irreducible representations given in terms of P and Q are equivalent iff* $||P - Q||^2_{H.S.} < \infty$. *here the H-S norm of K is defined as* $TrK^*K < \infty$.

2.5 Temperature States and KMS condition

For a finite quantization box (i.e. a discrete energy spectrum), finite temperature states on the CCR or CAR-algebra are described in terms of the Gibbs formula:

$$\rho = \frac{1}{Z} e^{-\beta H}, \quad Z = Tre^{-\beta H}, \quad H = H_0(\mu) + H_{int} \tag{2.86}$$

since $e^{-\beta H}$ is then a trace class operator. Here H_0 includes the chemical potential μ:

$$H_0(\mu) = \int \omega(\vec{p}) a^*(\vec{p}) a(\vec{p}) d^3p, \quad \omega(\vec{p}) = \frac{\vec{p}^2}{2m} - \mu \tag{2.87}$$

The box-enclosed version is of course a sum over discrete momenta where the latter result by extending the Laplace operator on smooth functions with support in the volum V in a selfadjoint manner to square integrable functions in V (the various ways of doing this correspond to the various boundary conditions). The physical role of the chemical potential is that the ground state energies for different particle numbers can be adjusted in such a way that the averaged particle n(x) and energy h(x) density:

$$\bar{n} = Tr\rho n(\vec{x}) \quad \bar{\epsilon} = Tr\rho h(\vec{x}) \tag{2.88}$$

remain finite in the thermodynamical limit $V \to \infty$ and hence can be expressed in terms of the two parameters β and μ. For the ideal Fermi or Bose gas ($H_{int} = 0$) the approach of the (quasifree) Gibbs state to the limit KMS state is obvious by explicite calculation:

$$\begin{array}{c} \lim_{V\to\infty} \omega_V(a^*(f)a(g)) = \omega(a^*(f)a(g)) \\ \omega_V(a^*(f)a(g)) = \frac{1}{Z} Tre^{-\beta \mathbf{H}}\, {}_V a^*(f)a(g) = (g, T_V f) \\ T_V = (\exp -\beta H_{0V})(1 + \exp -\beta H_{0V})^{-1} \end{array} \tag{2.89}$$

Here the (non-bold) H_0 are the one-particle operators acting on wave functions whereas $\mathbf{H_0}$ acts in Fock-space. The ω in the thermodynamic limit is also of the quasifree form with H_0 replacing H_{0V}. The simplest way of proving these relations is to use the KMS property:

$$\omega_V(\sigma_t(a^*(f))a(g)) = \omega_V(a(g)\sigma_{t+i\beta}(a^*(f))) \tag{2.90}$$

which for t=0 together with the (anti)commutation relation leads to:

$$\omega_V(a^*(f)a(g) \pm a^*(e^{-\beta H_{0V}} f)a(g)) = (g, e^{-\beta H_{0V}} f) \tag{2.91}$$

We used that the hamiltonian automorphism σ_t is of the Bogoliubov type. We rewrite this equation as

$$\omega_V(a^*((1 \pm e^{-\beta H_{0V}})f)a(g)) = (g, e^{-\beta H_{0V}} f) \tag{2.92}$$

Clearly this relation is solved by:

$$\sigma_t(a^*(f)) = a^*(e^{-\beta H_{0V}}(1 \pm e^{-\beta H_{0V}})^{-1} f) \tag{2.93}$$

After Fourier-transformation, $e^{-\beta H_{0V}}$ becomes a multiplication operator $exp - \beta(\frac{\vec{p}^2}{2m} - \mu)$ and hence

$$n(\vec{p}) = \exp -\beta\omega(\vec{p})(1 \pm \exp -\beta\omega(\vec{p}))^{-1}$$
$$\omega_V(a^*(\vec{x})a(\vec{y})) = \frac{1}{(2\pi)^3}\int e^{i\vec{p}(\vec{x}-\vec{y})}n(\vec{p}), \quad \omega(\vec{p}) = \frac{\vec{p}^2}{2m} - \mu \tag{2.94}$$

Since the + case belongs to the Fermions, one obtains for $\beta \to \infty$ the expected occupation for the finite density ground state:

$$\lim_{\beta\to\infty} n(\vec{p}) = \begin{cases} 1, & \text{if } \vec{p}^2 < \mu \\ 0, & \text{if } \vec{p}^2 > \mu \end{cases} \tag{2.95}$$

The main difference between the finite volum expression and the thermodynamic limit is that in the former case the p-values are discrete and that in the latter case the trace class property of $\exp -\beta H$ is lost and therefore the numerator and denominator in the Gibbs formula (and hence the Gibbs formula itself) become meaningless. It is not difficult to establish the thermodynamic limit for large classes of H_{int}.

With the help of the KMS condition one may avoid the finite quantization box and study statistical mechanics directly in the infinite system. It is interesting to note that the KMS condition is equivalent to the stability of the state under appropriately formulated local perturbations and to the second law of thermodynamics (see Haag's book). The GNS construction with a KMS state gives a GNS triple with a reference state Ω which, in addition of being cyclic, also has the separating property i.e. an operator from the algebra A which annihilates Ω must itself be zero. In fact the hamiltonian, or more generally the KMS automorphism, is the Tomita automorphism of the associated modular theory and vice versa, the Tomita modular automorphism is characterized by its KMS property. Let us illustrate this for quasilocal states on the CAR-algebra. Writing:

$$\langle\Omega\,|a(f)a^*(g)|\,\Omega\rangle = (f, Sg)\,, \quad 0 \le S \le 1 \tag{2.96}$$

the separability of Ω is garantied if S has no eigenvalues 0 and 1 and the representation of the CAR algebra is even factorial if the multiplicity of the eigenvalue $\frac{1}{2}$ is finite. The previous considerations suggest that the modular operator is related to S by:

$$S = \frac{\Delta_S}{1 + \Delta_S} \tag{2.97}$$

and the GNS representation may be most naturally be described by "doubling" i.e. in a Fockspace $\mathcal{H}_{double} = \mathcal{H}_F \otimes \mathcal{H}_F$ associated with the doubled one particle space $h_{double} = h \oplus h$:

$$\omega_S(A) = \omega_P^{double}(A)$$
$$P = \begin{pmatrix} S & S^{\frac{1}{2}}(1-S)^{\frac{1}{2}} \\ S^{\frac{1}{2}}(1-s)^{\frac{1}{2}} & 1-S \end{pmatrix} \tag{2.98}$$

The quasifree states in the doubled description are pure on the tensor product algebra (and its representation is irreducible) since P is a projector. But its restriction S to the first factor (which is the image of the original CAR algebra under the doubling) is impure and reducible. For this reason the doubling is also called "purification". In the application to KMS states of statistical mechanics, the second factor in the doubling is a "shadow world" i.e. a copy of the original one (corresponding to the algebra of the previously discussed right action) which has no spatial localization. Later we will also meet examples of the modular theory for which the commutant algebra has a complementary space-time localization. In those cases the modular theory has a deep relation to TCP symmetry (the particle-antiparticle issue) and the Hawking temperature.

2.6 The CCR- and CAR-Functors

In section 3 we introduced the CCR and CAR C^*-algebras as maps of Hilbertspaces of functions into C^*-algebras. In particular the Fock-representation of these C^*-algebras define functors from the category of Hilbertspaces into von Neumann algebras.

Let us first look at the CCR-functor. Starting from a Hilbert space (always complex unless stated otherwise) with a scalar product f,g$\rightarrow$(f,g), we first describe the associated bosonic Fockspace in the following way. Let e^f be the suggestive notation for the vector in the the Fockspace $\mathcal{H}_F^{sym} \equiv e^H$ associated to H with the following n-particle components and inner product:

$$e^f = 1 \cdot \Omega + \sum_n \frac{1}{\sqrt{n!}} \underbrace{f \otimes .... \otimes f}_{n}, \quad (e^f, e^g) = e^{(f,g)} \tag{2.99}$$

In this notation the vacuum is $\Omega = e^0$. These special vectors are linear independent as well as "total"(i.e. they form a dense set) in e^H. The Weyl operator $W(f)$ is defined on this dense set as:

$$W(f)e^g = e^{-\frac{(f,f)}{4}} e^{-(f,g)} e^{f+g} \tag{2.100}$$

The unitarity of W and hence the extension to the whole space follows from this formula. The isomorphic map $H \rightarrow e^H$ carries subspaces of H into subspaces of e^H and direct sum decompositions into tensor products decompositions. Furthermore linear densely defined maps A between one particle spaces $H \xrightarrow{A} K$ go over into $e^H \xrightarrow{e^A} e^K$ with the computational rules:

$$\begin{aligned} e^A e^h &= e^{Ah}, \quad (e^A)^* = e^{A^*} \\ e^A &= e^U e^{|A|}, \quad A = U|A| \end{aligned} \tag{2.101}$$

the latter describing the fate of the polar decomposition under the map.

In order to use the Weyl-operators W as a functor from the category of linear spaces to von Neumann algebras, we need to understand a particular family of

real subspaces of H. Let M be a set of vectors in H. Define the symplectic complement M':

$$M' = \{f \in H \mid \mathrm{Im}(f,g) = 0 \; \forall g \in M\} \tag{2.102}$$

Then M' is a closed real subspace (the use of the symplectic form $Im(f,g)$ requires the restriction to real linear combinations). The following list of properties follows directly from the definition:

$$\begin{aligned} M &\subset N \curvearrowright N' \subset M' \\ M \text{ dense in } H &\curvearrowright M' = \{0\} \\ (M+iM)' &= M' \cap iM' \end{aligned} \tag{2.103}$$

As for von Neumann algebras, the two-fold application of the '-operation i.e. $M \rightarrow M''$ gives the (in this case symplectic) completion i.e. the smallest closed real space generated by the set M. The following definition strengthens the analogy with von Neumann algebras.

Definition 1 *A real closed subspace M is called "standard" if $M+iM$ is dense and $M \cap iM = \{0\}$. Every standard M defines a "canonical involution" s via $s(f+ig) = f - ig$ where $f, g \in M$.*

In other words, standard M's are +1 eigenspaces of an (unbounded) involution s. We need its polar decomposition:

$$\begin{aligned} s &= j\delta^{\frac{1}{2}}, \quad j^2 = 1, \quad j\delta^{\frac{1}{2}} = \delta^{-\frac{1}{2}} j \\ sh &= h^* \quad \text{on dense set } \mathcal{D}_M = M + iM \end{aligned} \tag{2.104}$$

with * referring to the reality concept defined by M. The important relations:

$$j(M) = M', \quad \delta^{it} M = M \tag{2.105}$$

are a rather direct consequence of the definitions.

We now define a von Neumann algebra R(M) associated with the real subspace M:

$$R(M) = alg\,\{W(f) \mid f \in M\}'' \tag{2.106}$$

Note that although M is real, the von Neumann Algebras R are always complex. The map:

$$M \rightarrow R(M) \tag{2.107}$$

turns out to be an "orthocomplementary functor" from the category of Hilbertspaces H and their standard real subspaces into the $B(H_F)$ operator algebra and von Neumann subalgebras in standard position. Orthocomplementary means that the complement M' corresponds to the commutant R(M)'i.e. the validity of the following "duality":

$$R(M') = R(M)' \tag{2.108}$$

The importance of this functor in QFT in quantum physics results from the fact that the R(M)' describes all observables which are compatible (simultaneously measurable) with an observable from $R(M)$, where in important QFT

cases M describes a space of real (classical) functions localized in some region $\mathcal{O}$ in Minkowskispace and $M(\mathcal{O})' = M(\mathcal{O}')$ where $\mathcal{O}'$ denotes the causal disjoint region to $\mathcal{O}$. So the functor relates classical localization regions with the quantum notion of simultaneous measurability. The process of passing from classical functions with a symplectic structure to operator algebras is often referred to as "quantization". Since this word creates the misleading impression that quantum physics is founded on a parallelism to classical physics and in particular that localization needs a classical function space, we prefer to avoid it alltogether (Bohr's "correspondence principle" is the reverse, namely to recover classical physics in some special limiting situations). In some way algebraic QFT is the investigation of those structures which cannot be obtained by "quantization " methods as Lagrangian canonical- and pathintegral-methods.

The most interesting remaining problem is the connection between the properties of s, j and δ and their Fockspace counterparts $S = e^s, J = e^j$ and $\Delta^{it} = e^{\delta it}$. As a result of:

$$SW(f)\Omega = W(-f)\Omega = W(f)^*\Omega \tag{2.109}$$

S is Tomita's (unbounded) involution:

$$SA\Omega = A^*\Omega, \quad A \in R(M) \tag{2.110}$$

Here the star is the universal M-independent star of subalgebras of $B(H)$, and the M-dependence of S is solely encoded in its dense domain (whereas for s the star changes depends on M). It is the simple part of the Tomita-Takesaki theory that S and the operators J and Δ which result from polar decomposition thereof always exist for general von Neumann algebras $\mathcal{R}$ in standard position i.e. pairs $\{\mathcal{R}, \Omega\}$ with $\mathcal{R} \in B(H)$ and $\Omega \in H$ cyclic and separating. Tomita's deep theorem tells how these operators act on the algebra:

$$\begin{aligned} j(\mathcal{R}) &: = J\mathcal{R}J = \mathcal{R}' \\ \sigma_t(A) &: = ad\Delta^{it}A \in R \text{ if } A \in R \end{aligned} \tag{2.111}$$

J is the modular conjugation and σ_t the modular automorphism group implemented by the modular operator Δ^{it}. It is not difficult to see that $R(M) \cap R(M)' = \mathbf{C}1$ (i.e. $R(M)$ is a factor) iff $\bar{M} \cap M' = \{0\}$. This suggests the definition:

Definition 2 *A real subspace M is called factorial if $M \cap M' = \{0\}$.*

The family of standard von Neumann algebras which are in the range of this functor are a subset of all standard von Neumann algebras in $B(H_F)$.

There exists another functor which maps H into H_F^{antis} and the standard real subspaces M of H into von Neumann algebras generated by CAR operators:

$$\begin{aligned} \{a(g), a^*(f)\} &= (g, f)1 \\ \{a^\#(g)a^\#(f)\} &= 0 \end{aligned} \tag{2.112}$$

$$CAR(M) = alg\,\{A(f) = a^*(f) + a(f) \mid f \in M\} \tag{2.113}$$

where $a(f)$ is the Fockspace annihilation operator: $a(f)\Omega = 0$.

The functorial constructions of the CAR appear somewhat simpler (and more natural) if one follows Araki and interprets the complex Hilbertspace H as a "doubled" real Hilbertspace. This is achieved by taking two copies $H_\pm$ and introducing an antiunitary involution Γ:

$$\Gamma \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} \bar{f}_- \\ \bar{f}_+ \end{pmatrix}, \quad f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in K = H_+ \oplus H_-, \quad PK = (1+\Gamma)K \tag{2.114}$$

$$f_\pm \rightarrow \bar{f}_\pm \text{ conjugation in } H_\pm, \quad (f,g) = (f_+,g_+) + (f_-,g_-)$$

The selfconjugate subspace

$$\mathrm{Re}\,K = \{f \in K \mid \Gamma f = f\} \tag{2.115}$$

inherits on the one hand a real inner product and on the other hand this real subspace is is isomorphic with K_+ considered as a real space with the isomorphism being:

$$f \rightarrow \sqrt{2}Pf, \quad f \in \mathrm{Re}\,K$$

$\mathrm{Re}\,K$ admits the following complex structure (P as above):

$$if := iPf - i(1-P)f, \quad f \in \mathrm{Re}\,K \tag{2.116}$$

This description of one particle spaces K is the same for both functors. The only difference is in the interpretation: instead of the symplectic complement M' one uses the "i-symplectic" complement: $\tilde{M}' = iM'$. This could also be called the real orthogonal complement. The relation with the vanishing anticommutator is:

$$\{A(f), A(g)\} = 0, \forall g \in M, \curvearrowright f \in \tilde{M}' \tag{2.117}$$

An important distinction between the CCR and the CAR functor shows up if one looks at the Tomita-Takesaki theory. In the CAR case one finds:

$$S = J\Delta^{\frac{1}{2}}, \quad J = Te^j, \Delta = e^\delta \tag{2.118}$$

T is the so-called Klein twist, a transformation which is only defined in $\mathcal{H}_F^{antis}$ and not in H :

$$T = \frac{1 + ie^{i\pi \mathbf{N}}}{1+i}, \quad \mathbf{N} : \text{number op. in } \mathcal{H}_F^{antis} \tag{2.119}$$

The general setting does not tell which of the two functors one must take in concrete situations. In QFT this additional physical information is supplied by localization properties.

Lit. to Chapter 1 and 2 :

Rudolf Haag : "Local Quantum Physics", Fields, particles, Algebras. Springer-Verlag 1992

Ola Bratteli and Derek W.Robinson : "Operator Algebras and Quantum Statistical Mechanics" Vol.1 and 2 Springer-Verlag 1979

J.H.Roberts in "The Algebraic Theory of Superselection Sectors, Introduction and Recent Results" Ed. D.Kastler, World Scientific 1990.

The detailed presentation of the Weyl functor is taken from some unpublished notes of P. Leylands, J.Roberts and D.Testard, "Duality for Quantum Free Fields" CNRS, Marseille preprint 1978.

Chapter 3

Poincaré Symmetry and Quantum Theory

3.1 The Symmetry Concept of General Quantum Theory.

In quantum mechanics a symmetry operation is implemented by an hermitian operator ("charge"in case of inner symmetry) which commutes with the hamiltonian. Usually this operator has a geometric origin in terms of the quantization of a Noether "current".

In relativistic QFT one prefers a definition which does not use objects which depend on the reference frame as H and is more intrinsic to quantum theory than those symmetry concepts obtained through that parallelism to classical theory usually referred to as "quantization".

Let ψ a vector in a coherent subspace of a Hilbertspace of a quantum theory (example: an irreducible representation space of a CCR- or CAR-algebra.). The corresponding physical state (in the sense of expectation values as defined previously) corresponds to the unit ray:

$$\underline{\psi} = \{e^{i\alpha}\psi \mid \alpha \in [0, 2\pi], (\psi, \psi) = 1\} \tag{3.1}$$

The probability for a "source" state ψ containing a "measured" state φ is:

$$w(\underline{\varphi}, \underline{\psi}) = |(\varphi, \psi)|^2 \tag{3.2}$$

and does not depend on the representing vectors. A symmetry transformation $\underline{S}$ is defined to be a transformation of unit rays:

$$\underline{\psi} \longrightarrow \underline{\psi}' \qquad \text{with} \qquad w(\underline{\varphi}', \underline{\psi}') = w(\underline{\varphi}, \underline{\psi}) \tag{3.3}$$

The physical significance of such $\underline{S}$ only becomes evident via its action on local observables, an issue which we will take up in a later section.

It is comforting to know, that this projective definition may be reduced to the standard situation of (anti-)unitary operators in Hilbertspace:

Theorem 5 *(Wigner): Any ray representation $\underline{S}$ may be rewritten in terms of a (anti-)unitary vector representation S:*

$$\psi = S\psi' \quad \text{with } (\varphi',\psi') = \begin{cases} (\varphi,\psi) & \text{unitary} \\ (\psi,\varphi) & \text{antiunitary} \end{cases} \tag{3.4}$$

In the antiunitary case, S may be written in terms of any conjugation K (an antilinear operator which flips the bras and ket of a inner product) as S=UK with U unitary. Antiunitary operators appear in quantum theory exclusively in symmetry transformations which contain the operation of time reversal. For physical reasons one does not want a symmetric spectrum since the energy of systems at zero temperature should be bounded below in order to avoid instabilities due to transitions into arbitrarily negative energy eigenstates (the same reason why Dirac filled the negative energy Dirac sea).

The time reversal T flips the direction of time and therefore:

$$Te^{iHt}\psi = e^{-iHt}T\psi \tag{3.5}$$

Taking T unitary and ψ an energy eigenvector (the use of the spectral representation for H would be more rigorous), one would obtain a symmetric energy spectrum which is in conflict with the existence of a ground state (but not with the structure of finite temperature states).

If the symmetry S is part of a symmetry group whose group manifold is connected (i.e.every element is continuously deformable into the identity), evidently only unitary representers can occur.

Strictly speaking, the presence of superselection rules limits the previous consideration to coherent subspaces. In the total space

$$\mathcal{H} = \sum_{\oplus i} \mathcal{H}_i \tag{3.6}$$

the phases between the S's in the subspaces are arbitrary and without physical significance. Symmetries not related continuously with the identity, as the various reflections: P,T and PT as well as dicrete symmetries not related to space-time, as charge conjugation C, can in principle transform one subspace into another.

If we apply the above consideration to symmetriy groups, the two operators $U(g_2)U(g_1)$ and $U(g_2g_1)$ need only to be identical up to a phase factor :

$$U(g_2g_1) = e^{i\varphi(g_2,g_1)}U(g_2g_1) \tag{3.7}$$

The associativity of the threefold composition yields a consistency condition for the phase which depends on two group elements. It is called a 2-cocycle condition. It is important to know under what circumstances this phase may

be absorbed into a redefinition of the $U's$, i.e.under what circumstances the "cocycle is a coboundary". A physicist is familiar with two "obstructions": phase factors coming from the topology of groups (as the phase factor -1 in half-integer spin representations which becomes a projective representation if one considers SO(3) and not SU(2) the represented group.) and central extensions of Lie-algebras which after exponentiation also lead to unremovable phase factors in the associated groups. A famous physical illustration of the physical relevance of central extensions is the Galilei-group in Schrödinger theory.

Although there are other global mechanisms for cocycles, as far as I know, these two possibilities cover all known physical situations. In particular for semisimple groups (the Galilei- and the Poincaré- groups are not semisimple) as well as for the Poincaré-group the transition to the universal leads back to vector representation. for the latter group we have :

Theorem 6 *(Wigner, Bargmann) The projective unitary representations of the Poincare-group $\mathcal{P}$ are equivalently described by vector representations of its universal covering $\widetilde{\mathcal{P}}$.*

From a topological point of view the two-fold covering of $\mathcal{P}$ happens already inside the rotation subgroup SO(3) whose covering is SU(2) i.e. the phenomenon of halfinteger spin.

As usual in Lie-group theory, one describes representations in terms of infinitesimal generators fulfilling Lie-algebra relations. The best known case in physics is the unitary representation theory of the SU(2)=$\widetilde{SO(3)}$. If we characterize the rotation by an angle Θ and axis $\vec{n}$, we have:

$$U(\vec{n},\theta) = e^{i\theta\,\vec{n}\cdot\vec{J}} \tag{3.8}$$

where $\vec{J}$ is the quantum mechanical rotation operator with the Lie-algebra:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{3.9}$$

The unitary irreducible representations are all finite dimensional and are explicitely given by the following well-known matrices . For$\vec{J}^2 = s(s+1)$:

$$\begin{aligned}&\langle s,m\,|J_3|\,s,m\rangle = m\\ &\langle s,m+1\,|J_+|\,s,m\rangle = \langle s,m\,|J_-|\,s,m+1\rangle = [(s-m)(s+m+1)]^{\frac{1}{2}}\end{aligned} \tag{3.10}$$

Here $-s \leq m \leq s, J_\pm = J_1 + J_2$, and we have listed only the nonvanishing matrix elements in the 2s+1 dimensional representation space.

The distinction between a group and its covering does not show up in the Lie-algebra, but it can be seen in e.g. irreducible representations by looking at the values of the Casimirs (in the present case the distiction between halfinteger and integer spin in the eigenvalues of $\vec{J}^2$). The finite dimensional irreducible representations of the Lorentz group are constructed in complete analogy to the

rotation group. The following construction shows that the knowledge of the above formalism suffices. Choosing generators :

$$\Lambda = e^{\frac{1}{2}iM_{\mu\nu}\theta^{\mu\nu}} = e^{i\vec{M}\vec{n}+i\vec{N}\vec{e}} \text{ with}$$
$$\vec{M} \text{ and } \vec{N} \text{ related to } M^{\mu\nu} \text{ as } \vec{B} \text{ and } \vec{E} \text{ to } F^{\mu\nu} \tag{3.11}$$

and the Lie-algebra relations are :

$$[M_i^{\pm}, M_j^{\pm}] = i\epsilon_{ijk}M_k^{\pm}, \qquad [M^+, M^-] = 0 \qquad \text{with } M_j^{\pm} = \frac{1}{2}(M_j \pm iN_j) \tag{3.12}$$

Our previous SL(2,C) representation in the space of undotted two-component spinors is:

$$\alpha(\Lambda) = e^{i\frac{1}{2}\vec{\sigma}\vec{n}\theta+\frac{1}{2}\vec{\sigma}\vec{e}\chi} \tag{3.13}$$

Using the notation $\vec{J}$ and $\vec{K}$ for the representers of $\vec{M}$ and $\vec{N}$ in the respective representations we find for this spinor representation :

$$\vec{J} = \frac{\vec{\sigma}}{2}\,,\ \vec{K} = -i\frac{\vec{\sigma}}{2} \quad \text{or} \quad \vec{J^+} = \frac{\vec{\sigma}}{2}\,,\ \vec{J^-} = 0 \quad \text{with } \vec{J^{\pm}} = \frac{1}{2}\left(\vec{J} \pm i\vec{K}\right) \tag{3.14}$$

The standard notation for this fundamental SL(2,C) spinor representation is :

$$D^{[\frac{1}{2},0]}(\Lambda) = e^{i(\theta\vec{n}-i\chi\vec{e})\vec{J^+}} \qquad J^+ = \frac{\vec{\sigma}}{2} \tag{3.15}$$

Similarly for the dotted spinors :

$$D^{[0,\frac{1}{2}]}(\Lambda) = e^{i(\theta\vec{n}+\chi\vec{e})\vec{J^-}} \qquad J^- = \frac{\vec{\sigma}}{2}t \tag{3.16}$$

Note that in these representations (as well as in all finite dimensional representatios of the Lorentz-group) the generators K are not hermitean i.e. the associated D's are not unitary. The general irreducible finite dimensional representation are characterized in terms of two (half)integers $\frac{n_\pm}{2}$ which denote the formal $J_\pm$ angular momenta.

$$D^{[\frac{n_+}{2},\frac{n_-}{2}]}(\Lambda) = e^{i(\theta\vec{n}-i\chi\vec{e})\vec{J^+}} e^{i(\theta\vec{n}+i\chi\vec{e})\vec{J^-}} \tag{3.17}$$

Here $\vec{J^{\pm}}$ are the previously defined matrices of size $(2\frac{n_\pm}{2}+1)\times(2\frac{n_\pm}{2}+1)$. In the spirit of the spinor calculus, one should envisage these operators to act on tensor products of (un-)dotted spinors. The $J_\pm$ act as Pauli matrices on $\otimes_{n_\pm}$ $Mat_2(\mathbf{C})$:

$$\vec{J}_i = \sum_{i=1}^{n} 1 \otimes .... \otimes \frac{\vec{\sigma}_i}{2} \otimes .... \otimes 1 \Big|_{symm.} \tag{3.18}$$

The symmetrization in the $n_\pm$ spinorial indices assures the irreducibility.

3.2 One Particle Representations of the Poincaré Group.

We are now ready to study unitary representations of $\tilde{P}$and its subgroup $\widetilde{SO(3,1)}$. The infinitesimal generators for noncompact groups are necessarily unbounded operators. The domain problems for unbounded Lie-generators (common dese domain etc.) have been studied and we will ignore them unless they are of direct physical significance (as e.g. in the relation between the Tomita-Takesaki modular theory and symmetries described in a later section).

The commutation relations of the Poincaré generators follow from the composition property. for $\tilde{P}(a_2,\alpha_2)\cdot(a_1,\alpha_1)=(a_2+\alpha_2 a_1\alpha_2^{-1},\alpha_2\alpha_2)$ (3.19)
The second translational term is simply the Lorentz-transformed vector $\Lambda_2 a_1$. From the special case

$$(0,\alpha^{-1})\,(a,0)\,(0,\alpha)=(\Lambda^{-1}a,1) \tag{3.20}$$

one abstracts for infinitesimal translations:

$$U^{-1}(\alpha)P^{\mu}U(\alpha)=\Lambda^{\mu}_{\nu}P^{\nu}\quad\text{with}\quad U(a)=e^{iaP} \tag{3.21}$$

Analogously the transformation of the operator $U(\alpha)=e^{iM^{\mu\nu}\theta_{\mu\nu}}$ by another Lorentz transformation yields the tensor transformation property of $M^{\mu\nu}$

$$U^{-1}(\alpha)M^{\mu\nu}U(\alpha)=\Lambda^{\mu}_{\kappa}\Lambda^{\nu}_{\lambda}M^{\kappa\lambda} \tag{3.22}$$

In order to avoid clumsy notation, it is convenient to surpress the unimodulars α inside unitaries and write simply $U(\Lambda)$ with the understanding that Λ denotes an element of $\tilde{\mathcal{P}}$. Only for matrices (i.e.finite dimensional representations) the notational distinction matters. The Lie-algebra relations are otained from the above transformation laws by expanding $U(\alpha)\equiv U\Lambda)$ retaining only linear terms in $\theta_{\mu\nu}$.:

$$\begin{aligned}[M^{\alpha\beta},P^{\mu}]&=i(g^{\alpha\mu}P^{\beta}-(\alpha\longleftrightarrow\beta))\\ [M^{\mu\nu},M^{\rho\sigma}]&=i\{g^{\nu\sigma}M^{\mu\rho}-g^{\mu\rho}M^{\nu\sigma}-(\mu\longleftrightarrow\nu)\}\end{aligned} \tag{3.23}$$

The last relation is the tensor form of the previous $J^{\pm}$ commutation relations.

Approaching the Wigner theory via the infinitesimal generators P^{μ}, $M^{\mu\nu}$, one first looks for the Casimir (invariant) operators which take on characteristic values in irreducible representations :

$$\begin{aligned}P^{\mu}P_{\mu}&\quad\text{the mass operator}\\ W^{\mu}W_{\mu}&\quad\text{the Pauli-Lubanski invariant}\end{aligned} \tag{3.24}$$

The Pauli-Lubanski invariant is formed with the P.L vector $W_{\mu}=-\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta}P^{\gamma}$ whose commutation properties follow from those of the Poincare generators $M^{\mu\nu}$ and P^{μ} and read :

$$[W_{\mu},P_{\nu}]=0\qquad[W_{\mu},M_{\nu\kappa}]=i(g_{\nu\mu}W_{\kappa}-g_{\kappa\mu}W_{\lambda})\qquad[W_{\mu},W_{\nu}]=i\epsilon_{\mu\nu\kappa\lambda}W^{\kappa}P^{\lambda} \tag{3.25}$$

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$$\begin{aligned} [M^{\alpha\beta}, P^\mu] &= i(g^{\alpha\mu}P^\beta - (\alpha \longleftrightarrow \beta)) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i\{g^{\nu\sigma}M^{\mu\rho} - g^{\mu\rho}M^{\nu\sigma} - (\mu \longleftrightarrow \nu)\} \end{aligned} \tag{3.23}$$

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The Pauli-Lubanski invariant is formed with the P.L vector $W_\mu = -\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta}P^\gamma$ whose commutation properties follow from those of the Poincare generators $M^{\mu\nu}$ and P^μ and read :

$$[W_\mu, P_\nu] = 0 \qquad [W_\mu, M_{\nu\kappa}] = i(g_{\nu\mu}W_\kappa - g_{\kappa\mu}W_\lambda) \qquad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\kappa\lambda}W^\kappa P^\lambda \tag{3.25}$$

Since $P_\mu W^\mu = 0$, there is no nontrivial third invariant. The interpretation of W_μ and W^2 in terms of intrinsic angular momentum becomes visible, if we specialize to so called positive energy representations.

Wigner classified the irreducible representations according to their transitive p- space orbits (submanifolds of momentum space traced out by the action of $\mathcal{L}$ to a given vector):

(i) $p^\mu p_\mu = m^2 > 0 \quad p_0 > 0$

(ii) $p^\mu p_\mu = 0 \quad p_0 > 0$

(iii) $p_\mu = 0$

and the correponding orbits with negative energies $P_0 < 0$, as well as the spacelike orbit $p^\mu p_\mu < 0$. The first two exhaust the positive energy representations . In order to construct them explicitly, we look at the stability group ("little group") of a point on the orbit . Without loss of generality we may specialize to the stability group of a selected reference momentum, since the stability group for other momenta are equivalent (by Lorentz-boosts) In the case (i) we choose $p_R = (m, \vec{0})$ which yields the SO(3) resp.its covering SU(2) as the quantum theoretically relevant little group. The little group of the the light-like reference vector which is chosen to be turns out to be the euclidean group E(2) in two dimensions. Only the rotation around the 3-axis is geometrically obvious , the interpretation of the two euclidean "translations" is somewhat hidden and will be presented later. Let us now look in detail at the massive case (i). We start with a 2s+1 dimensional representation of the little group. This irreducible representation induces a unitary irreducible positive energy representation of the Poincarè group $\tilde{\mathcal{P}}$ as follows.We first chose the momentum in rest $p_R = (m, \vec{0})$ as the reference vector on the orbit $p^2 > 0, p_0 > 0$. The action on (improper, like plane waves) reference basis vectors is:

$$\begin{aligned} P^\mu \left| p_R, s_3; \gamma \right\rangle &= p_R^\mu \left| p_R, s_3; \gamma \right\rangle \ , \\ W_0 \left| p_R, s_3; \gamma \right\rangle &= 0 \ , \\ W_k \left| p_R, s_3; \gamma \right\rangle &= \tfrac{m}{2} \epsilon_{k\mu\nu 0} M^{\mu\nu} \left| p_R, s_3; \gamma \right\rangle \end{aligned} \tag{3.26}$$

The last relation connects the spatial components of W with the Wigner spin i.e. with the angular momentum in the rest frame:

$$W_k \left| p_R, s_3; \gamma \right\rangle = \frac{m}{2} \epsilon_{kij} M^{ij} \left| p_R, s_3; \gamma \right\rangle = m J_k \left| p_R, s_3; \gamma \right\rangle \tag{3.27}$$

Since an invariant operator can be evaluated on any vector, we have $W^2 = -m^2 \vec{J}^2$and therefore in an irreducible representation: $W^2 = -m^2 s(s+1)$. In this approach irreducibility just means the absence of an additional degenery label say γ (such labels, which go beyond spacetime caracteristics as momentum and spin, are related to internal symmetries and called charges.). One now uses a distinguished family of Lorentz-transformations which link p_R with a general point p on the $p^2 = m^2$ orbit. One chooses the family of rotational free Lorentz-

transformations ("boosts") to relate the p-eigenstates :

$$|p, s_3\rangle = U(L(p))\,|p_R, s_3\rangle\,, \qquad L(p) = \frac{1}{m}\begin{pmatrix} p^0, & \vec{p} \\ \vec{p}, & m\delta_{ik} + \frac{p^i p^k}{p^0+m} \end{pmatrix} = \Lambda(\vec{e}, \chi) \tag{3.28}$$

$$\text{with:}\quad \vec{e} = \frac{\vec{p}}{|\vec{p}|} \qquad ch\chi = \frac{p^0}{m} \tag{3.29}$$

We now are able to describe the $[m_+, s]$ Wigner representation in global terms as follows. The one- particle Hilbertspace is:

$$H^{(1)}_{[m,s]} = \left\{ \int \sum_{s_3} \tilde{\psi}(\vec{p}, s_3)\,|p, s_3\rangle\,\frac{d^3p}{2\omega}\;|\;\int \sum \left|\tilde{\psi}\right|^2 \frac{d^3p}{2\omega} < \infty \right\} \tag{3.30}$$

$$U(\Lambda)\,|p, s_3\rangle = U(L(p))U(R(\Lambda, p))\,|p_R, s_3\rangle \qquad \text{with } R(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{3.31}$$

$$\text{and}\quad U(R(\Lambda, p))\,|p_R, s_3\rangle = \sum_{s_3'} \left|p_R, s_3'\right\rangle D_{s_3' s_3}(R(\Lambda, p)) \quad \text{we obtain:} \tag{3.32}$$

$$U(\Lambda)\,|p, s_3\rangle = \textstyle\sum_{s_3'} \left|\Lambda p, s_3'\right\rangle D_{s_3' s_3}(R(\Lambda, p)) \quad \text{and for translations :}$$
$$U(a)\,|p, s_3\rangle = e^{ipa}\;|p, s_3\rangle \tag{3.33}$$

The sucsessive transformations by a boost Λ and inverse boost on the transformed momentum yields a transformation $R(\Lambda, p)$ which leaves p_R invariant and therefore is called the Wigner rotation. The appearance of p-dependent unitary matrices is typical for relativistic quantum theory. It prevents a simple minded transition to x-space covariant localizable functions via fouriertransformation. As well known, one can rewrite the transformations from the basis vectors in $H^{(1)}_{[m,s]}$ to the wave functions on which one finds the contragredient action :

$$\left(U(\Lambda)\tilde{\psi}\right)(\vec{p}, s_3) = \sum_{s_3'} D_{s_3' s_3}(R(\Lambda, \Lambda^{-1}p)\tilde{\psi}(\Lambda^{-1}\vec{p}, s_3') \tag{3.34}$$

Besides this Wigner "canonical" representation, there exists the closely linked "helicity" representation for which the spin quantization axis is identified with the direction of the spatial momentum of the particle. Calling the magnetic quantum number with respect to this direction λ wedefine :

$$|p, \lambda\rangle \equiv \sum_{s_3'} \left|p, s_3'\right\rangle D_{s_3' \lambda}(R_{p,\vec{p}}) \text{ with } R_{p\vec{p}} = Rot(\varphi, \theta) : \tag{3.35}$$

being the "minimal" rotation which changes the z-direction into $\vec{n} = \frac{\vec{p}}{|\vec{p}|}$ i.e.a rotation around the y-axis latitude θ followed by a φ-rotation around the z-axis. In the helicity basis the Wigner rotation is modified :

$$\check{R}(\Lambda, p) = R^{-1}_{\Lambda p, \Lambda\vec{p}} R(\Lambda, p) R_{p,\vec{p}} \text{ leaves } \check{p} = (\sqrt{p^2 + m^2}, 0, 0, |\vec{p}|) \text{ invariant.} \tag{3.36}$$

The evaluation of W_0 on the helicity reference state gives:

$$W_0\,|\check{p},\lambda\rangle = \vec{J}\cdot\vec{p}\,|\check{p},\lambda\rangle = |\vec{p}|\,\lambda\,|\check{p},\lambda\rangle \quad \text{or with} \quad h \equiv \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}\,, \quad h\,|\check{p},\lambda\rangle = \lambda\,|\check{p},\lambda\rangle \tag{3.37}$$

The column vectors of $D^{(s)}(R_{p,\beta})$ furnish a complete set of eigenstates of the helicity operator h,e.g. for s=$\frac{1}{2}$ we have.

$$\begin{aligned} w^{\pm} &= D(R_{p,\beta})\chi^{\pm}\,,\ \chi^{\pm} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ i.e. \\ D^{(\frac{1}{2})}(R_{p,\beta}) &= \begin{pmatrix} w_1^+ & w_2^- \\ w_2^+ & w_2^- \end{pmatrix},\ \tfrac{1}{2}\vec{\sigma}\cdot\vec{p}\,w^{\pm} = \pm\tfrac{1}{2}w^{\pm} \end{aligned} \tag{3.38}$$

The advantage of the helicity basis is that one may take the limit $m \to 0$.As expected,the helicity rotation matrix $D(\check{R})$ approaches a diagonal limit e.g.for s=$\frac{1}{2}$:

$$\begin{aligned} \lim_{m\to 0} D^{(\frac{1}{2})}(\check{R}) &= \begin{pmatrix} e^{i\frac{1}{2}\varphi(\Lambda,p)} & 0 \\ 0 & e^{-i\frac{1}{2}\varphi(\Lambda,p)} \end{pmatrix}, \\ e^{i\frac{1}{2}\varphi(\Lambda,p)} &= \sqrt{\frac{p^0}{(\Lambda p)^0}}\,\langle w^+(\Lambda p)\,|\alpha(\Lambda)|\,w^+(p)\rangle \end{aligned} \tag{3.39}$$

In the massless limit, the (2s+1)- component representation decomposes into 2s+1 one-component representations. A direct approach a la Wigner to the $m = 0$ case would start with the representation theory of the stability group of a light-like vector. In this situation there is no such natural choice as before. Chosing a light-like vector in the z-direction $p_R = (1,0,0,1)$ one obtains the following matrix realization of the 3-parametric euclidean group $E(2)$ in 2 dimensions :

$$G(\alpha,\beta) = \begin{pmatrix} 1+\frac{1}{2}\rho & \alpha & \beta & -\frac{1}{2}\rho \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \frac{1}{2}\rho & \alpha & \beta & 1-\frac{1}{2}\rho \end{pmatrix},\quad R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.40}$$

The first matrix is a Lorentz-transformation which leaves p_R invariant and transforms the time axis into $Gp = (1+\frac{1}{2}\rho,\alpha,\beta,\frac{1}{2}\rho)$, $\rho^2 = \alpha^2+\beta^2$. Any other transformation having this property can only deviate from $G(\alpha,\beta)$ by a transformation which leaves the two vectors p_R and the time axis invariant i.e.a x-y rotation $R(\theta)$. Therefore they generate the stability group which is easily checked to be isomorphic to E(2), the euclidean translations corresponding to $G(\alpha,\beta)$. To be more precise, since the euclidean group has to be considered as a subgroup of the covering the Poincaré group, only the two fold covering $\tilde{E}(2)$ is relevant. The unitary representation theory of such a noncompact group is somewhat more complicated than that of SU(2). But it is obvious that the representaton fall into two classes; the neutrino- photon class with $U(G(\alpha,\beta)) = 1$ i.e. trivial representation of the euclidean translations, and the remaining "continous spin" representations. The difference also shows up in the spectrum of the operator W^2. Wheras in the first case $W^2 = 0$ (in fact $W^\mu = hP^\mu$), the value of W^2 in

the second case can be any negative number which is responsible for the name .These representations of $\tilde{E}(2)$ are infinite dimensional. They are usually discarded as a result of the apparent absence of such particles in nature. We will later on pay some attention to these representations since a theoretician should use theoretical arguments. *It is also interesting to note that inventions like e.g. supersymmetry, which are certainly not as intimately related to the principles of QFT as the Wigner theory, are often not dismissed in this easy way.*

It is comforting to know that the $[m, s]$ representations admit an extension of the Poicaré group which includes the reflections, without enlarging the representation space. One obtains the well-known formulas for the parity P and the time reversal T:

$$\mathcal{P}\,|p, s_3\rangle = \xi_P\,|p_0, -\vec{p}, s_3\rangle \qquad \mathcal{T}\,|p, s_3\rangle = \xi_T \sum_{s_3'} D^{(s)}_{s_3, s_3'}(i\sigma_2)\left|p_0, -\vec{p}, s_3'\right\rangle \quad (3.41)$$

Here the $\xi's$ are undetermined phase factors.This result follows by first writing down the action of P and T on the reference vectors $|p_R, s_3\rangle$ (the antiunitarity of T brings in the spin-flip matrix $D(i\sigma_2)$) .The rest follows from the commutation relation of the reflections with the boost :

$$R_\lambda L(p) R_\lambda^{-1} = L(p_0, -\vec{p}), \quad R_\lambda = P\ ,\ T\ \ or\ PT \qquad (3.42)$$

The corresponding operator relation may contain phase factors D_λ i.e.

$$\mathcal{R}_\lambda U(\Lambda)\mathcal{R}_\lambda^{-1} = D_\lambda(\Lambda) U(R_\lambda \Lambda R_\lambda^{-1}) \qquad (3.43)$$

These phase factors must form a representation of the Lorentz group. But since there are no 1-dimensional representations and hence we have $D(\Lambda) = 1$. The above phases can be fixed. For unitary reflections we can achieve $R^2 = 1$, whereas for antiunitaries $R^2 = \pm 1$. In the above special case we find:

$$T^2 = (-1)^{2s} \qquad (3.44)$$

The formulae for the $[0, s]$ representations are different as a result of the differnt p_R and its stability group which contain the ad hoc z-direction :

$$\mathcal{P}\,|p, s\rangle = \xi_P e^{\pm i\pi s}\,|p, -\vec{p}, -s\rangle \qquad \mathcal{T}\,|p, s\rangle = \xi_T e^{\pm i\pi s}\,|p, -\vec{p}, s\rangle \qquad (3.45)$$

The $\pm$sign depends on the sign of p_y (see Weinberg), and this phase factor is only relevant if the states of opposite helicity are not separeted by a superselection rule.

The original motivation of Wigner was to classify relativistic wave equations up to physical equivalence. Disregarding the continous spin class, the classification of wave equations associated with finite energy epresentations is as follows. We first present the three special cases s=0, $\frac{1}{2}$, 1.m> 0

s=0

The Fourier transformation leads to covariant x-space wave function :

$$\psi(x) = \int e^{-ipx}\tilde{\psi}(p)\frac{d^3p}{2\omega} \quad \text{with } (U(\Lambda)\psi)(x) = \psi\left(\Lambda^{-1}x\right) \tag{3.46}$$

The x-space function is a positive frequency solution of the Klein-Gordon equation:

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi(x) = 0 \tag{3.47}$$

$s=\frac{1}{2}$

Here one has to convert the Wigner representation into a covariant one. This is achieve by:

$$\tilde{\Phi}_a(p) := \sum_{s_3}\alpha_{a,s_3}(L(p))\tilde{\psi}(p,s_3) \text{ with } \alpha(L(p)) = \sqrt{p^\mu\sigma_\mu}$$
a positive matrix (in the operator sense), in short:$\tilde{\Phi} = \alpha(L(p))\tilde{\psi}$ (3.48)

As the notation suggests, $\tilde{\Phi}$ transforms like a (undotted) spinor, a fact which follows by transforming the $\tilde{\psi}$ with the Wigner transformation and using its representation in terms of boosts :

$$\begin{aligned}\alpha(L(p))\alpha\left(R(\Lambda,L^{-1}p)\right) &= \alpha(L(p))\alpha\left(L^{-1}(p\,)\,\alpha(\Lambda)\alpha\left(L(\Lambda^{-1}p)\right)\right.\\ &= \alpha(\Lambda)\alpha(L(\Lambda^{-1}p))\\ i.e.\ \left(U(\Lambda)\tilde{\Phi}\right)(p) &= \alpha(\Lambda)\tilde{\Phi}(\Lambda^{-1}p)\end{aligned} \tag{3.49}$$

For later purpose it is helpful to rewrite the action of $\alpha(L(p))$ on $\tilde{\psi}$ in terms of the column vectors of the boost matrix:

$$\tilde{\Phi}(p) = \sum_{s_3} u(p,s_3)\tilde{\psi}(p,s_3)\,, \quad u_a(p,s_3) := \alpha_{a,s_3}(L(p)) \tag{3.50}$$

Both $\alpha(L(p))$ and u have the intertwining property between the Wigner and the covariant representation :

$$\alpha(L(p))D^{(\frac{1}{2})}(R(\Lambda,L^{-1}p)) = D^{[\frac{1}{2},0]}(\Lambda)\alpha(L(\Lambda^{-1}p) \tag{3.51}$$

A similar intertwining relation is valid between the conjugate complex of the Wigner representation D^* and the covariant D. In this case the intertwining matrix is $\alpha(L^{-1}(p))$ and its columns are called v-spinors.

Fourier transformation gives the x-space wave function:

$$\begin{gathered}\Phi(x) := \int\tilde{\Phi}(p)e^{-ipx}\frac{d^3p}{2\omega}\,, \quad (U(\Lambda)\Phi)(x) = \alpha(\Lambda)\,\Phi(\Lambda^{-1}x)\,,\\ \left(\partial^\mu\partial_\mu + m^2\right)\Phi(x) = 0\end{gathered} \tag{3.52}$$

In order to make contact with the Dirac theory one defines another spinor

$$\tilde{\chi}(p) := \frac{1}{m}p^\mu\tilde{\sigma}_\mu\tilde{\Phi}(p) = \alpha(L^{-1}p)\tilde{\psi}(p) \tag{3.53}$$

As indicated in the notation, $\tilde{\chi}^{\cdot}$ transforms as an upper dotted spinor i.e.with a matrix $\alpha(\Lambda^{-1})^{\dagger}$. This is a result of the relation:

$$p^{\mu}\tilde{\sigma}_{\mu}\alpha(\Lambda) = \alpha(\Lambda^{-1})^{\dagger}p^{\mu}\tilde{\sigma}_{\mu} \tag{3.54}$$

Defining a 4-component Dirac spinor, we immediatly read off its properties :

$$\tilde{\Psi}(p) = \sqrt{m}\begin{pmatrix} \tilde{\Phi}(p) \\ \tilde{\chi}^{\cdot}(p) \end{pmatrix}, \text{ with } (p^{\mu}\gamma_{\mu} - m)\,\tilde{\Psi} = 0 \text{ and } \gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \tilde{\sigma}_{\mu} & 0 \end{pmatrix} \tag{3.55}$$

The first two components of the Dirac equation are identical to the definition of $\tilde{\chi}^{\cdot}$ in terms of $\tilde{\Phi}$ and the remaining equation is the Klein Gordon identity for $\tilde{\Phi}$:

$$p^{\mu}\tilde{\sigma}_{\mu}\tilde{\Phi} = m\tilde{\chi}\ ,\ \ p^{\mu\cdot}\sigma_{\mu}\,(p^{\mu}\tilde{\sigma}_{\mu})\,\tilde{\Phi} = m^2\tilde{\Phi} \tag{3.56}$$

Rewriting the the inner product in terms of Ψ we obtain:

$$(\psi_2,\psi_1) = \frac{1}{2m}\int \tilde{\Psi}_2^{*}\tilde{\Psi}_1\frac{d^3p}{2\omega} = \frac{1}{2m}\int \overline{\tilde{\Psi}}\gamma_0\tilde{\Psi}\frac{d^3p}{2\omega} \quad \text{with } \overline{\tilde{\Psi}} := \tilde{\Psi}^{*}\gamma_0 \text{ the Dirac adjoint} \tag{3.57}$$

Since the gamma matrices transform as a 4-vector, the Dirac formalism permits to form tensors. In x-space we have:

$$\begin{aligned} &\Psi(x) = \textstyle\int e^{-ipx}\tilde{\Psi}(p)\frac{d^3p}{2\omega}\ ,\ \ (i\gamma_{\mu}\partial^{\mu} - m)\,\Psi(x) = 0\,, \\ &\overline{\Psi}(x)\Psi(x) = \textit{scalar},\ \overline{\Psi}(x)\gamma_{\mu}\Psi(x) = \textit{vector etc.} \end{aligned} \tag{3.58}$$

There are alltogether 16 independent tensorial densities which one can form in this way from products of $\gamma's$.

Dirac's inner product is conveniently expressed in terms of the conserved current :

$$j_{\mu} = \overline{\Psi}_2\gamma_{\mu}\Psi_1\,,\ \ \partial^{\mu}j_{\mu} = 0,\quad (\overline{\Psi}_2\Psi_1) := \int j_0 d^3x = 2m(\psi_2,\psi_1) \tag{3.59}$$

The 4-component description allows a local matrix realization of the parity symmetry:

$$(\mathcal{P}\Psi)(x) = \gamma_0\Psi(x_0,-\vec{x}),\quad \gamma_0\gamma_i\gamma_0^{-1} = -\gamma_i \tag{3.60}$$

It is helpful to define a fifth γ-matrix as the product of all four: $\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma$. This matrix is block- diagonal and behaves like a pseudoscalar under parity. Therefore densities involving γ_5 pseudo-scalars, -vectors etc. Finally we mention the u- and v-intertwiners:

$$\begin{aligned} &u(p,s_3) = S(L(p))u(p_R,s_3),\quad u(p_R,\pm\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ &S(L(p)) = \begin{pmatrix} \sqrt{p^{\mu}\sigma_{\mu}} & 0 \\ 0 & \sqrt{p^{\mu}\tilde{\sigma}_{\mu}} \end{pmatrix} \end{aligned} \tag{3.61}$$

$$v(p,s_3) = Cu^*(p,s_3)\,, \quad C = i\gamma_2 \tag{3.62}$$

It is easy to check that u and v intertwine the s=$\frac{1}{2}$ wigner representations $D^{(\frac{1}{2})}(R)$ resp. $D^{(\frac{1}{2})*}(R)$ with $D^{[\frac{1}{2},0]} \oplus D^{[0,\frac{1}{2}]}$ which is implemented by the matrices $S(\Lambda)$. The so-defined v fulfills :

$$\left(-p^\mu\gamma_\mu - m\right) v = 0; \;\; \bar{u}u = 2m, \;\; \bar{v}v = -2m, \;\; \bar{u}v = 0 \tag{3.63}$$

It is an interesting historical side remark that Dirac found his equation in a more formalistic way. In order to overcome what he considered as a serious shortcoming of the scalar Klein Gordon equation, Dirac searched for a first order matrix differential operator which is a kind of square root of the K.G. operator, i.e. $\left(i\partial^\mu\gamma_\mu - m\right)\left(-i\partial^\mu\gamma_\mu - m\right) = \partial^\mu\partial_\mu + m^2$. The nessessary and sufficient condition are the Clifford algebra conditions:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{3.64}$$

There is only one finite-dimensional irreducible representation, it has dimension equal to 4 (the Clifford algebra for a $2n$-dim. space has a 2^n-dim irr.representation). Our group-theoretical approach has provided us with the so called chiral representation in which γ_5 is diagonal and which for m$\rightarrow$ 0 decomposes naturally into the two Weyl equations:

$$p^\mu\tilde{\sigma}_\mu\Phi = 0, \;\; p^\mu\sigma_\mu\chi^{\cdot} = 0 \tag{3.65}$$

There are many equivalent representations which are useful for other purposes. We will mention two of them. There is the representation used first by Dirac:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3.66}$$

This representation is useful in calculations involving the nonrelativistic limit as in the hydrogen-problem. On the other hand for the field theoretec application to selfconjugate s=$\frac{1}{2}$ particles and fields the following Majorana representation is useful (with purely real $i\gamma_\mu$ i.e.a real Dirac operator).

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} \end{aligned} \tag{3.67}$$

s=1 In this case there are several low dimensional covariant intertwining possibilities:

$$D^{(1)}(R) \rightarrow \begin{cases} D^{[1,0]} \\ D^{[0,1]} \\ D^{[\frac{1}{2},\frac{1}{2}]} \end{cases} \tag{3.68}$$

the first two have three components and the last is the 4-component vector description which, if restricted to the rotation group decomposes as follows:

$$D^{[\frac{1}{2},\frac{1}{2}]}(R) \cong D^{(1)}(R) \oplus D^{(0)}(R) \tag{3.69}$$

For an explicit description we apply the boost to the three spatial coordinate vectors e_1, e_2, e_3 :

$$e_\mu(p,i) := L^\nu_\mu(p) e_\nu(0,i) \qquad e(0,i) := e_i \tag{3.70}$$

Remembering the definition of the Wigner rotation, the transformation law is (surpressing the vector indices):

$$\Lambda e_i(p) = L(\Lambda p) R(\Lambda, p) e_i = \sum_{i'} L(\Lambda p) e_{i'} R_{i'i}(\Lambda, p) = \sum_{i'} e_{i'}(\Lambda p) R_{i'i}(\Lambda, p) \tag{3.71}$$

The covariant vector like wave functions are then:

$$\tilde{V}_\mu(p) = \sum_i e_\mu(p,i)\tilde{\varphi}(p,i), \quad V_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ipx} \sum_i e_\mu(p,i)\tilde{\varphi}(p,i) \frac{d^3p}{2\omega} \tag{3.72}$$

As always ,they fulfill the Klein-Gordon equation but, as a result of the transversality $p^\mu e_\mu(p,i) = 0$ which ex presses the absence of the scalar component $D^{(0)}$, they also are divergenceless:

$$\left(\partial^\kappa \partial_\kappa + m^2\right) V_\mu = 0, \quad \partial^\mu V_\mu = 0 \tag{3.73}$$

Both equations can be combined into a so called Proca-Wentzel equation:

$$\left(\partial^\kappa \partial_\kappa + m^2\right) V_\mu - \partial_\mu \partial^\kappa V_\kappa = 0 \tag{3.74}$$

This covarinance of this equation incorporates the transformation properties of the field (just like for the Dirac equation) and is the Euler-Lagrange equation of the Proca-Wentzel Lagrangian. Although Euler-Lagrange fields exist for any spin (e.g. for s=$\frac{3}{2}$ the Rarita-Schwinger equations), the Wigner approach, in contrast to the canonical or functional integral approach, does not provide a preferential status to Lagrangian fields.

From the definition one reads off the completeness relation:

$$\sum_{i=1}^{3} e_\mu(p,i) e_\nu(p,i) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \tag{3.75}$$

A limit m$\rightarrow$ 0 does not exist i.e. there is no possibility to intertwine the $[m = 0, s = 1]$ Wigner representation with $D^{[\frac{1}{2},\frac{1}{2}]}$. The Maxwell description in terms of field strength $F_{\mu\nu}$ corresponds to $D^{[1,0]}$ or $D^{[0,1]}$. This restriction together with the demand that vector potentials are indispensable for desccribing the long-range electromagnetic interaction in the context of quantum theory (in classical physics vector potentials can be avoided) forces one to look for a compromise slightly outside the Wigner scheme which will be presented in the sequel.

s=1 m=0

In order to obtain a formalism similar to the previous case of vector mesons,one extends the two "polarization vectors" e_i i=1,2 in x-and y-direction by two orthogonal light-like vectors:

$$e^{\mu}_{\pm} = e^{\mu}_{0} \pm e^{\mu}_{3} = (1, 0, 0, \pm 1) \tag{3.76}$$

We choose e_+ as the reference vector k_R from which to start the boost $L(k, k_R)$. The latter consists of a rotation of the z-axis into the momentum direction $\vec{n} = \frac{\vec{k}}{\omega}$ (fixed uniquely by the standard prescription in terms of two Euler angles) and a subsequent L-boost along this direction :

$$\begin{pmatrix} 1 \\ \vec{n} \end{pmatrix} \rightarrow k = \omega \begin{pmatrix} 1 \\ \vec{n} \end{pmatrix} \tag{3.77}$$

The decomposition of a general Lorentz transformation Λ in terms of its little group $H(\Lambda, k)$:

$$\Lambda = L(\Lambda k, k_R) H(\Lambda, k) L^{-1}(k, k_R), \qquad H(\Lambda, k) \in \tilde{E}(2) \tag{3.78}$$

the twofold covering of the euclidean group in two dimensions which, as explained before, is generated by two translations α, β and one rotation θ , where all euclidean parameters are functions of Λ and k which can be computed from the previous formula. One defines two transversal polarisation vectors:

$$\epsilon(k, \lambda) = L(k, k_r) \begin{cases} \frac{1}{\sqrt{2}}(e_1 + ie_2), & \lambda = + \\ \frac{1}{\sqrt{2}}(-e_1 + ie_2), & \lambda = - \end{cases} \tag{3.79}$$

They are used as intertwiners in the attempt to define a vectorial wave function:

$$\tilde{A}_{\mu}(k) = k_{\mu}\tilde{\Phi}(k) + \sum_{\lambda=\pm} \epsilon_{\mu}(k, \lambda)\tilde{\varphi}(k, \lambda) \tag{3.80}$$

Here the logitudonal first component is not determined by the Wigner theory. We cannot consistently set it equal to zero, since the intertwiners generate such an additive term under the action of the $\tilde{E}(2)$ translations:

$$\Lambda(\alpha, \beta)\epsilon(k_R, \lambda) = \epsilon(k_R, \lambda) + \begin{cases} -\frac{1}{2}(\bar{\rho}, 0, 0, \bar{\rho}), & \lambda = + \\ +\frac{1}{2}(\rho, 0, 0, \rho), & \lambda = - \end{cases} \qquad \rho = \alpha + i\beta \tag{3.81}$$

whereas the behaviour under x-y rotations the ϵ picks up the standard Wigner phase factor. The polarization vectors are not invariant under the euclidean translations in $E(2)$, as one would have expected for a bona fide intertwiner from the $[0, s = 1]$ Wigner representation to the $D^{[\frac{1}{2},\frac{1}{2}]}$ covariant representation. Rather the intertwiner only has L-covariance up to additive gauge transformations i.e.up to longitudinal terms.This peculiar manifestation of the $[0, s = 1]$ little group $\tilde{E}(2)$ is the cause for the appearance of the local gauge issue in local quantum physics. Unfortunatly this quantum origin remains somewhat hidden

in the quantization approach, where it remains invisible behind the geometrical interpretation in terms of fibre bundles.

In the covariant quantization approach, contrary to the Wigner theory the gauge aspect becomes completely decoupled from the L-transformations. This close relation to classical fibre bundles is only obtained at the expense of leaving the realm of quantum physics by entering the world of "ghosts". If the reader wonders about the conceptual sense of a (quantization) formalism which overrides the quantum physical reason d'etre for the appearance of the additive boost gauge term, he is not alone. In fact it is our contention that this is one of the potential points of possible fruitful contradictions between the requirements of (classical) geometry and local quantum physics. Of course this observation must be pursued in the presence of electromagnetic interactions. Although we will try to sharpen this to the level of a paradoxon, time is not ripe to solve the associated problem.

Whereas the covariantization of the canonical Wigner ($m = 0, h = 1$) representation can be done in terms of covariant field strength, the requirement that the scalar product be expressible in terms of a local tensorial formula necessitates the introduction of the above vector potential A_μ. The Lorentz (gauge) invariant inner product for the $\tilde{A}_\mu$ is now only positive semidefinite on individual A_μ (but positive definite on gauge classes):

$$\left(A, A'\right) = -\int \tilde{A}^*_\mu \tilde{A}^\mu \frac{d^3p}{2\,|\vec{p}|} = \sum_{\pm} \int |\tilde{\varphi}|^2 \frac{d^3p}{2\,|\vec{p}|} \tag{3.82}$$

As we will show later, the *formal* (i.e. the words loose their physical meaning) *local covariant* formulation in *Fock-space* requires the already mentioned more radical introduction of *indefinite* metric and the use of a special perturbative procedure (e.g.the Gupta-Bleuler method) for the recovery of the perturbative quantum theoretical positivity for the gauge invariant physical quantities.

As pointed out by Weinberg, this gauge aspect is common to all $[0, s = n]$ representations for n$\geq$ 1. There are no intertwiners from this Wigner representation to $D^{[A,A]}$symmetric tensors, rather the possibility of intertwining is restricted to $D^{[A,B]}$ with $|A - B| = h$ (h the Wigner helicity). The vector potentials for s=1 and the symmetric tensor $g_{\mu\nu}$ for s=2 of the classical general relativity can at the quantum level only be introduced at the prize of a formal gauge principle.

The case of general $[m, s]$ intertwiners u is a routine exercise in Clebsch-Gordan gymnastics. One uses the intertwining relation for u:

$$u(p)D^{(s)}(R(\Lambda, p) = D^{[A,B]}(\Lambda)u(\Lambda^{-1}p) \tag{3.83}$$

for the caculation of the u's. Here we found it convenient to interprete the intertwiner u as a rectangular matrix with 2s+1 columns and $(2A + 1)(2B + 1)$rows. The first step consists in analyzing this equation for $p = p_R$ (Weinberg) with the result that the $u(\vec{0})$ is proportional to the Clebsch-Gordan coefficents:

$$u(\vec{0}) \sim C_{AB}(s, s_3; a, b) \tag{3.84}$$

The second step consists in an application of a boost:

$$u(p) = D^{[A,B]}(L(p))u(0) \tag{3.85}$$

For details we refer to Weinberg's first volume.

3.3 Wigner Theory and Free Fields

We now use the Wigner representation theory in order to construct fields in bosonic or fermionic Fock spaces. The creation operators in momentum space should transform in the same way as the one particle states since their application to the vacuum vector creates the the latter.

$$U(\Lambda)a^*(p,m)U^*(\Lambda) = \sum_{m'} a^*(\Lambda p, m')D^{(s)}_{m'm}(R(\Lambda,p) \tag{3.86}$$

For computational convenience we identify the Wigner rotation with its unimodular matrix representation:

$$R(\Lambda,p) \longrightarrow \sqrt{\frac{1}{m}(\Lambda p)^\mu \sigma_\mu}^{-1} \alpha(\Lambda)\sqrt{\frac{1}{m}p^\mu\sigma_\mu} \tag{3.87}$$

The corresponding relation for the annihilation operator contains the complex conjugate matrix D^* which is equivalent to D:

$$D(i\sigma_2)D^*(R)D(-i\sigma_2) = D(R) \tag{3.88}$$

If the particles are charged, there are also operators $b^\#(p,m)$ which describe annihilation and creation of particles with the same mass and spin and hence the same transformation property as $a^\#(p,m)$. In order to obtain covariant operators one uses the intertwiners u and v introduced in the previous section. Interpreting these intertwiners as p-dependent rectangular matrices of size $N \times (2s+1)$ with N= dimension of the representation space on which the matices $D^{[A,B]}(\Lambda)$ act, we have :

$$\begin{aligned} D^{[A,B]}(\Lambda)\ u(\Lambda^{-1}p) &= u(p)D^{(s)}(R^{-1}(\Lambda,p)) \\ D^{[A,B]}(\Lambda)v(\Lambda^{-1}p) = v(p)D^{(s)*}(R^{-1}(\Lambda,p)), &\quad v(p) = u(p)D^{(s)*}(i\sigma_2) \end{aligned} \tag{3.89}$$

Therefore we find the following covariant creation and annihilation operators :

$$\begin{aligned} A^{(*)}(p) &= \textstyle\sum_m v(p,m)a^*(p,m) \\ B^{(*)}(p,) &= \textstyle\sum_m v(p,m)b^*(p,m) \\ A(p) &= \textstyle\sum_m u(p,m)a(p,m) \\ B(p) &= \textstyle\sum_m u(p,m)b(p,m) \end{aligned} \tag{3.90}$$

We have added a bracket to the * in order to indicate that the covariant creation operator is not exactly the hermitian adjoint of the covariant annihilator. The

Fouriertransform preserves covariance :

$$\psi_A^{(-)}(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ipx} A(p) \frac{d^3p}{2\omega} \quad , \quad \psi_A^{(+)}(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ipx} A^*(p) \frac{d^3p}{2\omega}$$
$$\psi_B^{(-)}(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ipx} B(p) \frac{d^3p}{2\omega}, \quad \psi_B^{(+)}(x) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ipx} B^*(p) \frac{d^3p}{2\omega} \tag{3.91}$$

obey the covariant transformation law:

$$U(\Lambda)\psi(x)U^*(\Lambda) = D^{[A,B]}(\Lambda)^{-1}\psi(\Lambda x) \tag{3.92}$$

We want to construct "local" covariant fields i.e. covariant fields which (anti)commute for spacelike distances. The physical motivation is Einstein causality for local observables. Prominent local observables associated with charged fields are e.g. currents. Since they are typically second or higher even degree polynomials in the fields, the (anti)commutation of the fields is sufficient for the Einstein causality (spacelike commutativity) of the local observables. Fields which are themselves observables like the Maxwell field, must obey spacelike commutation relations.

It is well known that support properties in momentum space as the restriction to the forward light cone in the above formulas , prevents support properties of (anti)commutators in x-space. The former give rise to analytic properties of the latter. The standard example is the Fouriertransform of a function with support in the positive halfaxis which is the boundary value of a function analytic in the upper half plane. According to the Schwartz reflection principle such function cannot vanish in a dense real subset without vanishing identically. The above Fouriertransforms are multidimensional counterparts in which the halfline is replaced by the forward light cone and the upper half plane by a tube $z_\mu = x_\mu + iy_\mu$ with y in the dual cone i.e.the backward light cone. We therefore make the following Ansatz for local fields:

$$\psi_A(x) = \psi_A^{(-)}(x) + \psi_A^{(+)}(x), \quad \psi_B(x) = \psi_B^{(-)}(x) + \psi_B^{(+)}(x)$$
$$\psi(x) = \psi_A^{(-)}(x) + \psi_B^{(+)}(x) \tag{3.93}$$

Complex coefficients in this linear combination bring no gain in generality since they can be absorbed into redefinitions. The following calculations show that all these combinations between different frequency parts are local covariant fields. The first two combinations are only physically useful if A and B would be (accidentally equal mass and spin) selfdual particles. If on the other hand there is a charge superselection rule between A and B i.e. B is the antiparticle of A then we are forced to take the ψ combination because otherwise we would not be able to form local (Einstein-causal) neutral observables. In this sense causality and the superselection principle require the existence of anticharged particles of the same (m,s).

Returning to our notation for indicees of irreducible fite dimensional representations for the Lorentz-group, we find the following relation between the spin and the spacelike (anti)commutativity:

Theorem 7 *(Spin-statistics for free fields) :*

$$\left[\psi_{a,b}^{[A,B]}(x), \psi_{a',b'}^{[A,B]}(y)\right]_{\pm} = 0 \quad for \quad (x-y)^2 < 0 \tag{3.94}$$

where the +sign i.e. the anticommutator is to be taken for A+B=halfinteger. It is helpful to consider first free fields for s=0,$\frac{1}{2}$, 1 .

<u>s=0</u>

With $\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ipx} A(p) + e^{ipx} B^*(p)\right) \frac{d^3p}{2\omega}$ we obtain :

$$[\psi(x), \psi(y)]_{\pm} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ip(x-y)} \pm e^{ip(x-y)}\right) \frac{d^3p}{2\omega} = i\Delta^{(+)}(x-y) \pm i\Delta^{(-)}(x-y) \tag{3.95}$$

Here $i\Delta^{(-)}(\xi) := i\Delta^{(+)}(-\xi)$ and the momentum space integrals may be expressed in terms of Hankel functions. One first uses the fact that $i\Delta^{(+)}(\xi)$ is analytic in the tube $\xi \to \zeta = \xi - i\eta$ with $\eta \in \bar{V}^+$, the closed forward light cone as a result of the spectrum property p$\in V^+$. This means that the euclidean vector $(\xi_4 = i\xi_0, \vec{\xi})$ is in the analyticity region at least if $\xi_0 \geq 0$. This analytic continuation is part of the so called Wick-rotation. In this euclidean domain one now rewrites the integral for $i\Delta^{(+)}$ in terms of an euclidean conture integral :

$$i\Delta^{(+)}(x) = \frac{1}{(2\pi)^4} \int_C e^{ipx} \frac{d^4p}{p^2+m^2} \tag{3.96}$$

The conture C in the complex p_0−plane is the imaginary ξ_0-axis or the new (Wick-rotated) $\xi_4 = i\xi_0 + \epsilon$ axis. The proof of this claim follows simply by closing the conture by an infinitely large half-circle in the upper half plane on which the integrand vanishes sufficiently and the subsequent application of the residuum theorem to the pole at $p_4 = i\sqrt{\vec{p}^2 + m^2}$. Since the Minkowski metric has disappeared and there is no restriction on the Wick-rotated ξ_4(the euclidean representation achieved an analytic continuation to all real ξ_4), the remaining task is to perform a euclidean Fourier-integral with a rotational invariant rational integrand. The d-dimensional integration in polar coordinates requires the same amount of work as d=4.

$$\frac{1}{(2\pi)^d} \int e^{ip\xi} \frac{d^dp}{p^2+m^2} = \frac{1}{(2\pi)^d} \frac{2(\sqrt{\pi})^{d-1}}{\Gamma(\frac{d-1}{2})} \int_0^\infty \frac{p^{d-1}}{p^2+m^2} dp \int_0^\pi \sin^{d-2}\theta d\theta e^{ipr\cos\theta} \tag{3.97}$$

$$= \frac{2(\sqrt{\pi})^d}{(2\pi)^d} \int \frac{2^{\frac{d}{2}-1}}{(rp)^{\frac{d}{2}-1}} \frac{1}{p^2+m^2} J_{\frac{d}{2}-1}(pr) p^{d-1} dp \tag{3.98}$$

or the Bessel functions J as well as a formula linking the Hankel function of the first type to an integral over a Bessel function. The Hankel function $H_\nu(z)$is analytic in the cut z-plane with a cut running from $-\infty$ to zero i.e.K(z) has a cut for $z^2 \leq 0$. Specialyzing to d =4, we obtain the following representation of

the free field two-point function as a boundary value of an analytic function:

$$i\Delta^{(+)}(\xi) = \lim_{\epsilon\to 0} \frac{1}{4\pi^2} \frac{m}{\sqrt{-(\xi_0 - i\epsilon)^2 + \vec{\xi}^2}} K_1(m\sqrt{-(\xi_0 - i\epsilon)^2 + \vec{\xi}^2} \qquad (3.99)$$

As expected,the space-like points are (together with the euclidean points)in the analytic domain. The distributional boundary value prescription becomes important only in the time-and light-like region where the transcription of the Hankel (Kelvin K) function in terms of J- and N-functions and the subsequent performance of the $\epsilon - \lim it$ gives the distribution:

$$i\Delta^{(+)}(\xi) = \frac{1}{4\pi}\epsilon(\xi_0)\delta(\xi^2) + \frac{m}{\sqrt{-\xi^2}} \frac{i}{8\pi} \left(J_1(m\sqrt{\xi^2})\epsilon(\xi_0) + iN_1(m\sqrt{\xi^2}) \right), \quad \xi^2 \geq 0 \qquad (3.100)$$

The strength of singularity on the light cone (determined by the singularities of K_1 or N_1) is independent of the mass and given by the zero mass two-point correlation function:

$$iD^{(+)}(\xi) = \lim_{\epsilon\to 0} \frac{1}{4\pi^2} \frac{1}{-(\xi_o - i\epsilon)^2 + \vec{\xi}^2} \qquad (3.101)$$

It is even independent of the state, e.g.the two-point function in the ground state and that in any other vector or density matrix state (e.g.a temperature state) in Fock space have the same leading light cone behaviour. The next-to leading behaviour (the log-term) in K_1does however depend on the mass. The fact that correlation functions have analyticity properties in space-like regions is however very specific for the vacuum state; other states in the same "folium" of states (explained in a later section) have only smoothness but no analytic behaviour in their correlation functions. The dependence of the singularities on the space-time dimension follows from the properties of the K_ν functions. It is conveniently encoded into the notion of "operator dimensions"of the fields i.e.one says e.g. that dimA=1 (in mass units) for d=3+1 and dimA=$\frac{1}{2}$ for d=2+1 if the two-point function has the singularity $(-\xi^2)^{-\dim A}$.

It turns out that the correlation functions of the higher spin free fields can all be expressed in terms of $i\Delta^{(+)}$ and its zero mass limit $iD^{(+)}$.We again look at the the important special cases s=$\frac{1}{2}$,1 before we scetch the calculations for the general free fields.

s=$\frac{1}{2}$

The ansatz for the positive and negative frequency parts for the local spinor field (in analogy to the previous scalar field) is (using the condensed notation from the beginning of this section for A(p) and B(p)):

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ipx} A(p) + e^{ipx} B^*(p) \right) \frac{d^3p}{2\omega} \qquad (3.102)$$

For the (anti)commutators of the covariant creation and annihilation operators one needs to know the completeness relations for the u-and v-spinors:

$$\sum_{s_3} u(p,s_3)\bar{u}(p,s_3) = p^\mu\gamma_\mu + m = 2m\Lambda_+ \,, \quad \sum_{s_3} v(p,s_3)\bar{v}(p,s_3) = p^\mu\gamma_\mu - m = -2m\Lambda_- \tag{3.103}$$

where $\Lambda_\pm$ are projectors $\Lambda_+ + \Lambda_- = 1$ on the $\pm$ frequency subspaces in the 4-dim Dirac spinor space. With:

$$\langle \psi_\alpha(x)\overline{\psi_\beta}(y)\rangle = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}\sum_{s_3} u_\alpha(p,s_3)\bar{u}_\beta(p,s_3)\frac{d^3p}{2\omega} \tag{3.104}$$

$$= \left(-i\partial_x^\mu\gamma_\mu + m\right)_{\alpha,\beta} i\Delta^{(+)}(x-y) \tag{3.105}$$

$$\langle \overline{\psi}_\beta(y)\psi_\alpha(x)\rangle = \frac{1}{(2\pi)^3}\int e^{ip(x-y)}\sum_{s_3} \bar{v}_\beta(p,s_3)v_\alpha(p,s_3)\frac{d^3p}{2\omega} \tag{3.106}$$

$$= -\left(i\partial_y^\mu\gamma_\mu + m\right)_{\alpha,\beta} i\Delta^{(+)}(y-x) \tag{3.107}$$

we obtain with $\psi^\# = \psi \; or \; \overline{\psi}$

$$\left\{\psi^\#(x),\psi^\#(y)\right\} = 0 \;\; if \;\; (x-y)^2 < 0 \qquad \{\psi(x),\overline{\psi}(y)\} = \left(-i\partial_x^\mu\gamma_\mu + m\right) i\Delta(x-y) \tag{3.108}$$

whereas the commutator is nonvanishing for spacelike distances.We get the first glimpse at the spin-statistics connection.

A more general complex linear combination of the two A and B^* pieces would not lead to a more general situation since the constants can be absorbed into a redefinition of these operators (see Weinberg's book) and the same comment applies to the construction of the general local fields for arbitrary spin. The present construction of local $\psi's$ also sheds some light on the physical interpretation of the v-spinors in connection with the charge conjugation symmetry. The latter transformation is defined in Fock-space by :

$$CA(p,s_3)C^* = B(p,s_3) \tag{3.109}$$

Its action on the local fields is local and the transformation law involves a matrix C in Dirac space:

$$\psi^C := C\psi C^* = C\psi^* \tag{3.110}$$

In the helicity representation used here, the matrix is $C = \gamma_2$, whereas in the Majorana representation one finds C=1. This matrix transforms the u-spinors into the v's and vice versa and therefore is the image of $D^{(s)}(i\sigma_2)$ under the intertwining map into the Dirac spinor space. It is an additional fringe benefit that via the Dirac doubling all global Fock-space symmetries as P, T and C have local representations on Dirac spinors. Furthermore the Dirac description goes into the Weyl equation in the zero mass limit.

s=1,m≠ 0

The massive (vector meson) case is straightforward. The local field is :

$$V_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \sum_i e_\mu(p,i) \left(e^{-ipx} a(p,i) + e^{ipx} b^*(p,i)\right) \frac{d^3p}{2\omega} \qquad (3.111)$$

Its 2-point function results from the completeness relation of the polarization vectors :

$$\langle V_\mu(x) V_\nu(y)\rangle = \left(-g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}\right) i\Delta^{(+)}(x-y) \qquad (3.112)$$

It is obvious that only the commutator can vanish for spacelike distances. Different from the previous case,vector meson fields does not permit a zero mass limit. Therefore we should not be surprized to meet some pecularities in the vectorial description of photons.

s=1,m=0

For a formally local[1] description in terms of vector fields, the longitudinal part which the stability group transformation behaviour of wave functions demands (see previous section) is not enough; one also needs "scalar photons":

$$\tilde{A}_\mu(k) = e_\mu^{(+)}(k) c_+(k) + e_\mu^{(-)}(k) c_-(k) + \sum_{\lambda=\pm} \epsilon_\mu(k,\lambda) a(k,\lambda) \qquad (3.113)$$

$$A_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ikx} \tilde{A}_\mu(k) + e^{ikx} \tilde{A}_\mu^*(k)\right) \frac{d^3k}{2\left|\vec{k}\right|} \qquad (3.114)$$

Here the $e^{(\pm)}$ are obtained by boosting the light-like vectors $(1,0,0,\pm 1)$ i.e.e_+ is the old longitudinal part. We obtain the covariant two point function :

$$\langle A_\mu(x) A_\nu(y)\rangle = \frac{-g_{\mu\nu}}{(2\pi)^3} \int e^{-ik(x-y)} \frac{d^3k}{2\left|\vec{k}\right|} \qquad (3.115)$$

from the completeness relation of the four vectors:

$$\sum_{\lambda=\pm} \epsilon_\mu(k,\lambda) \epsilon_\nu^*(k,\lambda) + \frac{1}{2}\left(e_\mu^{(+)}(k) e_\nu^{(-)}(k) + e_\mu^{(-)}(k) \epsilon_\nu^{(+)}(k)\right) = -g_{\mu\nu} \qquad (3.116)$$

This is the case iff the $a^{\#'}s$ behave in the standard way but the c's have a nondiogonal inner product which corresponds to a genuine indefinite (not just semidefinite) metric :

$$\langle c_+(k) c_-^*(k')\rangle = 2\left|\vec{k}\right| \delta(\vec{k} - \vec{k}') \qquad (3.117)$$

The a's mix with the c's under L-transformations viz.the comments on gauge transformations in the wave function discussion of the last section. But whereas

[1]Local here means pointlike, i.e. fields which can be smeared with unrestricted Schwartz test functions. Without the unphysical components, we would have to restrict vector-valued test functions f_μ by demanding transversality.

in the wave function treatment of photons in terms of vectors only an extension by scalar "photons"was necessary, in our present Fock space description of formally local point like vector potentials we need in addition the negative metric scalar contribution. Only in the weak sense of matrix elements the condition of absence of scalar "photons"can be enforced in terms of a *local condition*:

$$\langle\psi\,|\partial^\mu A_\mu\,|\,\varphi\rangle = 0 \tag{3.118}$$

The alternative is to use a nonlocal condition in terms of the annihilation part of A: $\partial^\mu A_\mu^{(+)}\,|\varphi\rangle = 0$ In the interacting case one then must show that $\partial^\mu A_\mu$ fulfills the wave equation.

As we mentioned already in the previous section, all these problems are absent if we describe the photons in terms of field strength instead of vector potentials. In that case we only deal with physical photons:

$$F_{\mu\nu}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\sum_{\lambda=\pm}\left(e^{-ikx}u_{\mu\nu}(k,\lambda)a(k,\lambda)+h.a.\right), \tag{3.119}$$

$$u_{\mu\nu}(k,\lambda) = ik_\mu\epsilon_\nu - \{\mu\leftrightarrow\nu\} \tag{3.120}$$

But in order to formulate interacting QED with its specific long range interaction[2] throgh the local coupling of free fields, the vector potential has been indispensible. This fact is the origin for certain complications in local gauge theories. In the standard indefinite metric method, the descend from the unphysical vectorial description defined by a free field with the two-point function [?] to the physical photons in the sense of Wigner is done with the help of the Gupta-Bleuler method. By the above transversality constraint one eliminates the scalar c_+"photons". This step leads from the indefinite metric "Fock"-space to a positive semidefinite subspace $\mathcal{H}$ which still contains the zero norm longitudinal "photons". The elimination of the latter can only be accomplished through descend to a factor space (defined by equivalence classes):

$$\mathcal{H}_{phys.} = \frac{\mathcal{H}}{\mathcal{H}^{(0)}}, \quad \mathcal{H}^{(0)} = \textit{nullspace of zero norm vectors} \tag{3.121}$$

The Gupta-Bleuler method (as well as its BRST generalization) has a certain geometrical elegance in renormalized perturbation theory but its conceptual physical aspects leave a lot to be desired. Of course the mathematical troubles start only with interations. I do not know any controllable mathematics for indefinite metric algebras which could be used for structural investigations i.e. spin&statistics, localization etc. The Gupta-Bleuler method or the BRST formalism is no match or substitute for the powerful theory of von Neumann algebras. The alternative namely to stay in physical space and introduce nonlocal potentials has not been seriously considered because it requires a more profound and difficult conceptional investment. In the net approach this problem seems to be related to finding a natural algebraic analogon of semi-infinite

[2] i.e. the quantum counterpart of the minimal external electromagnetic coupling.

"axial gauges" ??. More remarks on such ideas will appear in a later section 3.5.

The higher spin cases are treated analogously.We only give a brief scetch, the details may be found in Weinberg's book. Using the completeness relations of the general $[m,s]$ intertwiners one finds a two point function of the form:

$$(\psi_{ab}(x)\psi^*_{a'b'}(y)) = P_{ab,a'b'}(i\partial)i\Delta^{(+)}(x-y) \qquad (3.122)$$

Here P is a covariant polynomial in the derivatives. Again one finds the possibility of a matrix realization of P;T and C if one uses the "doubling" $D^{[A,B]} \oplus D^{[B,A]}$. The requirement of locality leads to the spin-statistics connection which generalizes the previous special observations.

The zero mass case leads to a severe restriction between A, B and the helicity h=s namely $|A-B|$=h. For h=2 the analogy with classical general relativity and the long range nature ot the graviton interaction again demands to side step this rule by using a gauge theoretic description in terms of a symmetric tensor $g_{\mu\nu}$. in analogy (but more complicated) with the vector potential for h=1 .The massive s=0, s=$\frac{1}{2}$ and s=1 fields as well as their massless helicity counterparts are "Eulerian" i.e. the transformation property is a consequence of the matrix form of the differential operator which is the 4×4 Dirac or the 4×4 s=1 Proca-Wentzel operator. Also for higher spins there are such Eulerian operators e.g the Rarita-Schwinger operator for s=$\frac{3}{2}$. But most of the "covariantizations of the Wigner representations are not Eulerian and can not be used for Lagrangians and canonical quantization procedures (in particular all minimal i.e. 2s+1 component descriptions for s> 0). But this does not make them less physical ore useful.

Finally we make the following important observation. Despite the fact that the Wigner theory gives a unique description for each mass and spin, we completely loose this uniqueness on the level of local fields. We obtain a countable covariant local family of fields which all share the same Fock-space operators but differ in their u and v intertwiners. This is true for any spin; even in case of s=0 we may use vectors or tensors which of course turn out to be just derivatives of the standard scalar field. In the next section we will show that these different fields generate the same local algebras. With respect to those algebras they behave like different coordinates in geometry. The intrinsic physical information is in the "net" of local algebras. As in geometry it is of course not wrong to use coordinates.

3.4 The Equivalence Class of a Free Field

We have seen that the Wigner representation theory together with the locality principle leads to a multitude of (m,s) fields. Actually the set of physically equivalent desccriptions is even much larger. Let us understand this first in the case of a neutral scalar field:

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left(e^{-ipx}a(p) + e^{ipx}a^*(p)\right) \frac{d^3p}{2\omega} = A^{(-)}(x) + A^{+}(x) \qquad (3.123)$$

Such operator-valued distributions cannot be pointwise multiplied as classical functions can. In order to find a substitute for classical pointwise multiplication, one studies first the matrix elements of products of A at different points e.g.

$$\langle \Omega \,|A(x_1)A(x_2)....A(x_n)|\,\Omega\rangle \tag{3.124}$$

Clearly the terms which become singular for coalescent points (more generally if one of the difference vectors $x_i - x_j$ becomes light like) results from "Wick-contractions" :

$$A^{(-)}(x_i)A^{(+)}(x_j) = i\Delta^{(+)}(x_i - x_j) + A^{(+)}(x_j)A^{(-)}(x_i) \tag{3.125}$$

$i\Delta^{(+)}$ which are generated by commuting the annihilation components $A^{(-)}$ through the $A^{(+)\prime}s$ to the right vacuum. The resulting terms in which the annihilators are on the right of creators i.e. operator products of the form:

$$A^{(+)}(x_{i_1})....A^{(+)}(x_{i_k})A^{(-)}(x_{i_{k+1}})....A^{(-)}(x_{i_n}) \tag{3.126}$$

have vanishing vacuum expectation values and finite matrix elements between finite (but arbitrarily large) particle number vectors in Fockspace. In those "Wick-ordered" products the limit $x_i \to x$ of colliding points can be taken without peril. Therefore one defines local functions of the local field A(x) in the sense of Wick-ordering as:

$$:A^n(x):= \sum_{k-partitions} A^{(+)}(x)....A^{(+)}(x)A^{(-)}(x)....A^{(-)}(x) \tag{3.127}$$

i.e. the terms which result by simply ignoring the contractions. These are the equal point limits of "split-point" Wick-products:

$$:A(x_1)....A(x_n):= \sum_{k-partitions} A^{(+)}(x_{i_1})....A^{(+)}(x_{i_k})A^{(-)}(x_{i_{k+1}})....A^{(-)}(x_{i_n}) \tag{3.128}$$

The usefulness of the Wick-ordering results from the fact that despite their nonlocal origin in terms of frequency separation, the resulting operators are local resp. multilocal. This is because the above definition is equivalent to the following obviously local inductive definition:

$$A(x_1)....A(x_n) =: A(x_1)....A(x_n) : + \sum_{m=1}^{[\frac{n}{2}]} \sum_{m\ contr.} :A(x_1)\,\underbrace{..\,...\,.\,...\,.}\,..A(x_n): \tag{3.129}$$

where the lower brackets represent the Wick- "contracted" pairs and the sum goes over all m-pairings and finally over all m. Clearly this formula provides an inductive definition of :.....: ordering (the right hand sum only involves ordered products with a lower number of operators). The proof that the previous frequency-ordering definition leads to this inductive formula is elementary and left to the reader. The multi-localized (at $x_1....x_n$) product obviously approaches

the one-fold localized Wick power of the free field. Here the word "local" has a classical as well as a quantum meaning. Classically it means that one only has to know the A's around the spacetime point x in order to compute $: A^n(x) :$. whereas the operational quantum meaning is that this pointlike composite commutes with all the A's whose localization is spacelike with respect to x (locality in the sense of Einstein causality, which in local Quantum Theory means simultaneous measurability). In order to get a feeling for the properties of these local composites, let us look at their two point functions.

Here the calculation of spectral representations for Wick-powers is missing.

The family of pointlike Wick-ordered composites is bigger than the above illustrations; also derivatives as $: \partial_\mu A(x)\partial_\nu A(x) :$ are included. It is very gratifying that also the inverse is true: the set of fields in Fockspace which commute for spacelike distances with the free field A(x):

$$BC(A) := \left\{ B \mid [B(x), A(y)] = 0 \quad for \quad (x-y)^2 < 0 \right\} \tag{3.130}$$

is called the Borchers equivalence class $BC(A)$ and consists precisely if the local composites. The equivalence class aspects will be discussed in a later chapter in the context of interacting fields. At the end of this section we will give a proof of this theorem. It is important (e.g. for the derivation of the Feynman rules) to be able to Wick-order products of local composites of free fields. Let us look at examples:

$$\begin{aligned} :A^4(x)::A^4(y) :=: A^4(x)A^4(y) : +4^2 i\Delta^{(+)}(x-y) : A^3(x)A^3(y) : + \\ +4^2 3^2 \left(i\Delta^{(+)}(x-y)\right)^2 : A^2(x)A^2(y) : + (4!)^2 \left(i\Delta^{(+)}(x-y)\right)^3 : A^3(x)A^3(y) : + \\ + (4!)^2 \left(i\Delta^{(+)}(x-y)\right)^4 \end{aligned} \tag{3.131}$$

$$\begin{aligned} :\bar\psi(x)\gamma_\mu\psi(x)::\bar\psi(y)\gamma_\nu\psi(y) :=: \bar\psi(x)\gamma_\mu\psi(x)\bar\psi(y)\gamma_\nu\psi(y) : \\ + : \bar\psi(x)\gamma_\mu iS^{(+)}(x-y)\gamma_\nu\psi(y) : \\ +Tr\left\{iS^{(-)}(y-x)\gamma_\mu : \psi(x)\bar\psi(y) : \gamma_\nu\right\} + Tr\left\{iS^{(-)}(y-x)\gamma_\mu iS^{(+)}(x-y)\gamma_\nu\right\} \end{aligned} \tag{3.132}$$

For a good understanding of the Wick-formalism of local functions a knowledge of the following statements is indispensible.

Statement 1: *Powers of the two-point functions are well-defined distributions (singular functions)* e.g. $F(x) = (i\Delta^{(+)}(x))^n$ is again a distribution with momentum space support properties. This is a multidimensional generalization of the well-known statement that singular functions F in one variable, whose Fouriertransform $\tilde F$ have support on the half line, can be freely multiplied. The reason is that (as a result of the support property) F is the boundary value of on the real axis of an analytic function holomorphic in the upper half plane and therefore this property is inherited by F^n. Equivalently the convolutions of $\tilde F$ only extend over over a compact region. In the multidimensional version the half lines are to be replaced by conic regions. In standard QFT momentum space correlation functions are well behaved functions, which at most have singularities at small momenta (infrared problems). Their asymptotic increase is responsible for the x-space singularities on the light cone.

Statement 2: *The Noether conservation laws of classical field theory also hold for the corresponding Wick-orderd objects in the free field Borchers class.*

We provide two typical illustrations:

$$\partial^\mu j_\mu(x) = 0, \quad j_\mu(x) =: \bar{\psi}(x)\gamma_\mu\psi(x): \quad \psi = \text{Dirac-field}$$
$$\partial^\mu T_{\mu\nu}(x) = 0, \; T_{\mu\nu}(x) =: \partial_\mu A(x)\partial_\nu A(x) : -g_{\mu\nu}\tfrac{1}{2} : (\partial^\kappa A(x)\partial_\kappa A(x) - m^2A^2(x)) : \tag{3.133}$$

As in the classical case the covariant divergence hits both of the fields and lead to operations on the u-and v-intertwiners which thanks to certain identities (e.g. the vanishing of the momentum space Dirac operator on these intertwiners) give the desired conservation law. In no stage of the argument does one need the canonical formalism or the Euler-Lagrange form of equation of motions, on only needs identities on intertwiners u and v which are an immediate consequence of their definition.

Statement 3: *In the relation between local "currents" and global "charges":*

$$Q = " \int d^3x j_0(x)", \quad P_\mu = " \int d^3x T_{\mu 0}(x)" \tag{3.134}$$

the phenomenon of vacuum polarization enforces a nonclassic subtlety which is explained in the following.

A composite of a free field is more singular than the free field. In particular for d≥2+1 it does not fit into the framework of canonical equal time (anti)commutation relation, but rather has to be smeared with test functions in d dimensions (in our case d=3+1). This can already be seen by using the previously calculated two-point function of the composite current operator j , e.g.

$$\langle j_\mu(x)j_\nu(y)\rangle = \int (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)i\Delta^{(+)}(x-y,\kappa^2)\rho(\kappa^2)d\kappa^2 \tag{3.135}$$

Since $\int \rho(\kappa^2)d\kappa^2 = \infty$, the smearing with test functions supported on a spacelike hypersurface i.e. of the form $f(x) = \hat{f}(\vec{x})\delta(t)$ does not give a finite answer, one rather needs smoothness in time as well. As in the classical case, one tries to obtain the global charge $Q = \int (a^*(p)a(p) - b^*(p)b(p))\frac{d^3p}{2\omega}$ as a limit of "partial" charges referring to a finite region:

$$Q(g.h) := \int j_0(x)g(\vec{x})h(t)d^4x, \quad \text{supp g} \subseteq V + \delta V, \; \text{supp h} \subseteq \{|t| \leq \epsilon\}$$
$$g \equiv 1 \text{ in 3-volum V}, \quad \int h dt = 1 \tag{3.136}$$

In words: g is a characteristic function of the 3-dim. volum region which has been smoothened outside, whereas h(t) is a smoothened δ-function. It is easy to see that:

$$[Q(g,h), B] = [Q, B], \quad \text{for loc. B in compl.V} \tag{3.137}$$

i.e. for operators B localized in the causal completion of V (smearing functions with support in compl.V) the commutator is already independent of g,h (and

identical to the global charge). However on the vacuum vector Ω the partial charge has such strong vacuum fluctuations (resulting from the presence of $a^* - b^*$ terms) that:

$$\lim_{V\to\infty} ||Q(g,h)\Omega||^2 = \infty, \quad \text{but } \lim_{V\to\infty} (\psi, Q(g,h)\Omega) = 0 \tag{3.138}$$

Here ψ is from the dense domain on which the local functions of the free field are defined i.e. the polynomial domain. The vacuum fluctuations were discovered in the early days of QFT by Heisenberg and their physical significance was studied by Weisskopf. One such manifestation is a contribution to the Lamb-shift (see next chapter). This quantum phenomenon has no counterpart in quantum mechanics and it has far going structural consequences, e.g. it makes the local algebras of QFT essentially different from the quantum mechanical Heisenberg-Weyl algebras (the former admit no pure state pure states or minimal projectors.

Here the calculation of the free Borchers class based on the multi-variable Cauchy propagation problem is still missing.

3.5 A First Look at Modular Localization

Recently it turned out that the Wigner representation theory contains information on localization which allows a direct access to the local algebras avoiding the use uf nonunique field coordinates [3]. The starting point is the abelian subgroup of Lorentz boosts belonging to a wedge, say the standard t-x wedge $x > |t|$. The Wigner theory also provides an anti-unitary operator which reflects the standard wedge into its opposite wedge. In the simplest case of irreducible representation for scalar neutral particles, this reflection j differs from the TCP operation by a π-rotation around the x-axis:

$$(\Theta\varphi)(p) = \bar{\varphi}(p) \quad (j\varphi)(p) = \bar{\varphi}(p_0, p_1, -p_2, -p_3,) \tag{3.139}$$

Define now an unbounded positive closed operator δ by functional calculus from the selfadjoint standard (x-t)boost generator K:

$$\delta = e^{-K}, \quad \delta j = j\delta^{-1}, \quad \text{since } e^{iKx} j = j e^{iKx} \tag{3.140}$$

With the help of the Tomita-like unbounded involutive operator $s := j\delta^{\frac{1}{2}}$ we define a closed "real" subspace H_R of the Wigner representation space H:

$$H_R = \{\varphi(p) \in H \mid s\varphi = \varphi\}, \quad s = j\delta^{\frac{1}{2}} \tag{3.141}$$

The $\pm$ eigenspaces (since s is antilinear, only real lineat combinations are possible) of the closed operator s can easily be shown to form a dense set in H and the above definition is also the unique polar decomposition of s. To be more specific, s acts as:

$$s: \quad h + ik \to h - ik, \quad h, k \in H_R \tag{3.142}$$

[3] B.Schroer "Notes on the Wigner Representation Theory of the Poincaré Group, Localization and Statistics".

A more explicite description of H_R is obtained by introducing the wedge-affiliated "rapidity" θ :

$$p = m(qcosh\theta, qsinh\theta, n_2, n_3), \quad q = \sqrt{1 + n_2^2 + n_3^2} \tag{3.143}$$

The domain of the operator $\Delta^{\frac{1}{2}}$ in terms of rapidity-dependent wave functions consists of boundary values of analytic functions which are holomorphic in the θ-strip $0 < \theta < i\pi$ and H_R is the closed real space of wave functions fulfilling the boundary condition:

$$\overline{\varphi(\theta + i\pi)} = \varphi(\theta) \tag{3.144}$$

where we surpressed the dependence on n_i. Let us call this the "$s-$reality property". It is somewhat surprizing that this concept did not seem to have appeared in mathematical physics, e.g. it is absent in the various books including those by Reed and Simon. For massive spin s representations the $s-$ reality property reads:

$$D^{(s)}(i\sigma_2) lim_{\chi \to i\pi} D^{(s)}(R(p, \Lambda^{st}(\chi)) \overline{\varphi(\theta + i\pi)} = \varphi(\theta) \tag{3.145}$$

If particles are not selfconjugate, the 2s+1 component φ must be doubled and the action of J on the direct sum involves a flip-operation on the two Hilbert spaces. For zero mass, the rapidity parametrization for the standard wedge is defined by $k = \Lambda^{st}(\theta)k_0$ with $k_0 = (1, \vec{n})$ and the Wigner rotation $R(k, \Lambda)$ is to be replaced by the helicity representation in terms of the Wigner phase factor of the euclidean group $E(2)$.

Looking at the geometric interpretation of this construction, one conjectures that the subspace H_R of these momentum space wave functions has something to do with localization in the standard wedge (or in the opposite wedge in case of the -subspace). This idea can be confirmed by studying coherence properties of the net of real wedge spaces generated via Poincaré transformations g on the standard wedge :

$$\begin{aligned} H_R(W) \quad &: \quad = U(g)H_R, \quad W = gW^{sta} \\ H_R \quad &= \quad H_R(W^{sta}) \end{aligned} \tag{3.146}$$

For localization in the quantum sense, one needs a a concept of "outsideness". In Schrödinger theory as well as in the relativistc work of Newton and Wigner, one uses the orthogonality in wave function space: one calls f localized in a 3-dim. region R if a spatial translation which carries R into its geometric complement transforms the wave function into the orthogonal complement. For relativistic wave functions this idea unfortunately (much to the dismay of Wigner) cannot be extended from equal time localizaton to spacelike localization (apart from localization in an "effective" sense i.e. modulo Compton tails). For our purposes we need a precise localization in the following symplectic sense:

$$H'_R = \{h' \mid Im(h', h) = 0\}, \quad H'_R(W) = U(g)H'_R \tag{3.147}$$

It then follows that:

$$H'_R(W) = H_R(W)', \quad \text{and } H'_R(W) = H_R(W'), \quad W' \equiv W^{opp} \tag{3.148}$$

where the last nontrivial equality is a consequence of:

$$s^*_W = s_{W'} \quad \longleftrightarrow \quad j_W = j_{W'}\ , \ \delta^{-1}_W = \delta_{W'} \tag{3.149}$$

which in turn follows from the commutation relation of the standard (x-t) reflection j (which sends the wedge W into W^{opp}) and the Lorentz-boost δ^{it}. Again one ends up with real Hilbert spaces which are standard and factorial in the sense of 2.6.

Thus we arrive at a covariant net of wedge spaces and now we want to show that this net is isotonous i.e. that if a wedge is contained in another one, the same is true for the associated spaces. But in such a situation the second wedge is obtained from the first by two lightlike translations which carry it inside.so we have to show isotony for lightlike translations. For such translations we have:

$$s_{\hat{W}} \subset s_W, \quad \hat{W} = g(\lambda l)W \subset W, \quad \lambda > 0 \tag{3.150}$$

where $g(\lambda l)$ is a translation along the lightlike vector l. In order to show that s_W extends $s_{\hat{W}}$ we rewrite this relation as :

$$U(\lambda l) j_W \delta^{\frac{1}{2}}_W U(\lambda l)^* \subset j_W \delta^{\frac{1}{2}}_W \tag{3.151}$$

For the bounded antilinear operator j_W this gives the covariance law, whereas for unbounded δ the required relation results from the commutation relation of the lightlike translation with the standard Lorentz-boost $U(\chi)$:

$$U(\lambda l)U(\chi) = U(\chi)U(e^{\chi}\lambda l) \tag{3.152}$$

One can show that the isotony is quite generally equivalent to the positivity of the energy.

Wedge localization is too weak for a physical interpretation of the theory (e.g. for the derivation of statistics and scattering theory). The localization underlying standard (e.g. Lagrangian) theory is compact localization which in our context means (K stands for double cone):

$$H_R(K) = \bigcap_{W \supset K} H_R(W) \tag{3.153}$$

$$H_R(K) + iH_R(K) \quad \text{dense in } H, \quad H_R(K) \cap iH_R(R) = \{0\} \tag{3.154}$$
$$H_R(K) \cap H'_R(K) = \{0\}$$

This property which previously was called standard and factorial 2.6 can be shown for all Wigner (m, s) representations and even for $m = 0$ (with the exception of the continuous spin representations which do not permit such a

localization, since in this case the $H_R(K)$ spaces turn out to be trivial). As a consequence the spaces fulfill the following duality property:

$$H'_R(W) = H_R(W'), \quad H'_R(K) = H_R(K') \tag{3.155}$$

In case of (m, s)-representations one can prove this even for disconnected and non simply connected regions in Minkowski-space. This is not so for the zero mass representations. For example in the case of photons ($m = 0, h = 1$) one finds a violation for the toroidal "corona" region $\mathcal{T}$. Let T be the causal completion of a spatial torus which we call the "corona". The size of the corona is chosen in such a way that the causal complement $\mathcal{T}'$ of $\mathcal{T}$ consists of a double cone of diameter r and a "double cone at infinity": $|\vec{x}| \geq R + |t|$ causally separated from the former by the $\mathcal{T}$ with width: $R - r \geq 0$ region in between . Then one obtains the following proper corona- inclusion:

$$H^R(\mathcal{T}) \subset H^R(\mathcal{T}')' \tag{3.156}$$

$$\curvearrowright \mathcal{A}(\mathcal{T}) \subset \mathcal{A}(\mathcal{T}')'$$

where $H^R(\cdot)'$ denotes the previously defined symplectic complement and the A's denote the corresponding von Neumann-algebras as obtained from the $H^R(\cdot)$ by the Weyl construction.

This "classical" obstruction, formally related to the appearance of δ' in the E-H canonical commutation relation, can be physically understood in terms of a (suitably regularized) magnetic flux through a surface which stretches from a circle inside the torus into the space-like separated region inside. Such a flux does not change if one passes through another surface subtended from the same circle. Hence such a flux, also not being localizable within the 4-dim toroidal region nevertheless belongs to the symplectic complement of the spacelike complement of the corona consisting of two spacelike seperated pieces. This entails the above violation of Haag duality for the corresponding algebras.

A more systematic approach in the spirit of Wigner consists in rewriting the inner product in terms of a would be vectorial object. This time, unlike the massive case, there are no covariant intertwiners which lead to a nondegenerate inner product which is expressible in terms of a momentum space integral over an invariant momentum space integrand. The best one can do is to introduce a partially covariant inner product (invariant after integration) associated (by polarization) to the norm:

$$\begin{aligned} \int |\psi(k,\pm)|^2 \frac{d^3k}{2\omega} &= \int \tilde{A}_\mu(k,n,\pm) A^\mu(k,n,\pm) \frac{d^3k}{2\omega}, \quad \omega = \left|\vec{k}\right| \\ \tilde{A}_\mu(k,n,\pm) &= \frac{n^\nu \tilde{F}_{\nu\mu}(k,\pm)}{n\cdot k - i\varepsilon}, \quad \tilde{F}_{\nu\mu}(k,\pm) = A(k_\nu \varepsilon_\mu(k,\pm))\psi(k,\pm) \end{aligned} \tag{3.157}$$

where A denotes the antisymmetrization in μ, ν and $\varepsilon_\mu(k,\pm)$ the polarization vectors. The singularity in k-space corresponds to the semiinfinite line integral

along n in x-space.

$$\begin{aligned} A_\mu(x,n) &= \int_0^\infty n^\nu \tilde{F}_{\nu\mu}(x+ns)ds \\ &= \int (e^{-ikx}\sum_{i=\pm}\tilde{A}_\mu(k,n,i)+h.c.)\frac{d^3k}{2\omega} \end{aligned} \tag{3.158}$$

This vector potential has the following obvious properties:

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= F_{\mu\nu} \\ (U(\Lambda)A)_\mu(x,n) &= \Lambda_\mu^\nu A_\nu(\Lambda^{-1}x,n') \\ &= \Lambda_\mu^\nu A_\nu(\Lambda^{-1}x,n)+\partial_\mu G(x) \\ G(x) &= \lim_{\epsilon\searrow 0}\int e^{ikx}\frac{1}{(kn-i\epsilon)(kn'-i\epsilon)}n\cdot\tilde{F}(\Lambda^{-1}k)\cdot n'\frac{d^3k}{2\omega} \end{aligned} \tag{3.159}$$

i.e. the Lorentz transformation which acts on the Wigner wave function resp. on the $F_{\mu\nu}$ tensor, transforms the potential covariantly except an additive gauge term. The nonlocality of the vectorpotential is made manifest by this noncovariant transformation law. This peculiar "gauge" behaviour is a consequence of the nonfaithful helicity representation of the noncompact "little group" $E(2)$. In particular as we already emphasized in section 2 and 3 of this chapter *the quantum origin of gauge and gauge invariance has no direct relation with the notion of classical fibre bundles* as most of the books make believe. This is one of the more interesting and potentially fruitful clashes between quantization and an intrinsic quantum-based approach. The chance to use it for the conceptual advancement of QFT should not be lost by enshrouding it with a barrage of differential geometry or topology. The quantization method (from the viewpoint of the Wigner method) as explained before would trade the physical nonlocality of A_μ with (physically artificial) formal elegance by the introduction of an indefinite metric (Gupta-Bleuler). In this way the additive term would loose its significance related to Lorentz transformations and become the gauge concept of the mathematicians and of classical Maxwell theory. This is the method in which covariant renormalized perturbation theory is carried out. One profits from the formal elegance at the prize of a conceptionally questionable return to quantum physics. An electromagnetic interaction density W_{em} to be used in a perturbative Bogoliubov-Shirkov $S(g)$-approach (see next chapter in particular section 5) has the form (supp$g$ $\subset$double cone):

$$W(g) = \int_{supp g} g(x)A^\mu(x) : \bar{\psi}(x)\gamma_\mu\psi(x) : d^4x \tag{3.160}$$

Here ψ and A_μ are free feelds in Fock space. But whereas in the indefinite metric description A_μ contains ghosts and the locality properties are not the physical ones, the A_μ in [?]lives in the physical Hilbert space. If one now performs a Lorentz-transformation on $W(g)$ or $S(g)$ one realizes that the additive gauge contribution in[?] produces via partial integration with the conserved current a

nonlocal surface term which spoils an important structural property of thr perturbative $S(g)$ approach. Whereas usually $W(g)$ is partially L-invariant (inside the supp$g$) this is not so in this case. The difficulty propagates into the time ordered products. On the other hand if the standard Gupta-Bleuler approach is correct than there should be a local way of expressing positivity. For anybody who has not totally succumbed to (in this case unmathematical) formalism, this must appear as an interesting conceptual clash. Its resolution could lie in the realization that A_μ is a string-like (spacelike cone) semiinfinitly localized object. In that case it would be unwise to use a g with a double cone support and a noncompact support $supp g \subset spacelike\ cone \subset wedge$. For such noncompact regions the field algebras are still unitarily equivalent to the free field algebras i.e. Haag's theorem prohibiting the existence of dressing transformations and the existence of the interaction picture is not effective for such regions. This apparent locality clash as well as the previous obstruction against the corona duality are in my view windows into as yet unsufficiently known aspects of local quantum physics.

The obstruction against equality in 3.156contains the following interesting conceptional message. Whereas violations of Haag duality for simply connected regions are the hallmark of spontaneous symmetry breaking (in fact they may be used for a model independent definition of that concept in the setting of algebraic QFT, see last chapter), the violation for not (simply) connected regions has two different physical explanations. The most common one is the mechanism of "charge split" into causally disjoint regions. In this case the commutant is bigger than the geometric complement suggests because the charge split mechanism on a neutral observable algebra is not incorporable into a geometric picture. Of course one is always invited to enlarge the observable algebra to the field algebra for which there is harmony with the geometrical picture and no duality obstruction. The second mechanism is the one at hand: a "quantum-topology"-caused mismatch between the geometrical complement and the quantum theoretical opposite in the sense of local commutativity resp. of symplectic structure. The defect dimension of the two real Hilbert spaces in 3.156is:

$$dim\,[H^R(\mathcal{T}')' : H^R(\mathcal{T})] = 1$$

The obstruction is caused by the presence of just one object:

$$\oint_{C\subset\mathcal{T}} A^{reg}_\mu(x,n)dx^\mu$$

$$A^{reg}_\mu(x,n) = \int \rho(\vec{x}-\vec{y})A_\mu(\vec{y},x_0,n)d^3y$$

The integration is over a closed path C inside $\mathcal{T}$ and we regularized the vector potential with a smooth function of small support $supp\rho \in B_\epsilon$ so that one maintains normalizability 3.157and remains inside $\mathcal{T}$. The line integral represents the class of expressions of this kind, any two such elements differ only by field strength localized in $\mathcal{T}$. The line integral is a L-invariant and may be expressed in terms of a magnetic flux through any surface S with the C boundary.

It is precisely this floating surface stretching beyond C, which in the quantum setting of commutativity (or symplectic orthogonality) prevents the affiliation with $H^R(T)$ and makes it a member of the nongeometric $H^R(T')'$. This is of course an intrinsic property of the theory which cannot be removed by the indefinite metric formalism.

It is a much more difficult question as to what becomes of this topological obstruction in the presence of interactions. The reader can find some remarks in the last section. It is tempting to interpret this obstruction as indicating the necessity of an interaction [4] i.e.of the presence of non-vanishing electric or magnetic (or both) currents.

$$\partial^\mu F_{\mu\nu}(x) = j_\nu(x), \;\; \partial^\mu \tilde{F}_{\mu\nu}(x) = \tilde{j}_\nu(x) \tag{3.161}$$

The idea is that interactions are necessary to restore perfect Haag duality which is violated in the free theory. Such a point of view would attribute a very distinguished role to electromagnetic duality i.e. those superselection rules which originate from the quantum version of the Maxwell structure and may well be the physical concept behind the semi-classical "gauge principle". This issue of problematizing the notion of "magnetic field" on the same level of depth as the notion of "charge" in the DHR superselection theory is presently ill-understood in QFT.

In low dimensional QFT the analogous issue of order-disorder duality and the connection with Haag duality is much better understood. There, even in free theories (see last section of this chapter), it is not possible to have *no* charge sectors with both order and disorder the realization of both charges being related in d=1+1 to the zero mass limit. The previous idea of maintaining corona duality would bring the interacting Maxwell-like theories closer to the 2-dim. situation.This analogy is another reason to believe that the free Maxwell situation is peculiar. The remaining three non-peculiar cases, namely the appearance of objects with e.-, m.- or e.m.-charges have an infrared structure, whose implications for localization properties are outside the present scope of understanding. A better understanding of the connection between these properties and the modular theory (in the vein of the remarks about interactions in the last section) seems to be essential for future progress..

The corona inclusion may be constructed solely in terms of the Wigner theory supplemented by the modular theory for wedges avoiding covariant amplitudes like $F_{\mu\nu}$ alltogether. Helicities h$\geq$ 1 present similar corona structures.

Let us make some more remarks about the massive case. The principle of locality requires to study intersection of wedges. Intersecting the translated standard wedge W_a^{st} with the opposite wedge W_{opp}^{st} leads to x-t double cone which is cylindrically extended in the y-z direction. Since the modular localization in W_{opp}^{st} corresponds to a $s-$reality conditon in the negative θ-strip, the intersection of both gives rise to a new "edge of the wedge" problem i.e. a Hilbert space $H_R(W_a^{st} \cap W_{opp}^{st})$ of analytic functions which are meromorphic in

[4] This a speculative remark is taken from the book of R.Haag "Local Quantum Physics" page 147.

both strips and fulfil a matching condition on the real $\theta-$axis in which the translation enters. Again $H_R(W_a^{st} \cap W_{opp}^{st})$ is standard in the sense of the definition given in the section on CCR and CAR functors.

The analytic situation for intersections of non coplanar wedges as one needs them for double cones in ?? becomes very rich and is essentially unexplored. In d=3+1 theories with halfinteger spin QFT of free fields indirectly yield the information that the corresponding real subspaces are standard and factorial 2.6.

If we apply this localization concepts to halfinteger spin, we find a very interesting discrepancy by a factor i between the action of j and that of the $\pi-$rotation of the wedge caused by the SU(2) transformation law of the spin. Through this obstruction the Wigner theory takes notice of the Fermi statistics.

We now explain the direct conversion of the net of Wigner subspaces into a net of CCR- and CAR-algebras 2.6.

Consider first the case of integral spin. The application of the Weyl-functor to the subspace $H_R(W)$ gives the von Neumann-algebra:

$$\mathcal{F} : H \to \mathcal{A}(H), \quad f \longmapsto W(f) \tag{3.162}$$

$$\begin{aligned} \mathcal{A}(W) &= \text{v.Neumann Alg.}\{W(f) | f \in H_R\} \\ &= \mathcal{F}(H_R(W)) \end{aligned} \tag{3.163}$$

which inherits the following properties from the Hilbert spaces:

$$\begin{aligned} \text{isotony} &: \quad \mathcal{A}(W) \subset \mathcal{A}(\tilde{W}), \quad \text{for } W \subset \tilde{W} \\ \text{Haag duality} &: \quad \mathcal{A}(W') = \mathcal{A}(W)', \quad W' = W^{opp} \\ \text{covariance} &: \quad U(g)\mathcal{A}(W)U^*(g) = \mathcal{A}(gW), \quad g \in \mathcal{P} \end{aligned} \tag{3.164}$$

In the halfinteger spin case we take the CAR functor $\psi^\#(f)$:

$$\mathcal{F} : H \to \mathcal{A}(H), \quad f \to \psi(f) \in B(\mathcal{H}_F) = \mathcal{A}(H) \tag{3.165}$$

$$\begin{aligned} \mathcal{A}(W) &= \text{v.Neumann Alg.}\left\{\psi^\#(f) \mid f \in H_R(W)\right\} \\ &= \mathcal{F}(H_R(W)) \end{aligned} \tag{3.166}$$

Different from the bosonic case, the operators J and S of this algebra are not given by the application of the previous functor $\mathcal{F}$ but the J contains the famous Klein twist K which changes geometrically causally disjoint localized operators into commuting ones which one needs in the T.T. theory:

$$J = K\mathcal{F}(j), \quad \Delta^{it} = \mathcal{F}(\delta^{it}), \quad S = J\Delta^{\frac{1}{2}} \tag{3.167}$$

This is the T-T modular theory for wedge subalgebras of the CAR-algebra.

The same modular formalism can be used in order to construct relativistic KMS states on free field algebras. In complete analogy to chapter2 section5 the thermal two-point functions have the form ($z=e^{\beta\mu}$):

$$\langle\psi(x)\bar{\psi}(y)\rangle_{e,\beta,\mu} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{-ip(x-y)}\sum_{s_3} u(p,s_3)\bar{u}(p,s_3)\frac{1}{1\mp ze^{-\beta p\cdot e}}\frac{d^3p}{2\omega} + \\ + \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{ip(x-y)}\sum_{s_3} v(p,s_3)\bar{v}(p,s_3)\frac{ze^{-\beta p\cdot e}}{1\mp ze^{-\beta p\cdot e}}\frac{d^3p}{2\omega}$$

Here e is a time-like vector which characterizes the rest frame of the heat bath and the $\mp$sign corresponds to Boson/Fermion statistics. Mistakes in the combinations of signs in front of the integrals can be easily corrected by remembering that the thermal correlation functions must have the same (anti)commutator functions as the standard free field correlation functions (in addition to the KMS property). These relativistic correlation functions have rather interesting analytic properties; they are analytic in $x-y=:\xi\to z$, $z\in\mathcal{T}_{\beta e}$ where $\mathcal{T}_{\beta e}$ is the tube $\mathcal{T}_{\beta e}=\left\{z\in C^d : Imz\in V_+\cap(\beta e+V_-)\right\}$. The boundary values at the two edges fulfill as expected the KMS condition:

$$\exists F(z) \text{ analytic in } \mathcal{T}_{\beta e} \text{ s.t.}$$
$$\langle\psi(x)\bar{\psi}(y)\rangle = \lim_{z\to\xi}F(z),\ \langle\bar{\psi}(y)\psi(x)\rangle = \lim_{z\to\xi+i\beta e}F(z)$$

where the boundary values are taken from inside the analytic tube region. All the statements are easily checked by explicite computations. Although the boundary KMS conditions is the standard one which relates the boundary values on the two sides of the temperature strip, the relativistic aspect generates a larger analytic tube in x-space which contains the strip in the e-direction. The temperature can be directly introduced as an extension of the Wigner theory. It should be interesting to combine the modular localization aspect with the heat bath temperature within the Wigner setting.

Lit. The first presentation of the $(m=0,h=1)$−obstruction against Haag duality for the corona region was given in the unpublished work of Leyland et al. mentioned at the end of the previous chapter.

3.6 Rindler Wedges and Hawking Temperature

QFT of a uniform electric field in a half space or of a family of uniformly accelerated observers. in both cases a wedbe form part of space time becomes seperated by a horizon. The ensuing loss of information converts the original vacuum state into a Hawking-Unruh temperature state. This situation supplies the nicest physical illustration of the modular concepts with deep relations to antiparticles, the TCP-theorem and crossing symmetry as a consequence of the KMS temperature cyclicity property (treated in a later section).

Lack of time prevented the writing of this section.

3.7 Fields associated with Free Fields

In the following we will give an example for a field whose Borchers class is associated with but not equal to that of a free field. This construction is part of the d=0 duality (order- disorder) construction.

Let us start from a complex free complex Dirac field in d=1+1:

$$\begin{aligned}\psi(x) &= \int \frac{dp}{2\omega}(e^{-ipx}u(p)a(p) + e^{ipx}v(p)b^*(p)) \qquad (3.168)\\ u(p) &= \sqrt{\frac{m}{2}}\begin{pmatrix} e^{\frac{\theta}{2}} \\ -e^{-\frac{\theta}{2}}\end{pmatrix}, \quad v(p) = \sqrt{\frac{m}{2}}\begin{pmatrix} -e^{\frac{\theta}{2}} \\ -e^{-\frac{\theta}{2}}\end{pmatrix}\\ v(p) &= u^C(p) = Cu(p)i\gamma_0, \quad p = m(\cosh\theta, \sinh\theta)\end{aligned}$$

Here we took the following realization of the Dirac equation:

$$\begin{aligned}(i\gamma_\mu\partial^\mu - m)\,\psi &= 0,\ \gamma_0 = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} \qquad (3.169)\\ C &= -i\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\end{aligned}$$

This field is U(1) covariant and the local generator is the conserved current $j_\mu =: \bar{\psi}\gamma_\mu\psi :$. This d=1+1 current has (relatively to ψ) nonlocal pseudo-potential:

$$\begin{aligned}j_\mu(x) &= \varepsilon_{\mu\nu}\partial^\nu\phi(x) \quad \phi(x) =: F_x(a^\#, b^\#) := \int_{-\infty}^{x} \varepsilon^{\mu\nu} j\nu d\xi_\mu = \qquad (3.170)\\ &= \int d\theta_p d\theta_q \left\{ \begin{array}{l} \frac{-1}{\sinh\frac{1}{2}(\theta_p - \theta_q + i\varepsilon)} e^{i(p-q)x}\left[a_p^* a_q - b_p^* b_q\right] \\ + \frac{1}{\cosh\frac{1}{2}(\theta p - \theta_q)}\left[e^{i(p+q)}a_p^* b_q^* + e^{-i(p+q)}a_q b_p\right] \end{array} \right\}\end{aligned}$$

As naively expected, the ϕ is a local field which although relatively local with respect to the observables (generated by the current) fails to be local relative to the field ψ but instead fulfills:

$$\begin{aligned}\phi(x), \psi(y) &= \theta(x_0 - y_0)\psi(y)\phi(x) \qquad (3.171)\\ \lim_{x^2\to-\infty, x^1\to\infty} \phi(x) &= \sqrt{\pi}Q, \quad Q : charge\end{aligned}$$

Formally:

$$U_<(2\pi\lambda) = \exp -2\pi i \int_{-\infty}^{x_1} j_0(x_0, y_1)dy_1 = \exp -2\sqrt{\pi}i\lambda\phi(x) \qquad (3.172)$$

is the representation for "halfspace" rotation i.e. $U_<$ implements:

$$\psi(x) \to \left\{ \begin{array}{ll} e^{-2\pi i\lambda}\psi(x) & x_1 < 0 \\ \psi(x) & x_1 > 0 \end{array} \right. \qquad (3.173)$$

Such halfspace transformations may be viewed as the point limit of Bogoliubov transformations. The correct normal product which is necessary in order to convert $U_<$ into a well defined expression is the triple ordering. This is also recursively defined but different from the ϕ-Wickproduct on subtracts all connected correlation functions and not just the two-point function. Formally we have the following simple exponential formula:

$$:e^{ia\phi(x)}: = \frac{e^{ia\phi(x)}}{\langle e^{ia\phi(x)}\rangle} \tag{3.174}$$

Reexpressed in terms of the original ψ-Wickproduct we obtain a nonlocal looking expression, which is best written in terms of rapidities:

$$\mu(x) = :\exp -2i\sqrt{\pi}\lambda\phi(x): =: \exp L_\lambda(x) : \tag{3.175}$$

$$L_\lambda(x) = \frac{\sin\pi\lambda}{2\pi}\int d\theta_p d\theta_q \left\{ \begin{array}{l} \frac{e^{-\lambda(\theta_p-\theta_q)}}{\cosh\frac{1}{2}(\theta_p-\theta_q)}\left[e^{i(p+q)x}a_p^*b_q^* + h.c.\right] \\ \left[-\frac{e^{-\lambda(\theta_p-\theta_q)}}{\sinh\frac{1}{2}(\theta_p-\theta_q+ic)}\right]e^{i(p-q)x-i\pi\lambda}a_p^*a_q \\ \left[\frac{e^{-\lambda(\theta_p-\theta_q)}}{\sinh\frac{1}{2}(\theta_p-\theta_q-ic)}\right]e^{-i(p-q)+i\pi\lambda}b_q^*b_p. \end{array} \right\} \tag{3.176}$$

Although L_λ is represented in terms of nonlocal kernels in rapidity space and is itself nonlocal, μ is a bosonic local field in the quantum sense which is however nonlocal relative to ψ i.e. outside the ψ-Borchersclass. It is easy to see that our special solution μ of the halfspace commutation relation with ψ belongs to a whole family of solutions. We may modify the μ by any bosonic local function of the $\psi^\#$ without change in the relative commutation relations. Within our construction method this is made manifest by the "quasi-periodicity" in λ mod 1With the help of μ one can now construct another field $\sigma(x)$ which carries the same charge as ψ, but has quite different spacelike commutation relations. Through the definition:

$$\begin{aligned} \sigma(x) &= N[\mu\psi](x) = \lim_{y\to x}\mu(x)\psi(y) \\ &= \frac{1}{\sqrt{4\pi}} : \int (e^{-ipx}a_p + e^{ipx}b_p^*)\mu(x)d\theta_p : \end{aligned} \tag{3.177}$$

one obtains the same soliton like relative commutation relations with μ as those between ψ and μ :

$$\mu(x)\sigma(y) = \left\{ \begin{array}{ll} e^{i\alpha}\sigma(y)\mu(x) & x > y \\ \sigma(y)\mu(x) & y > x \end{array} \right. \tag{3.178}$$

However the σ carries a fractional spin and "statistics"(see later comments). Instead of dual commutation relations one finds symmetric commutation relations associated with abelian representations of the braid group i.e.

$$\sigma(x)\sigma(y) = e^{-2\pi i\lambda}\sigma(y)\sigma(x) \tag{3.179}$$

These commutation relations appear as a interpolating continuous generalization of Fermions and Bosons and are called "anyonic". Their relation to particle statistics will be discussed later. The bosonic field as well as the anyonic field are living in the same Fockspace generated by the free field ψ, but they are not members of the ψ-Borchersclass. As a local field μ generates its own Borchersclass (it is an irreducible field in its own Hilbertspace cyclically generated from the vacuum). The question of whether the notion of equivalence classes of fields can be generalized to anyonic fields will not be persued here.

A physically more relevant illustration of duality and non-free Borchers classes is obtained by starting from a Majorana (selfconjugate) spinor field (a=b). In this case the symmetry is the discrete Z_2 and the previous method based on a conserved Noether current not applicable. There are however several alternative methods which lead to the following result:

$$\begin{aligned} \mu(x) &= :e^{\Lambda(x)}:, \quad \sigma(x) =: \hat{\psi}(x)\mu(x): \\ \Lambda(x) &= \frac{i}{4\pi}\int d\theta_p d\theta_q \left\{ \begin{array}{c} 2\coth\tfrac{1}{2}(\theta_p-\theta_q+i\varepsilon)\cdot e^{i(p-q)x}c_p^*c_q \\ +\tanh\tfrac{1}{2}(\theta_p-\theta_q)\cdot(e^{i(p+q)x}c_p^*c_q^* - h.a.) \end{array} \right\} \\ \hat{\psi}(x) &= \frac{1}{2\pi}\int d\theta_p (e^{-ipx}c_p + e^{ipx}c_p^*) \neq \psi(x) \end{aligned} \tag{3.180}$$

Whereas μ and σ fulfill the relative Z_2-duality relation, now both fields are bosonic. Hence σ generates a new Borchersclass in $\mathcal{H}_F$ which is inequivalent to the Fermion Borchersclass. It is quite straightforward to show that the Ising lattice model can be described in terms of lattice Fermions which in the scaling limit (for fixed correlation length) become Majorana Fermions. In addition the lattice (dis)order goes over into $(\mu)\sigma$ if one takes that limit from the disorder side ($T \to T_c + \varepsilon$). So we are justified to call our σ-fields the (order) "Ising fields". Let us compare the free Majorana field with the Ising field Borchers class. Consider the modular objects for the wedge algebras of the two classes. The modular operators Δ^{it} are identical and equal to the wedge-based Lorentz transformations. However the modular reflections J are different. For the free Fermion algebra the Wigner theory gave $J_F = K\mathcal{F}(j)$ with j being the antiunitary wedge reflection, $\mathcal{F}$ the CAR-functor and K the Klein transformation in Fockspace. The Boson algebra generated by σ on the other hand has $J_B = KJ_F$ because its commuting structure for space-like distances requires the absence of the twist. This can also be read off directly from the TCP transformation property of σ under the TCP in Fockspace. Note that the Klein factor is a global operator whose halfspace version is the the disorder field μ (the Jordan-Wigner transformation in lattice theory). In our Ising example ($\mathcal{N}_F$: fermion#):

$$K = \frac{1+iU}{1-iU}, \quad U = e^{i\pi \mathbf{N}_F} \tag{3.181}$$

Since the TCP symmetry θ of σ is related to the free field θ_0 TCP of the Majorana Fermion ψ in the same way as the above J's, namely by:

$$\theta = K\theta_0 \tag{3.182}$$

and since (as will be shown in the section on scattering theory) the unitary S-matrix is related to the antiunitary $\theta's$ by $S = \theta\theta_0$, we conclude $K = S$. this means that the S of σ is energy independent and $S^{(2)} = -1$ for two particles. On a somewhat formal level we can understand this through:

$$\lim_{t\to\infty} \mu(x) = \begin{cases} U \\ 1 \end{cases}, \quad \lim_{t\to\pm\infty} \sigma(x) = \begin{cases} U\psi \\ \psi \end{cases} \tag{3.183}$$

Writing $U\psi = K\psi K^*$, we read off: $S = K$ i.e. the Jordan-Wigner transformation approaches the global symmetry whose square root is the Klein transformation (which in this model coalesces with the S-matrix).

Returning for a moment to the λ-halfspace rotation in the previous complex free Dirac field, we find by the same method in case of rational $\lambda = \frac{1}{N}$ (Z_N symmetry):

$$\begin{aligned} U &= \sum_n e^{-2\pi i \frac{n^2}{N}} E_n \\ K &= \sum e^{-i\pi i n \frac{n^2}{N}} E_n \end{aligned}$$

The quadratic n-dependence of the spin-statistic phase on the charge eigenvalues $\sim n^2$ is characteristic for anyonic commutation relations. The "exotic"nature of the σ commutation relation shows up in the deviation of the σ modular reflection J Since this case is outside standard scattering theory, it is not so clear if K could be interpreted as a scattering operator. An alternative interpretation (which turns out to be the appropriate one) is to say that the fields are free in the sense of $S_{scat} = 1$, and that the deviation from the J_0 (or θ_0) of the free Dirac field is due to a change in statistics: $S_{stat} = K$. However statistics in d=1+1 massive QFT is a somewhat ambiguous notion. Consider the $a^{\#}(p)$ of a free Dirac field and write:

$$\begin{aligned} c(p) &= e^{i\pi\lambda \int_{-\infty}^{p} n(p)dp} a(p) \\ n(p) &= a^*(p)a(p) \end{aligned} \tag{3.184}$$

The commutation relation of the c's is now "anyonic":

$$\begin{aligned} c(p)c^*(q) &= -e^{i\pi\lambda} c^*(q)c(p) + 2\omega\delta(p-q) \\ c(p)c(q) &= -e^{i\pi\lambda} c(q)c(p), \quad etc. \end{aligned} \tag{3.185}$$

The c's live in the same Fockspace, are covariant with respect to the same representation of the Poicaré-group and (as it should be) create the same one particle states, even though the c's have "anyonic"(for $\lambda = 1$ bosonic) commutation relations. This is a special instance of a general phenomenon: particles in massive d=1+1 theories are statistical "schizons"i.e. the nature of their charges (fusion laws etc.) does not determine their statistics (i.e. a Mendeleev table in a d=1+1 world allows for a bosonic description in terms of long range interactions). This is very different from all other situations, including chiral conformal QFT for

which the field commutation relations (the "exchange algebra") is uniquely determined by the charges carried by the fields. Warning: the statistical schizon phenomenon should not be confused with "bosonization" in chiral conformal QFT (see next section). The schizon phenomenon is related to the fact that the natural framework for massive d=1+1 theories is the "soliton" framework in which the concept of braid-group statistics is replaced by the more general "exotic"(or solitonic) commutation relations which can be changed at will (within certain limitations) without effecting the superselection structure. The most interesting new phenomenon in the Algebraic QFT of solitons is that the problem of multiple vacuum states, even in situations where this cannot be blamed on spontaneous symmetry-breaking, becomes related in a profound way with the superselection structure. We will return to these problems in the sections on algebraic QFT (the net approach).

We conclude this section by some comments on the algebraic description (independent of fields) of (dis)order. If one assumes that the theory is given in terms of a field net $\mathcal{F}(\mathcal{O}), \mathcal{O} \in \mathcal{K}$ (family of double cones). As usual the observable algebra is related to the field algebra by the invariance principle with respect to a symmetry group G:

$$\mathcal{F}(\mathcal{O})^G \mid_{H_{vac}} = \mathcal{A}(\mathcal{O}) \tag{3.186}$$

Whereas for d$\neq 1+1$ the so defined $\mathcal{A}(\mathcal{O})$ generically (apart from spontaneous symmetry breaking presented in a later section) is Haag dual if F had this property (for fermionic F the duality must be twisted):

$$\mathcal{F}^{tw}(\mathcal{O})' = \mathcal{F}(\mathcal{O}') \curvearrowright \mathcal{A}'(\mathcal{O}) = \mathcal{A}(\mathcal{O}') \tag{3.187}$$

However in massive d=1+1 theories this conclusion is incorrect for double cones but remains correct for wedges $\mathcal{O} = W$. The alternative definition in terms of $\mathcal{A}(W)$:

$$\mathcal{A}^d(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W), \quad \mathcal{O} \in \mathcal{K} \tag{3.188}$$

gives a bigger algebra (equal to the dual net) which is Haag dual. It comes as a bit of a pleasant surprise that the issue of $\mathcal{A}$ versus $\mathcal{A}^d$ is inexorably linked with the (dis)order structure. In any massive two-dimensional QFT with an internal group symmetry G (i.e. not just for free fields) there is a canonical way to introduce halfspace transformations $U_l^{\mathcal{O}}(g)$ which implement the full g-transformation on the spacelike left of the double cone $\mathcal{O}$ and is equal to the identity on its right. This construction uses the "split-property" (equivalent to the "nuclearity" i.e.a good phase-space behaviour of QFT) and will be explained in the chapter on the algebraic net framework. Assume for the moment that G=Z_N i.e. an abelian group which leads to one halfspace generator $U_l^{\mathcal{O}}$, $(U_l^{\mathcal{O}})^N \sim 1$. We then extend the field algebra $\mathcal{F}(\mathcal{O})$ by the disorder operators $U_l^{\mathcal{O}}$:

$$\tilde{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee U_l^{\mathcal{O}} \tag{3.189}$$

The map $\mathcal{O} \to \mathcal{F}(\mathcal{O})$ is still an isotonous net but it lost locality. The application of the invariance principle yields:

$$\mathcal{F}(\mathcal{O})^{Z_n} = \mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}) \vee U_1^{\mathcal{O}} = \hat{\mathcal{A}}(\mathcal{O})$$

and we may arrage our result in form of the following "commuting square" of inclusions:

$$\begin{array}{ccc} \mathcal{A}^d(\mathcal{O}) & \subset & \mathcal{F}(\mathcal{O}) \\ \cup & & \cup \\ \mathcal{A}(\mathcal{O}) & \subset & \mathcal{F}(\mathcal{O}) \end{array} \tag{3.190}$$

Is there an invariance principle which describes the entire commuting square, in particular of $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O})$? In the above abelian Z_N- illustration the halfspace transformations U commute with $G = Z_N$ but they suffer a nontrivial action of the dual group $\hat{Z}_N$ ($\sim Z_N$):

$$\begin{aligned} \alpha_\chi(U(g)) &= \chi(g)U(g), \quad \mathcal{A}^d(\mathcal{O})^{Z_N} = \mathcal{A}(\mathcal{O}) \\ g &\in Z_N, \quad \chi \in \hat{Z}_N \end{aligned} \tag{3.191}$$

Whereas on $\mathcal{F}$ only G acts, $\mathcal{F}$ is a natural domain for the action of the "double" $G \times \hat{G}$:

$$\mathcal{A}(\mathcal{O}) = (\mathcal{F}(\mathcal{O}))^{G \times \hat{G}}, \quad \mathcal{F}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{\hat{G}} \tag{3.192}$$

It turns out that this has an interesting counterpart for nonabelian G's. In that case the "double" has to be taken in the in the sense of Drinfeld which is the cross product Hopf algebra which was introduced in the third section of the first chapter:

$$D(G) = C(G) \bowtie_{adj} G \tag{3.193}$$

However in contradistinction to ordinary group symmetry, the double is always spontaneously broken and maximally only G survives as an unbroken symmetry. This mathematical manifestation of the (dis)order structure is presently the only case for which Hopf algebras emerge naturally from physical principles.

The expert reader will have recognized the close relation with the global Kramers-Wannier Duality of statistical mechanics of lattice systems. In fact in the formal scaling limit near a critical point towards a continous QFT maintains the local (the Kadanoff-)form of the K-W symmetry[5]. The presentation of this section shows the close relation of the stat. mech.duality concepts to the Haag duality of algebraic QFT. But where the former (in its relation to charge sectors) is limited to d=1+1, Haag duality (and its controlled breaking) does not suffer such a limitation.

As already mentioned, all this rich structure (including the statistical "schizon" aspect of particles) may be subsumed into the algebraic QFT framework for soliton sectors which will be presented in a later section.

[5]The temperature becomes traded for the mass, but there is no "dual mass" which could substitute for the dual K-W temperature. Rather the dual symmetry becomes a pure algebraic concept in the sense of Kadanoff. It is always spontaneously broken, except in the conformal scale invariant zero mass limit.

Finally it is worthwhile to remark that all of the known d=1+1 (dis)order models allow for a euclidean functional (Feynman-Kac)representation involving an external Aharonov-Bohm potential or (in the case of additive symmetries i.e. translations in field space) external "electric and magnetic" sources. Some more details can be found in the section on functional methods.

3.8 Special Features of m=0, d=1+1 Fields

It is well-known that the zero mass limit of massive free fields enhances the space-time symmetry to the conformal group symmetry. In addition to the general well understood peculiarities of such an extension (Einstein causality "paradoxon" as the result of a continuous link through infinity of the space- and time-like), there is a very surprising phenomenon which only happens for d=1+1: there are continuously many local quantum theories in the holomorphy region between the real and imaginary time boundary values. Let us verify this for the massless Dirac field which results from the formula3.168:

$$\begin{aligned} \psi_r(v) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty dp(e^{-ipv}a_l(p) + e^{ipv}b_l^*(p)), \quad v = t - x \qquad (3.194) \\ \psi_l(u) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty dp(e^{-ipu}a_r(p) + e^{ipu}b_r^*(p)), \quad u = t + x \end{aligned}$$

where the right and left movers $a_r^\#(p), b_r^\#(p)$ and $a_l^\#(p), b_l^\#(p)$ anticommute with each other i.e. the chiral fields $\psi_{r,l}$ define independent chiral theories. Therefore from now on we will select one chirality and omit the r,l subscript.

As expected the two-point function can be rewritten in compact circular coordinates:

$$\begin{aligned} \left\langle \psi(u)\psi^*(u') \right\rangle &= \frac{1}{u - u' + i\varepsilon} \curvearrowright \langle \psi_c(z)\psi_c^*(z') \rangle = \frac{1}{z - z' + \varepsilon} \qquad (3.195) \\ \psi_c(z) &= z = e^{i\varphi} \end{aligned}$$

From this one reads off the spatial invariance group. It is the 3-parametric Moebius-group $SL(2R)$ in u or $SU(1,1)$ in z. The analytic continuation of this two-point function has a positive definite restriction onto any (Jordan) curve C circulating around zero i. e.

$$(f,g) = \oint_C \oint_C \frac{1}{z(\gamma(t)) - z'(\gamma(t'))} \bar{f}(t)g(t')dtdt' \qquad (3.196)$$

is a (positive) scalar product, leading to a Hilbertspace. Since the higher point correlation functions are products of the two-point functions, we obtain positivity. The algebra is still a CAR-algebra, but the quasifree state defined through C differs from the vacuum state. The unitary equivalence of the representations is easily checked with the Hilbert-Schmidt criterion of the first chapter. This means that the transformation of the circle (the "living space" of the original real time theory) to the parametrized curve C is unitarily implemented. This

emergence of the automorphism associated to the $Diff(S^1)$ group and their unitary implementation is the special property which fails in any other dimension. The infinitesimal manifestation is the well-known Lie-field structure of the energy-momentum tensor, in our case [6]:

$$\begin{aligned} T(z) &= :j^2(z):, \quad j(z) =: \psi^*(z)\psi(z): && (3.197)\\ [j(z_1), j(z_2)] &= i\delta'(z_{12}) && (3.198)\\ [T(z_1), T(z_2)] &= -i\frac{1}{12}(\delta'''(z_{12}) + \delta'(z_{12})) + i(T(z_1) + T(z_2))\delta'(z_{12}) \end{aligned}$$

The generic $T-$algebra is obtained from this special case by replacing the factor $\frac{1}{12}$ by $\frac{c}{12}$ with a positive c which turns out to take on any value above $c = 1$ and is quantized below this free value. The (nonlocal) Fourier components with repect to the rotation group lead to Kac-Moody and Virasoro algebras. The latter have commutation relations:

$$[T_n, T_m] = (n-m)T_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \qquad (3.199)$$

Since all irreducible L-representations are one dimensional, the Lorentz spin s (halfinteger) fields can all be represented in one bosonic (or fermionic) Fockspace generated by bosonic (or fermionic) $a^\#, b^\#$ which are independent of s. It is also easy to see that it is not possible to generalize this to arbitrary L-spin s i.e. to construct fields $\phi(u)$ with anyonic commutation relations:

$$\begin{aligned} \phi(u)\phi(u') &= e^{2\pi i\lambda}\phi(u'), \phi(u), \quad u > u' && (3.200)\\ \phi(u)\phi^*(u') &= e^{-2\pi i\lambda}\phi^*(u')\phi(u), \quad u > u' \end{aligned}$$

within the setting of Fourier transforms of creation/annihilation operators. Only anyonic momentum space operators as in 3.184 can be constructed in this way.

The rich world of general chiral QFT begins to open if one realizes the peculiar role of the scalar free field:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int (e^{-ikx}a(k) + h.a.)\frac{dk}{2\omega} = \varphi(t+x) + \varphi(t-x) \qquad (3.201)$$

Due to the infrared divergence in this representation, the pointlike $\varphi(x)$ does not exist, only $\varphi(f)$ with $\tilde{f}(0) = 0$ defines an operator in Fock space. In order to maintain well defined local generators in Fock space, we consider the infrared finite first derivative $j(u) = \partial_u\varphi(u)$, u=t+x. A simple calculation shows that j can also be obtained as the chiral current of a free fermion field 3.194.Its commutation relations define the abelian current algebra 3.198 We now use the Weyl functor 2.6 :

$$\begin{aligned} W(f) &= e^{i\int j(x)f(x)}, \quad W(f)W(g) = e^{i\frac{1}{2}\sigma(f,g)}W(f+g) && (3.202)\\ \sigma(f,g) &= \frac{1}{2}\int \{f(x)\partial g(x) - g(x)\partial f(x)\}\, dx && (3.203) \end{aligned}$$

[6] The triple dot denotes Wick ordering according to the frequency decomposition of j whereas the double dot refers to Fermion Wick ordering.

In order to make the Möbius-covariance of this algebra manifest, one uses the angular parametrization for the compactified line:

$$u \to z = \frac{i-u}{i+u} \qquad z = e^{i\varphi}$$

In terms of this compact description, the above symplectic form σ becomes:

$$\begin{aligned} \sigma(f,g) &= \int \frac{dz}{2\pi i} f'(z)g(z) = \sum n f_n g_{-n} \qquad (3.204) \\ f(z) &= \sum_n f_n z^n, \quad f_n^* = f_{-n} \end{aligned}$$

Thanks to the aforementioned infrared property which forced us to define the Weyl algebra in terms of currents instead of fields, the symplectic inner product3.204 is degenerate since it vanishes on constant functions ("zero modes"). These are carried into the center of the abstract C*-algebra which is generated freely from the $W's$, subject to the Weyl relation 3.203 The center defines an abelian charge algebra and there are continuously many superselected charge sectors obtained by diagonalization of the center.

In order to come to a more interesting situation one must extend the Weyl algebra by a lattice so that the extended algebra is not only characterized by the linear space of functions, but in addition has an underlying lattice. In mathematical terms the linear spaces are extended by "noncommutative tori". In order to allow for sufficient generality, we start from a multi-component abelian current algebra:

$$[J^i(z_1), J^j(z_2)] = -g^{ij}\delta'(z_1 - z_2), \quad i,j = 1...N \qquad (3.205)$$

where $\delta'(z_1 - z_2)$ is the appropiate circular δ–function:

$$\int \frac{dz'}{2\pi i} f(z')\delta'(z' - z) = -f'(z)$$

The symplectic form which now lives in LV i.e. smooth loops in the N-dimensional vector space V is given by:

$$\sigma(f,g) = \int \frac{dz}{2\pi i} \langle f'(z), g(z) \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in V given in terms of the positive definite metric g^{ij}. The Moebius-group acts on LV as:

$$(u(g)f)(z) \equiv f_g(z) := f(g^{-1}z), \quad g \in PSU(1,1)$$

and leaves σ invariant.

We are interested to classify all positive energy representations.

Theorem 8 *Every covariant positive energy representation (π, H_π) of the C* algebra $\mathcal{U}$ generated by the Weyl-operators $W(f)$ associated with σ is a direct sum of irreducible ground state representations i.e. $H = \sum_i H_i$, $H_i = \overline{U\Omega_i}$, Ω_i =ground state in H_i*

We may recover the current algebra fields 3.205if we restrict to regular representations i.e. those which are related to states ω fulfilling continuity of the function $\lambda \to \omega(W(\lambda f)$. This suggests the question whether the irreducible components can be created by applying (smeared out) covariant fields to the vacuum i.e. if the net point of view and the more standard field point of view are not only based on the same physical pictures but are even mathematical equivalent. Here we will only quote the result and leave the proof up to the last part on algebraic QFT.

Theorem 9 *Every regular ground state representation of the (abelian or non-abelian) current- or the energy momentum tensor- algebra is generated by charge-carrying localized fields. The currents and the energy momentum tensor commute with the charge-carrying fields for noncoalescing points whereas the latter can be chosen in such a way that they obey braid group commutation relations (special case: permutation group) among themselves.*

The representation theory of the above multicomponent Weyl-algebra is not very interesting since it pocesses a continuous set of representations labeled by additive (multicomponent) charges. They are generated by the following localized automorphisms γ_ρ :

$$\begin{aligned} \gamma_\rho(W(f)) &= e^{i\rho(f)}W(f) \\ \rho(f) &= \oint \frac{dz}{2\pi i}\rho(z)f(z) \end{aligned}$$

Here the N-component function ρ is local with support $\subset S^1$ so that γ_ρ acts as the identity if f and ρ have disjoint supports. The total charge $q = \rho(1)$ labels automorphism classes which are "inner equivalent" i.e. for which there are unitaries $u(\sigma,\rho) \in \mathcal{U}$ which intertwine between the two automorphisms:

$$\begin{aligned} \gamma_\sigma(W) &= u(\sigma,\rho)\gamma_\rho(W)u^*(\sigma,\rho) \quad W \in \mathcal{U} \\ u(\sigma,\rho) &= W(f_{\sigma,\rho}) \quad if'_{\sigma,\rho} = \sigma - \rho \end{aligned} \tag{3.206}$$

These equivalence classes of automorphisms are also referred to as abelian (superselection)sectors of $\mathcal{U}$. They exhaust the locally generated sectors of $\mathcal{U}$

The properties of the automorphism immediately translate into properties of the associated representations $\pi_\rho(W) := \pi_0 \cdot \gamma_\rho(W)$ where π_0 denotes the vacuum representation. So the charge distribution $\rho(z)$ "measures" the local deviation from the vacuum. The representation formalism is more close to the standard formulation of QFT. Vice versa any representation of $\mathcal{U}$ which deviates only locally from the vacuum (suitably defined) can be shown to allow a representation of the above form in terms of a local automorphism. This is part of a general theory, the so-called DHR (Doplicher, Haag, Roberts) theory, which we will meet in the section on algebraicQFT. These representations of the abelian current algebra was essentially known (though in a more conventional field theoretic language which is less precise) by Swieca and collaborators[?, ?].

The C^*-Weyl algebra $\mathcal{U}$ may be used as building blocks of structurally richer and more interesting C^*-algebras. The first step in this direction is the process of extensions by incorporating local sectors into the algebra. The naturalness of the so-called lattice-(or noncommutative torus-)extensions is best understood by looking first at subalgebras of $\pi_0(\mathcal{U})$ belonging to disconnected localization regions:

$$\begin{aligned} \mathcal{A}_L(I_1 \cup I_3) &= Alg\,\{\pi_0(W(f)) \mid f \in \mathcal{S}_L(I_1 \cup I_3)\} \\ \mathcal{S}_L(I_1 \cup I_3) &= \{f \in \mathcal{S} \mid f = const.\, in\ I_2, I_4,\ \ f(z_2) - f(z_4) \in 2\pi L\} \end{aligned}$$

Here Alg stands for the generated von Neumann algebra, $\mathcal{S}$ is the Schwartz space of smooth test functions on the circle, $z_{2,4}$ are two arbitrary points from $I_{2,4}$ and L is an even lattice in V. The commutant of this operator algebra acting on the vacuum Hilbert space is not, as one could expect by a naive application of Haag duality equal to $\mathcal{A}_L(I_2 \cup I_4)$, but it rather equals the bigger algebra:

$$\begin{aligned} \mathcal{A}_L(I_1 \cup I_3)' &= \mathcal{A}_{L^*}(I_2 \cup I_4) \\ L^* &= dual\ of\ L \end{aligned}$$

The reason for this state of affairs becomes clearer if, one looks at the physical interpretation of these algebras. The $I_{1,3}$ localized algebra contains, in addition to the naively expected operators which are separately neutral in I_1 and I_3(zero values of f in $I_{2,4}$), also operators which are only globally neutral but locally charged with I_3 containing the compensating (anti-)charge to that in I_1. The dual charge (being described by the dual lattice $L^* = V/L$) consists precisely of those values which lead to relative local commutativity:

$$\begin{aligned} W(f), W(g) &= e^{i\sigma(f,g)} W(g)W(f) \quad f \in \mathcal{S}(I_1 \cup I_3),\, g \in \mathcal{S}(I_2 \cup I_4) \\ \sigma(f,g) &= 2\pi\,\langle l \cdot .l^* \rangle = 2\pi \cdot integer \end{aligned}$$

The existence of these dual subalgebras of the vacuum representation of $\mathcal{U}$ suggests to look for extensions of $\mathcal{U}$ by lattices in V. For this purpose it is convenient to introduce homogeneous charge-transfer operators $\Gamma_\alpha, \alpha \in V$ in a subspace $H_L \in H_{uni}$ defined in the following. H_{uni} is the (nonseperable) universal representation space which is simply the direct sum of all charged q representations for all real values of q. $H_L \in H_{uni}$ contains only those charges lying on the $L-$lattice and Γ_α creates a chage $\alpha \in L$. as the charge q representation we simply take for H_q a copy of the vacuum representation Hilbert space H_0 but with $\mathcal{U}$ acting through $\pi_0(\gamma_\rho(W))$. Hence the universal representation is:

$$\begin{aligned} (\pi_{uni}(W)\phi)_\alpha &= \pi_0(\gamma_\rho(W))\phi_\alpha \quad \phi_\alpha \in H_0 \\ (\Gamma_\alpha \phi)_\beta &= \phi_{\beta-\alpha} \end{aligned}$$

and the restriction to L means that this formula is restricted to $H_L = \sum_{\alpha \in L} H_\alpha$ i.e. all charge indices $\alpha, \beta \in L$. In particular the vacuum considered as a vector in $H_{\beta-\alpha}$ is mapped into the vacuum but this time considered as a vector in H_β. We will denote the universal representation restricted to H_L as $\hat{\pi}$. In order

to speak about the ground state in each charge sector we need a hamiltonian. In conformal field theory there are two: the time translation and the rigid S^1 rotation generator. For the present discussion we only need the action of the rigid rotations:

$$(R(\tau)\phi)_\alpha := e^{\frac{i}{2}\langle\alpha,\alpha\rangle} e^{iH_0\tau}\phi_\alpha$$

Then ground states are mapped into ground states and Γ_α commutes with $R(\tau)$ (rotational homogenuity) and the ground state energy in the sector α is $\frac{1}{2}\langle\alpha,\alpha\rangle$.Γ_α implements a nonlocalized automorphism:

$$\begin{aligned}\Gamma^*\hat{\pi}(W)\Gamma &= \hat{\pi}(\gamma_\alpha(W)) \quad \gamma_\alpha(W(f)) = e^{i\langle\alpha,f_0\rangle}W(f)\\ f_0 &= \oint \frac{dz}{2\pi i}\frac{1}{z}f(z) \; i.e.\; \rho_\alpha(z) = \frac{1}{z}\alpha\end{aligned}$$

Localized charge carrying operators in the same charge class with prescribed support properties for $\rho_\alpha(z)$ may be obtained by modifying Γ_α with a Weyl operator:

$$\psi^\zeta_{\rho_\alpha} = \eta_\xi(\rho_\alpha)\hat{\pi}(W(\bar{\rho}_\alpha)\Gamma_\alpha \tag{3.207}$$

One easily checks that the necessary test function $\bar{\rho}_\alpha$ solves the first order differential equation:

$$\begin{aligned}\frac{d}{dz}\bar{\rho}_\alpha(z) &= i(\rho_\alpha(z) - \frac{\alpha}{z})\\ \bar{\rho}_\alpha(z) &= i\sum_{n\neq 0}(\rho_\alpha)_{-n}\frac{z^n}{n} - i\int\frac{dz}{2\pi i}\rho_\alpha(z)ln_\zeta(z)\end{aligned}$$

Here $\zeta \in S^1$ denotes the direction of the cut along the line $\{\lambda\zeta \mid \lambda \geq 0\}$ which is necessary in order to define the branches of the logarithm. Remember that $\zeta = -1$ corresponds to infinity in the original $\mathbf{R}-$parametrization. Here and in the followind statement the lattice restriction is not yet needed, everything holds within the universal representation for an arbitrary charge α. The numerical factor η will be determined later. One finds:

Theorem 10 *If $\zeta \notin supp\rho$ the charged operators $\psi^\zeta_{\rho_\alpha}$ 3.207 are local with respect to the observables U and fulfill the following relations:*

$$\begin{aligned}(i)\quad \psi^\zeta_{\rho_\alpha}\psi^\zeta_{\rho_\beta} &= e^{\pm i\pi\langle\alpha,\beta\rangle}\psi^\zeta_{\rho_\beta}\psi^\zeta_{\rho_\alpha}, \quad if\ supp\rho_\alpha \cap supp\rho_\beta = \emptyset\\ (ii)\quad \psi^{\zeta_1}_\rho(\psi^{\zeta_2}_\rho)^* &= e^{-\sigma i\pi\langle\alpha_\rho,\alpha_\rho\rangle}e^{2\pi i\langle Q,\alpha\rangle}, \quad if\ supp\rho \subset S^1\setminus\{\zeta_1,\zeta_2\}\\ (iii)\quad R(\tau)\psi^\zeta_\rho R^*(\tau) &= \psi^\zeta_{r(\tau)\rho}, \quad if\ \eta_\zeta = e^{\frac{i}{2}\frac{d\alpha}{2\pi i}\langle\alpha_\rho,\rho\rangle ln_\zeta(z)}\end{aligned}$$

The sign in (i) is coupled to the orientation of the path going from I_1 to I_2 through ζ. The σ in (*ii*) denotes $0, \pm 1$ according to whether the path which connects ζ_1 with ζ_2 and runs through $supp\rho$ in addition runs through -1 ($\sigma = 0$) or not ($\sigma = \pm 1$, depending on the $\pm$ orientation. The charge measuring operator Q is defined by:

$$(\langle Q,\beta\rangle\,\Phi)_\alpha := \langle\alpha,\beta\rangle\,\Phi_\alpha$$

The theorem is easily verified by explicite calculation.

Property (i) tells us that that bosonic local fields correspond precisely to even lattices:

$$\langle \alpha, \beta \rangle = 2n, \quad n = 0, \pm 1, \pm 2, ...$$

Restricting to such lattices L, the right hand side of (iii) applied to H_L is equal to one and therefore independent of ζ i.e. those fields live on S^1 (and not a covering thereof).

Now we change our standpoint by considering the von Neumann algebra generated by the extended operators the new (neutral) vacuum algebra $\mathcal{A}_L$ and the representation in H_L the vacuum representation. It turns out that the new net $\mathcal{A}_L$ has only finitely many positive energy representations. They are labelled by points on the dual lattice $L^* \supset L$ modulo L i.e. in L^*/L. Lattices with $L^* = L$ are called selfdual. They only have one sector (the vacuum sector) and they fulfill the "split" Haag duality:

$$\mathcal{A}(I_1 \cup I_2) = \mathcal{A}(I_2 \cup I_4)'$$

A famous illustration is the root lattice of E_8 as well as the Leech lattice Λ_{24}.

The charge sectors of $\mathcal{A}_L$ corresponding to the abelian group L^*/L can again be described in a manner similar to the previous formula:

$$\phi^{\zeta}_{\rho_\alpha} \; : = e^{i\pi(Q,\nu_\alpha)} \psi^{\zeta}_{\rho_\alpha} \, |_{H_{L^*}} \quad \nu_\alpha := \lambda_\alpha + \sum_\beta \langle \lambda_\beta, \mu_\alpha \rangle \, \lambda_\beta \qquad (3.208)$$

$$\mu_\alpha \; : = \sum_{i=1}^{N-1} \langle \alpha, \alpha_i \rangle \, \lambda_i \quad i = 2...N, \quad \mu_1 = 0$$

i.e. ν_a is a linear combination of the dual lattice basis vectors $\lambda_j : \langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ i,j=1...N. The first factor in ϕ is a Klein factor which plays a similar role as previously namely it adjusts certain commutation relations to standard form, in this case relative commutation with the observables $\mathcal{A}_L$ for disjoint localization. Again the unitary $\phi's$ implement localized automorphisms. It is easy to see that these sectors exhaust the possibilities of finite energy sectors. The restriction to L^*−charges results from the requirement that the action of L^* exhaust the possibility of leaving th set of L− charges invariant. The phenomenon of charge quantization by charge extension is a special case of the very general phenomenon of decrease in the number of superselection sectors with increasing size of algebraic extension. Note that the generators of $\mathcal{A}_L$ are in physical terms loops which close modulo $2\pi L$ (and hence lead to univalued phase factors) instead of the "Weyl loops" f in $W(f)$ for which f is strictly periodic. Mathematically they consist of "affine Hilbert spaces" i.e. multicomponent functions on the intervall $[0, 2\pi]$ which fulfill lattice boundary conditions i.e. a combination of two well studied objects: Weyl algebras over vector spaces with a (possibly degenerate) symplectic form and Weyl-like algebras over (not necessarily even) lattices. The noncommutative tory of the mathematicians as well as the external magnetic field problems of Hofstedter are illustration of the

latter. Whereas the von Neumann uniqueness applies to regular representations of over finite dimensional space with a nondegenerate σ, the tory algebras are never simple and therefore have several representations.

Having constructed all the charge sectors of the extended observable algebra $\mathcal{A}_L$ one may look for a field algebra $\mathcal{F}_{L^\bullet}$ generated by all the charge carrying fields 3.208. It is easy to establish the following theorem[?].

Theorem 11 $\mathcal{F}_{L^\bullet}(I) = \mathcal{F}_{L^\bullet}(I')^{tw}$

Here the twist tw is a generalization of the fermionic twisted commutant. As in that case one must " twist" the von Neumann commutant with a Klein transformation which also in this case is a "square root" of the unitary operator which represents the 2π rotation $e^{-i2\pi L_\bullet}$??.This deviation of quantum physics and geometry increases with increasing amount of non-commutativity (from Fermions to Anyons) and naturally also makes the modular theory of anyonic field algebras for the wedge regions (in chiral conformal theories just intervals) less geometric than that of the observable algebras.

Besides the above extensions there is one other mechanism namely that of factorizing the observable net $\mathcal{U}$ by a subgroup of its symmetry group. In the case of one current there is just the charge conjugation: $j \to -j$, whereas in the multicomponent case there are more possibilities. One finds new representations for the fixed point algebras. Some of these representations are not extendable to representations of the original $\mathcal{U}(S^1)$ but only to the noncompact $\mathcal{U}(R)$. These are called soliton representation because their charge distributions behave differently for $x \to \pm\infty$. If one prefers vague analogies with differential geometry to concrete expressions from physics, one may also call them "orbifolds".

The crucial remaining question is whether there exists a purely field theoretical systematics which also leads to the more interesting representations of algebras in which the charge sectors have branched fusion laws (as current algebras associated to nonabelian groups and W-algebras). The characteristic feature of those algebras is that have representations π with nontrivial statistical (or quantum) dimensions $d_\pi > 1$ and endomorphisms instead of automorphisms. Since both concepts are far removed from standard QFT (Lagrangians etc.), their explanation will be postponed to the last chapter. Here we will only scetch how by "amplification" and "reduction" one may get away from the lattice-extended Weyl algebras.

By amplification we mean tensor products and in particular our interst is to study nets formed by diagonal tensor products of extended observable algebras:

$$\Psi^{(k)}(\rho) = \psi(\rho) \times \psi(\rho) \times ....\psi(\rho), \quad \alpha_\rho \in L$$

where the tensor factors are of the form 3.207 which we now write as $\psi(\rho)$. If we would follow the logic of loop-groups, we would chose L = root lattice of e.g. SU(n) and $exp\rho \in$loop-group. Technically speaking one is dealing with a tensor product of k level one loop-group representations. It is well known that by reduction one obtains the higher level representations (with nontrivial

branching laws) of the loop-group. There are also arguments by which W-algebras are relate with current algebras through an invariance priniple.

On the other hand a classification of admissable statistics by methods of algebraic QFT (exchange algebras with braid-group commutation relations) leads to 4-point functions which exactly match the two families of current- and W-algebras[?] This strongly suggests that the two families exhaust the possible plektonic (nonabelian braid group) commutation relations with finitely many ("rational"). A direct proof that the amplification and reduction procedure leads to a family of irreducible nets among which the nets with a finite number of plektonic charge sectors (rational theories) are exhausted by nonabelian current algebras end W-algebras is still missing.

Chapter 4

Elementary Approach to Perturbative Interactions

4.1 Kinematical Decompositions

Before presenting an elementary approach to interactions and perturbation, it is helpful to have a closer look at those observable quantities which one wants to compute. Since among local "field coordinates" only currents have a distinguished physical meaning, one is naturally interested in matrix elements as:

$$\Gamma_\mu(p',p) = \langle p' | j_\mu(0) | p \rangle , \quad W_{\mu\nu}(p,x) = \left\langle p \left| j_\mu(\frac{x}{2}) j_\nu(\frac{-x}{2}) \right| p \right\rangle_{conn.} \tag{4.1}$$

The first quantity (where possible spin quantum numbers have been surpressed) is called the (electromagnetic) form factor of the p-particle and its static limit $(p-p')^2 \to 0$ can be probed by external magnetic fields and is related to the (anomalous) magnetic moment. The second (diogonal) matrix element of two currents (connected part means removal of the ill-defined vacuum contribution $\langle p' \mid p \rangle \langle 0 | j_\mu j_\nu | 0 \rangle$ before the limit p'→p, which however does not influence the structure of the covariant decomposition) gives rise to the notion of "structure function" of the p-particle and appears in the description of high-energy electron (more general: lepton) inclusive scattering on nucleons (scattering in which one does not observe the created outgoing hadrons).

Important energy shifts as the Lamb shift cannot be expressed in an elegant form in terms of such matrix elements (only if one defines "off-shell" extrapolations). The most important observables are the particle matrix elements of the S-matrix (or scattering operator) from which the cross sections can be obtained:

$$\langle p'_1, p'_2 .... p'_m | S-1 | p_1, p_2 .... p_n \rangle = \delta(\sum_{i=1}^{m} p'_i - \sum_{k=1}^{n} p_k) T(p'_1, ... p'_m, p_1, ... p_n) \tag{4.2}$$

where we used the fact that S is a Poincaré invariant operator in the Fockspace of

incoming particles (the energy-momentum conserving δ−function results from translation invariance).

Kinematical properties means the decomposition of covariant into invariant functions and the specification of the invariant variables on which the latter depend. For the formfactor of $s = \frac{1}{2}$ particles one finds the following decomposition (with k=p'-p) :

$$\Gamma_\mu(p',p) =$$

$$\frac{1}{(2\pi)^3}\bar{u}(p',s_3')\left(\gamma_\mu F(k^2) - \frac{i}{2m}(p'+p)G(k^2) + \frac{1}{2m}k_\mu H(k^2)\right)u(p,s_3) \quad (4.3)$$

The fastest way to see this is to first use the free field formalism to compute the matrix elements of the free current by "Wick-gymnastics":

$$\langle 0 \mid a(p',s_3') : \bar{\psi}(0)\gamma_\mu\psi(0) : a^*(p,s_3) \mid 0\rangle = \frac{1}{(2\pi)^3}\bar{u}(p',s_3')\gamma_\mu u(p,s_3)$$

Then one has to construct the most general vector object from the γ−matrices and two mass shell momenta p and p'.modulo the identity $\gamma_\mu p^\mu - m = 0$ which is valid on the intertwiner u(p). This leaves besides γ^μ itself, which appears already for a free current, only the above two momentum vectors (or linear combinations thereof). Current conservation $k_\mu\Gamma^\mu = 0$ gives $H \equiv 0$ (because H is a nonsingular function) and the value 1 of the total charge $Q = \int j_0(x)d^3x$ between the one-particle states requires $F(0) = 1$.

Due to kinematical identities of the u and v intertwiners, there are many different forms of covariant decompositions. For example the identity:

$$\bar{u}(p')i\sigma_{\mu\nu}q^\nu u(p) = \bar{u}(p')(2m\gamma_\mu + i(p'+p)_\mu)u(p)$$

may be used to eliminate the γ_μ term in favour of the (p'+p)$_\mu$ and the $\sigma_{\mu\nu}$ terms:

$$\bar{u}(p')\Gamma_\mu u(p) = \bar{u}(p')\{\frac{i}{2m}(p'+p)_\mu(F(q^2)+G(q^2)) + \sigma_{\mu\nu}q^\nu F(q^2)\}$$

In this form the leading contribution for small spatial momenta p and p' comes solely from the second term. The physical interpretation of F (which as G can only depend on k^2 since this is the only invariant which one can form two mass shell vectors) becomes clear if one rewrites the canonical coupling of the current to an external (classical) vector-potential as follows:

$$\left\langle p' \left| -e\int d^3x\vec{j}(x)\vec{A}(x)\right| p\right\rangle = -e\frac{1}{(2\pi)^3}\int e^{i(\vec{p}-\vec{p}')\vec{x}}\bar{u}(p',s_3')\vec{\Gamma}(p',p)u(p,s_3)\vec{A}(x)d^3x$$

$$\approx -eF(0)\frac{1}{(2\pi)^3}\frac{p_0}{m}\int d^3x e^{i(\vec{p}-\vec{p}')\vec{x}}(\vec{A}(x) - \left[(\vec{p}-\vec{p}')\times\vec{J}\right]) \quad (4.4)$$

Here the last line is the static approximation of $\bar{u}\vec{\Gamma}u$ in first order of $\vec{p}$-$\vec{p}'$ which brings in the angular momentum operator $\vec{J} = \frac{\vec{\sigma}}{2}$ (just γ−"gymnastics between

u-intertwiner). The last step is to use $\vec{B} = \vec{\nabla} \times \vec{A}$ and to take B constant (static limit):

$$\left\langle p' \left| -e \int d^3x \vec{j}(x)\vec{A}(x) \right| p \right\rangle \approx -\frac{e}{2m} F(0) 2p_0 \delta(\vec{p} - \vec{p}') \vec{B}\vec{\sigma}_{s'_3,s_3} \qquad (4.5)$$

i.e. we obtain the magnetic moment interaction $-\vec{\mu} \cdot \vec{B}$ with ($\mu = |\vec{\mu}|$)

$$\begin{aligned} \mu &= \tfrac{e}{2m} F(0) = \tfrac{e}{2m} + \mu_{anom.} \\ \mu_{anom.} &= \tfrac{e}{2m}(F(0) - 1) = \tfrac{e}{2m} G(0) \end{aligned} \qquad (4.6)$$

In a similar fashion one decomposes the structure function:

$$\begin{aligned} &\int W^{\mu\nu}(p,x) e^{iqx} d^4x \doteq : W^{\mu\nu}(p,q) = \\ &-(\tfrac{q_\mu q_\nu}{q^2} - g^{\mu\nu}) W_1(\nu, q^2) + \tfrac{1}{m^2}(p^\mu - \tfrac{p\cdot q}{q^2} q^\mu)(p^\nu - \tfrac{p\cdot q}{q^2} q^\nu) W_2(\nu, q^2) \end{aligned} \qquad (4.7)$$

The invariant structure functions W_i i=1,2 depend on two variables (q is not on shell) q^2 and $\nu = \frac{p\cdot q}{m}$ (m=target mass). Again the number of invariants has been reduced by using current conservation for the two currents. their experimental significance will be discussed in a later chapter.

The S-matrix is not measured directly, but rather through the ensuing scattering cross sections. The relevant formulas in most textbooks are derived by inventing a "box-quantization"(in order to solve the problem of "squaring the energy-momentum δ–function"). In order to avoid the impression that concepts which require $V = \infty$ for their formulation, as the previous temperature states for the thermodynamic equilibrium or the present problem of scattering theory (for which walls of a vessel would create a conceptual obstruction), cannot be delt with adeaquate mathematics we will in the following derive the cross section formulas by spacetime wave packet techniques. But first some physical remarks about the S-matrix.

Still missing: more comments about the S-matrix, a wave-packet derivation of cross sections and life times without quantization box.

Some remarks on the decomposition of the scattering amplitude T and some examples of practical relevance: Compton-scattering, Möller-scattering, pair-creation etc. are also still missing.

4.2 Elementary Notion of Interaction and Perturbation

In section of the first chapter we solved the simple problem of a perturbation by an external source on a free bosonic system and found that there are two ways, one via the method of unitary transformations (the so called "dressing" transformations) and the other by the use of the interaction picture in the form of a *time-ordered* exponential:

$$S(j) = Te^{i \int A(x)j(x)d^4x}, \quad A(x,j) = S^*(j) T A(x) e^{i \int A(y)j(y)d^4y} \qquad (4.8)$$

By Wick-ordering these expressions, we saw that they agree with the more rigorous dressing transformation method up to a phase (Feynman's famous "vacuum phase") and that phases show up in the form of cocycle factors in the composition law of the $S(j)$.

In nonexternal translation invariant problems, there is a famous obstruction against the existence of such a unitary dressing operator, the *Haag Theorem*. It says that in a translation invariant theory the ground state of an interacting system cannot be described in the space of vector states of the free system. The traditional way out is to start with a system which fullfills only "partial translational invariance" (similarly to the partial charges in the free field theory of the previous chapter). We start by defining (here A stands generically for the would-be Heisenberg field which corresponds to A_0) :

$$S(g) = Te^{i \int g(y)W(x)d^4x}, \quad A(x,g) = S^*(g)TA_0(x)e^{i \int g(y)W(y)d^4y} \tag{4.9}$$

Here W is an invariant Wick-ordered polynomial in terms of (not necessarily identical) free fields which implements the notion of interaction. For g we choose a smooth function with compact support in Minkowski space which can be thought of as a smooth version of the characteristic function of e.g. a double cone (with support in large double cone and $g(x) \equiv 1$ in a smaller cone placed inside the bigger). Before we show some remarkable properties of these formal operators in Fockspace, some comments are in order.

(i) Haag's theorem is not applicable to the $S(g)$ formalism (no translation invariance), and we are allowed to do our calculations in Fockspace. One of the remarkable properties is that the local observables localized within the smaller double cone fulfill *partial* translation invariance in the sense explained later.

(ii) The standard derivation of the above formula for $A(x)$ (more precisely for the vacuum expectation values of time ordered products of A) goes through the canonical formalism and is known under the name of *Gell-Mann-Low formula*. Such derivations suffer from two conceptual weaknesses. On the one hand they give (physically unmotivated) preference to special field coordinates (only "Eulerian" free fields among the class of (m,s) Wigner fields are canonical) and on the other hand they rely on assumption that the fields A to be constructed are not more singular for short distances than the corresponding canonical free fields. These assumptions are only valid in certain very special low-dimensional models.

(iii) The interaction density W(x) is a local function of free fields which (without the existence of a dressing transformation) has no direct (outside infinitesimal deformations) physical interpretation. This means that there is (apart from external perturbations and some very special low dimensional models) no general physical reason to believe that after a certain necessary repair ("renormalization"), one obtains a mathematically existing theory. Contrary to popular believes, it is not just the singular short distance behaviour as such which endangers the existence of the theory, but rather the standard framework of the time-orderd formalism (i.e. its use in local relativistic QFT) [1] which cre-

[1] The so called Bogoliubov axiomatics may have no solution in higher dimensions and

ates an obstruction against an intrinsic understanding of interactions. There is a very interesting lesson in this respect from the d=1+1 "bootstrap" constructions which show that short distance singularities can be worse than any given inverse power of the Minkowski distance without the existence of the theory being threatened.

Let us now show that on a formal level the Fockspace operator $A(x,g)$ fulfills some remarkable formal properties. Suppose that we restrict the x to the double cone K in which $g \equiv 1$ i.e. we consider A(f,g)

$$A(f,g) = \int d^4x f(x) S^*(g) T A_0(x) S(g) \tag{4.10}$$

with *supp.f* $\subset K$. Then as a generalization of the composition operator $S(j)$ in our old source model we find:

$$\begin{aligned} &S(g_2 + g_1) = S(g_2)S(g_1), \quad \text{supp} g_2 \geq \text{supp} g_1 \\ &A(f,g) = A(f,g'), \quad \text{if supp}(g - g') \subset V_-(K)^\perp \end{aligned} \tag{4.11}$$

where the notation means that the points in supp g_2 are either spacelike or timelike from those in supp g_1 and $V_-(K)^\perp$ is the complement of the smallest backward light cone which contains the double cone K. Furthermore any change of g to g' localized in $V_-(K)^\perp \setminus K$ can be implemented by a unitary ("partial dressing") transformation U(g) which is independent of f, i.e.the same for all operators in the algebra A(K):

$$A(f,g') = U(g',g) A(f,g) U^*(g',g) \tag{4.12}$$

Formally this unitary has the same form as S(h) where the smooth function h is compactly supported in the intersection of $V_-(K)^\perp \setminus K$ with a double cone K which contains the support of both g's . For the study of the net of double cone algebras localized in K inside K the common U is irrelevant since nets which are related by one common unitary are identical (isomorphic families define identical nets by definition) i.e. it is only the relative positions of these algebras and not the absolut position in the ambient space which counts. Hence even the limit K$\rightarrow \infty$ the net of algebras may be described within Fockspace. However this Fockspace is purely auxilary. Physical states strictly speaking are to be obtained as states on the net of operator algebras with suitable localization properties. This would be the scenario for the construction of interacting theories within the setting of time ordered exponentials of free field "interaction densities" W(x).

Before we look at the lowest nontrivial perturbative evaluation of these formal operators, let us briefly notice that A(x,g) fulfills Einstein causality wihin K:

$$[A(x,g), A(y,g)] = 0 \quad (x-y)^2 < 0 \quad \textit{and} \quad x, y \in K \tag{4.13}$$

The formal reason is that for spacelike separartions the product can be written in terms of one (cancellations between S's!) time-ordered free field expression:

$$\begin{aligned} S^*(g)T(A_0(x)A_0(y)S(g)) \quad &= \quad S^*(g)T(A_0(x)S(g)) \cdot S^*(g)T(A(y_0)S(g)) \\ \text{holds for } (x-y)^2 \quad &< \quad 0 \end{aligned} \tag{4.14}$$

therefore the time-ordering method may be not appropiate for introducing interactions.

(remember: the T only acts on all the A_0's to the right). The (Bogoliubov-Shirkov, Glaser-Epstein) renormalization approach allows to show that these formal relations are valid at least in every order of perturbation theory(expansion in W).

Another remark, whose importance can only be fully appreciated later, is the statement that the local algebras of a net are all unitarily equivalent and there is (outside of perturbation theory) no relation between the particle structure of the ambient Fockspace and the physical content of local algebras: the interaction (generically speaking) wrecks the one to one correspondence between particles and fields which existed in the free theory. For local observables desribed in terms of local nets of algebras, the Hilbertspace description allows great flexibility and the chosen massive Fockspace of the above formalism is not to be interpreted as a commitment about physical parameters. This picture is unfortunately somewhat blurred by perturbation theory which maintains an unrealistic rigid correspondence between fields and particles (apart from the mentioned flexibility of chosing the Fockspace mass parameter different from the physical mass). This (among other things) has created the misleading impression that QFT is nothing more than a relativistic made form of quantum theory. Although it is a *quantum theory* and it is *relativistic*, it is primarily a *new physical realm* whose deep and unexpected concepts (despite its 70 years of existence) still await exploration. This will become much more evident in the later chapter on modular localization and the bootstrap-formfactor approach than on the leval of perturbation theory.

4.3 Second Order Perturbation and the Adiabatic Parametrization

The naive expectation (i.e. by analogy to the external source problem in chapter1) about $S(g)$ would be that the limit of the theory for $K \to R^4$ exists and describes the physical S-matrix. Even in perturbative evaluation this picture needs two corrections. One is related to the infrared divergence problem in certain theories involving zero mass as QED, a somewhat special phenomenon whose physical basis will be reserved for a later discussion. The other is of a completely general nature related to the phenomenon of selfinteraction, well-known already from classical field theory where it leads to the famous problems of constructing consistent particle models within a classical field theory, as studied by Poincaré and Lorentz . As a result of selfinteraction, parameters with a physical name as mass, charge etc. which entered the construction of $S(g)$ and $A(x,g)$ do not represent the true measured value. Whereas for fields A and their correlation functions this does not matter (the true physical values can be recovered from asymptotic properties of correlation functions, see later), the large volume limit of S(g) for $|K| \to \infty$ represents the physical S-matrix for the scattering of A-particles only if the true physical mass is used. The same applies to any quantity which is partially "on shell" i.e. contains particles

states as e.g. the electromagnetic formfactor. The reason is that the adiabatic switching on and off by multiplying W(x) with $e^{-\alpha|t|}$ and then lim t$\rightarrow \infty$ is physically harmless only if W includes the effect of persistent selfinteraction "counter-terms" which maintain the mass used in the Fockspace in every order of W at their physical value. In case of a neutral scalar $W = g : A_0(x)^4 :$ model the modification is:

$$\begin{aligned} W_{adiab}(x) &= W(x) + \frac{1}{2}\delta m^2 Z : A_0(x)A_0(x) : \\ +\frac{1}{2}(Z-1)(&: \partial_\mu A_0(x)\partial^\mu A_0(x) : -m^2 : A_0(x)A_0(x) :) \end{aligned} \tag{4.15}$$

The "selfmass" δm^2 is chosen in every order to maintain m as the physical mass and $m_0^2 = m^2 - \delta m^2$ is an auxilary unphysical mass (which loosely speaking corresponds to the mass without the stablelyzing counterterm which changes in every order of W). The second Z-counterterm has been added in order to obtain a nicer form of the adiabatic principle which is the following requirement:

$$\lim_{g\rightarrow 1} \langle 0|A(x,g)|p\rangle = \langle 0|A_0(x)|p\rangle \tag{4.16}$$

By adjusting δm^2 and Z in every order such that this identity holds we took all selfinteractions into account. A subsequent adiabatic change of W_{adiab} i.e.

$$W_{adiab} \rightarrow e^{-\alpha|t|}W_{adiab}, \quad \alpha \rightarrow 0 \text{ at end of calculation} \tag{4.17}$$

will not cause any harm i.e. does not change the one particle characteristics. In theories without selfinteraction e.g. in Schrödinger theory, this is automatically fulfilled. Using our formal time ordered expressions we may rewrite the above requirement in second order:

$$\frac{(\delta m^2 Z)^{(2)} + (Z-1)^{(2)}(p^2 - m^2)}{p^2 - m^2 + i\varepsilon} = \frac{1}{2}\int\int \langle 0|TA_0(0)W(x_1)W(x_2)|p\rangle \tag{4.18}$$

since the zero order terms agree and the second order term of the above requirement:

$$\lim_{g\rightarrow 1} \langle 0|A(0,g)|p\rangle^{(2)} = 0 \tag{4.19}$$

consists of a WW contribution and the lowest counterterm contribution (which we wrote on the left hand side). The evaluation of the right hand side (omitting combinatorical factors) gives:

$$\tilde{}g^2 \int\int \Delta_F(0-x_1)\Delta_F^3(x_1-x_2)\frac{e^{ipx_2}}{(2\pi)^{\frac{3}{2}}}\tilde{}\frac{1}{p^2-m^2+i\varepsilon}\int e^{ip\xi}\Delta_F^3(\xi)d^4\xi \tag{4.20}$$

Therefore $\delta m^{2(2)}\tilde{}g^2 \int e^{ip\xi}\Delta_F^3(\xi)d^4\xi$ and $Z^{(2)}$ is the second Taylor coefficient in the expansion of the integral around $p^2 = m^2$(note that $Z = 1 + Z^{(2)} + ...$).

Later we will (via time dependent scattering theory) meet a formalism which relates off- shell quantities (correlation functions of interacting fields) in a natural way with distinguished free fields which have the correct physical mass (the

"incoming" /outgoing fields). This relation is independent of the mass of the Fock space which we use for their perturbative construction. Scattering theory may be viewed as an extension of the adiabatic principle to multiparticle states. It is applied to off-shell correlation functions and requires the introduction of the physical mass i.e. a reparametrization which relates the mass of the auxilary Fockspace to that of the in and out Fockspace of scattering theory. Since (Haag-Ruelle or LSZ-) scattering theory (and a fortiori) the above adiabatic principle are consequences of the general framework of QFT, a more systematic and conceptually clearer approach would have presented the perturbative treatment of interaction *after* an introduction to the general framework. However a physical minded reader prefers to see some important results and understand them at least partially with a modest amount of mathematics and concepts *before* making a large formal investment.

Even though there are *no physical reasons* to introduce counterterms *for off-shell* quantities, the fact that the time-ordered products of $W's$ via Wick's theorem yield ill defined (formally infinite) expressions as e.g. $i\Delta_F^3(x_1 - x_2)$ forces "renormalization" for *mathematical reasons*. With other words our starting formula in Fockspace was incorrect, but not byond redemption. The integrand in S(g) i.e.$TW(x_1)....W(x_n)$, although not defined on all (Schwartz) testfunctions, is well defined on the big class of testfunctions $f(x_1,....x_n)$ which vanish of sufficient high order for coalescent points. If W is a polynomial with $dimW \leq 4$, the order does not increase in n. This means that the (Hahn-Banach) extension to all testfunctions will lead to time-ordered distributions which, although lacking uniqueness, have well-controlled ambiguities whose space-time dependence is given in terms of δ–functions and derivatives up to a maximal finite order. With other words, different extensions differ by finite local counterterms. These counterterms may be used in order to achieve certain normalization conditions (as in the case of the adiabatic principle), but there is no mathematical necessity to take the ambient Fockspace with a mass equal to the physical mass. The mass of a particle does not belong to the set of observables which can be measured locally. Globally e.g. two free fields with different masses cannot be unitarily equivalent. In the following we will use the more pedestrian regularization methods rather than the extension method (which will only be used in the later Curved Space Time problems where it is the only renormalization method). Whereas for structural arguments we mostly use the A^4model, the explicite second order calculations will be done in Quantum Elecrodynamics. We now specialize to the W describing QED (first without the counterterms):

$$W(x) = -ej_{0\mu}(x)A^{\mu}(x), \quad j_{0\mu}(x) =: \bar{\psi}_0\gamma_\mu\psi_0(x) : \quad \psi_0, A_{0\mu} \text{ free fields} \quad (4.21)$$

and consider S(g) in second order:

$$S^{(2)}(g) = \frac{e^2}{2!}\int\int g(x)g(y)Tj_{0\mu}(x)j_{0\nu}(y)A_0^{\mu}(x)A_0^{\nu}(y)d^4xd^4y \qquad (4.22)$$

with the Wick-reordering from the previous chapter we obtain for the formfactor:

$$\langle p' \,|j_\mu(0)|\, p\rangle^{(2)} \;=\; \langle 0\,|a(p',s_3')S^*T(j_{0\mu}(0)S)a^*(p,s_3)|\,0\rangle^{(2)} \qquad (4.23)$$

$$= \langle 0\,|a(p',s_3')T(j_{0\mu}(0)S)a^*(p,s_3)|\,0\rangle^{(2)}_{v.c.} \qquad (4.24)$$

"vacuum-connected"(v.c.) has the same meaning as before: leave out the S^* in front of the T and ignore the vacuum "bubble"contributions. The evaluation of the right hand side amounts to look for the $\bar{\psi}-\psi$ contribution in the Wick reordering of:

$$Tj_{0\mu}(0)j_{0\nu}(x_1)j_{0\kappa}(x_2)A_0^{\nu}(x_1)A_0^{\kappa}(x_2) \qquad (4.25)$$

In order to keep track of the combinatorical possibilities, it is customory to draw graphs with vertices and connecting lines. In our case their are three interaction points 0,$x_1$and x_2 , one connecting photon line (one photon contraction) and two e-lines so that one uncontracted $\psi_0, \bar{\psi}_0$ remain. One easily sees that there are three combinatorical distinct contributions according to whether the remaining pair may come from $j_{0\mu}(0)$, from the j(x)'s or if it is of mixed origin i.e. one from j(0) and the other from a j(x). The first case only contributes in zero order since :

$$\langle p'\,|j_{0\mu}(0)|\,p\rangle\,\langle 0\,|S^*S|\,0\rangle^{(2)} = 0 \qquad (4.26)$$

This cancellation is a general feature of all "vacuum bubble" contributions of S (which only give a phase factor in S and the opposite in S^*. More interesting are the terms in which both of the $\psi-\bar{\psi}$ "legs" are contracted with legs in $S^{(2)}$. This constitutes the famous "vacuum polarization" contribution $\Gamma_{\mu.pol}$ to the form factor and the so called "one particle irreducible" form factor $\Gamma_{\mu,loop}$. The vacuum polarization contribution contains the "fluctuation" of the zero order current:

$$i\Pi_{\mu\nu}(x) = e^2\,\langle 0\,|Tj_{0\mu}(x)j_{0\mu}(0)|\,0\rangle \qquad (4.27)$$

The $\Gamma_{\mu,loop}$-contribution originates from a contraction in which one leg goes to one W-vertex and the other to the second. The remaining "electron selfenergy contribution" arises from the mixed contraction. It contain the electron selfenergy Σ and therefore is called $\Gamma_{\mu,e.s.}$. The three types of terms are conveniently pictured in terms of Feynman diagrams.(Fig.)

Inserting now the Fourier representation of the time ordered electron and photon propagators we obtain:

$$\langle p'\,|j_{\mu}(0)|\,p\rangle = \bar{u}(p',s_3')(\Gamma_{\mu,pol}+\Gamma_{\mu,loop}+\Gamma_{\mu,e.s.})u(p,s_3) \qquad (4.28)$$

where the vacuum polarization-, vertex-loop- and electron-selfenergy-contributions to the matrix-valued Γ_{μ} is

$$\Gamma_{\mu,pol}(p',p) = \frac{-1}{(p'-p)^2+i\varepsilon}\Pi_{\mu\nu}(k)\gamma^{\nu} \qquad (4.29)$$

$$\text{with } \Pi_{\mu\nu}(k) = \frac{-ie^2}{(2\pi)^4}\int d^4q \frac{Tr([-i\,\not{p}+m]\,\gamma_{\mu}\,[-i(\not{q}-\not{k})+m]\,\gamma^{\nu})}{(q^2-m^2+i\varepsilon)((q-k)^2-m^2+i\varepsilon)} \qquad (4.30)$$

$$\Gamma_{\mu,loop}(p',p) = \frac{ie^2}{(2\pi)^4}\int d^4q\gamma^{\rho}\frac{-i(\not{p}'-\not{q})+m}{(p'-q)^2-m^2+i\varepsilon}\gamma_{\mu}\frac{-i(\not{p}-\not{q})+m}{(p-q)^2-m^2+i\varepsilon}\gamma_{\rho} \qquad (4.31)$$

$$\Gamma_{\mu,e.s.} = \{i\Sigma(p')S_F(p')\gamma_\mu + \gamma_\mu S_F(p)i\Sigma(p)\} \tag{4.32}$$

$$\text{with } i\Sigma(p) = \frac{e^2}{(2\pi)^4}\int d^4q \frac{1}{q^2+i\varepsilon}\frac{\gamma^\kappa(i\not{p}-i\not{k}+m)\gamma_\kappa}{(p-q)^2-m^2+i\varepsilon} \tag{4.33}$$

The electron selfenergy $i\Sigma$ is not an observable quantity and fortunately its contribution drops out by the adiabatic principle. This is because the two momentum variables p' and p are on the physical (mass m) mass shell and therefore the adiabatic principle forces us to work with W_{adia} instead of W and fix the counterterm in such a way that the one particle matrix element of ψ equals that of ψ_0. This is easily seen to be identical (in second order)to: $i\Sigma(p)_{adia}u(p,s_3)\,|_{\not p=m} = 0$ with $i\Sigma_{adia} = i\Sigma +$ counter terms. The on shell vanishing of the selfenergy is just the mathematical expression that the persistent selfenergy contribution to the large time asmptotics (equivalent to the momentum space mass shell limit) has been correctly taken into account and not "switched off". On the other hand as already mentioned if we were to compute the off shell 3-point function $\langle T\psi\bar{\psi}A_\mu\rangle$ we have the option to either use free fields with the "bare" mass m_0 and W or we can use free fields with the physical mass and W_{adia}. This observation is well known from the Gell-Mann Low representation or the Feynman-Kac representation of correlation functions (naturally off shell). For a more systematic treatment we refer to an appendix.

In passing we mention that the Gell-Mann Low representation for the correlation function of Heisenberg fields (for a scalar neutral selfinteracting field) has the form:

$$\langle TA(x_1)....A(x_n)\rangle = Z^{-1}\left\langle TA_0(x_1)....A_0(x_n)e^{i\int W(A_0(x))d^4x}\right\rangle_{Fock} \tag{4.34}$$

$$\text{with } Z = \left\langle e^{i\int W(A_0(x))d^4x}\right\rangle$$

we note that this is a special case of the previous formalism if we extend it to products of fields and consider formally the adiabatic limit $g\to 1$. The denominator Z is a phase factor and represents the (volum-dependent) "vacuum-bubbles" which cancel against similar contributions from the numerator. The volum structure is completely analogous to the thermodynamical limit of the Gibbs representation of thermal correlation functions (at the end of chapter 1), with the only structural difference that there is no equivalent to the KMS condition. For physical reasons the correctly normalized ground stae expectations should result from the KMD theory for $\beta \to \infty$.

Looking formally at the momentum space representation for the second order vacuum representation one would expect (by power counting in p) a quadratic divergence. Invoking current conservation (or gauge invariance) only a logarithmic divergence remains in $\pi_{\mu\nu}$. A closer look at the electron selfenergy term $i\Sigma$ reveals that the divergence is also logarithmic and the same is obviously true (by power counting) for $\Gamma_{\mu,loop}$. In the remaider of this section we present and explain the result of the renormalization on the second order formfactor. The presentation of the techniques and the actual calculation will be deferred to the

next section. We collect the results (omitting tildes in Fouriertransforms):

$$\begin{aligned} \Pi_{\mu\nu}(k) &= i(k_\mu k_\nu - g_{\mu\nu}k^2)\pi(k) \quad \text{with:} \\ \pi^{(2)}(k) &= -\frac{\alpha}{3\pi}\left\{+\frac{1}{3} + 2(1+\frac{2m^2}{k^2})[x\,arccotx - 1]\right\} - Z_3^{(2)} \\ \text{where } x &= (\frac{4m^2}{k^2} - 1)^{\frac{1}{2}} \text{ for } k^2 < 4m^2 \text{ and anal.cont.,} \\ Z_3^{(2)} &= \frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2} \end{aligned} \tag{4.35, 4.36}$$

$$\Sigma^{(2)}(p) = \frac{\alpha}{2\pi}\left\{\begin{array}{l}(\not p - m)(1 + \ln\frac{\mu^2}{m^2}) + 2m\frac{m^2-p^2}{p^2}\ln(1-\frac{p^2}{m^2}) \\ \quad - \not p\left[\frac{3}{2}\frac{m^4-(p^2)^2}{(p^2)^2}\ln(1-\frac{p^2}{m^2}) + \frac{m^2-p^2}{p^2}\right]\end{array}\right\} + \delta m + Z_2^{(2)}(\not p - m) \tag{4.37}$$

$$\text{with } \delta m = \frac{3\alpha}{4\pi}m(\ln\frac{\Lambda^2}{m^2} + \frac{1}{2}),\ Z_2^{(2)} = \frac{\alpha}{2\pi}(\frac{1}{2}\ln\frac{\Lambda^2}{m^2} + \ln\frac{\mu^2}{m^2} + \frac{9}{4} + O(\frac{\mu}{m})) \tag{4.38}$$

$$\begin{aligned} \Gamma^{(2)}_{\mu,loop}(q^2) &= \gamma_\mu F^{(2)}_{loop}(\theta) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu G^{(2)}_{loop}(\theta) + \gamma_\mu B \\ F^{(2)}_{loop}(\theta) &= \frac{\alpha}{\pi}\left\{(\ln\frac{\mu}{m} + 1)(\theta\coth\theta - 1) - 2\coth\theta\int_0^{\frac{\theta}{2}}\chi\tanh\chi d\chi - \frac{\theta}{4}\tanh\frac{\theta}{2}\right\} \\ G^{(2)}_{loop}(\theta) &= \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta} \quad \text{with } q^2 = -4m^2\sinh^2\frac{\theta}{2} \quad \theta : \text{"rapidity"} \end{aligned} \tag{4.39}$$

Here Λ is a cutoff i.e. a formal device which cuts off certain divergent momentum space integrals in a Lorentz-invariant manner. Although Λ carries no direct physical significance (and will be removed shortly), it is important that the Λ-dependent terms have at most a polynomial p-dependence i.e. they are of the form of δ-functions and derivatives thereof. The infrared cutoff μ on the other hand has a physical origin. The interaction of charged particles with "soft" ($\omega \to 0$) photons is very strong and changes the character of the one particle states. Its projection on Wigner particles is zero and strictly speaking we must abandon our formulation of the adiabatic principle (and the standard forms of scattering theory) and think about "infraparticles". In order to not drift too far away from elementary treatments we followed standard practice and introduced a "photon-mass" μ into the A_μ propagator retaining at the end only the leading contribution for small μ. Note that the Λ-dependent (unrenormalized) Σ is infrared-finite (B contains a compensating contribution). In Γ there is no such compensation. We have separeted the Λ cutoff dependent δm^2 and B terms in Σ because the adiabatic principle fixes the counter terms in W_{adiab} to be :

$$W_{c.t.} = \delta m^{(2)}\bar\psi\psi - B\bar\psi\ \not\beta\psi \tag{4.40}$$

This leads to a modification (renormalization) of Σ. We already noted that the resulting Σ_{adia} is the Λ-independent content of the curly bracket. Insertion into

the form formula (..) gives $\bar{u}(p')\Gamma^r_{\mu,e.s}u(p) = 0$ i.e. the renormalized contribution vanishes on the mass shell $p = m^2$. Enforcing the charge normalization for the diagonal matrix element:

$$\bar{u}(p,s_3)\Gamma^r_\mu u(p,s_3) = \bar{u}(p,s_3)\gamma_\mu u(p,s_3) \tag{4.41}$$

we also eliminate the $\Lambda-$dependence in $\Gamma_{\mu,pol}$ and $\Gamma_{\mu,loop}$. The result is:

$$\Gamma^r_\mu = \gamma_\mu F^{(2)}(\theta) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu G^{(2)}(\theta) \tag{4.42}$$

with F and G given by the previous formulae in the regime $q^2 < 0$ and everywhere by analytic continuation (as $\pi(q^2)$ and $\Sigma(q)$ they can be represented by analytic functions with a cut on the real axis. According to the previous section we obtain for the anomalous contribution to the magnetic moment:

$$\mu_{an} = \frac{e}{2m}(F(0) + G(0)) = \frac{e}{2m}\frac{\alpha}{2\pi} \tag{4.43}$$

Note that only the infrared-finite G contributes to the zero Taylor term.

The calculation of the Lamb-shift is more complicated computationally (since atomic physics enters) as well as conceptionally. Here one is interested in the $2s_{\frac{1}{2}} - 2p_{\frac{1}{2}}$ energy split which the Dirac theory with external fields cannot explain. The idea is to study the electron selfenergy term Σ in the presence of an external field. Wheras it is not difficult to represent the resulting energy shift in a stationary A^{ex}_μ as:

$$\delta E_N = -\int\int d^3p'd^3p\bar{u}_N(p')\Sigma^r(p',p,E_N;A^{ext}_\mu)u_N(p) \tag{4.44}$$

the evaluation of the low energy part of the renormalized external field dependent selfenergy part Σ^r requires significant skills and intuition. Since the atomic wave functions u_N prevent the electro to become free, one expects δE_N to be a "good" variable i.e. $\mu-$independent. In distinction to the previous case this independence is not manifest and can be used as a check on the approximation methods needed. The only managable approximation method involves a different treatment of high and low energy contributions(see Weinberg). For the low energy part one uses the above formula. It turns out that the main modification consists in replacing the $S_F(x-y)$ function by $S_F(x,y,A^{ext}_\mu)$ as well as a so-called "tadpole" term involving:

$$\langle 0\,|j_{0\mu}(x,A^{ext}_\mu)|\,0\rangle = Tr\gamma_\mu S_F(x,x,A^{ext}_\mu)$$

The latter occurs because the charge conjugation invariance, which prohibits any vacuum expectation with an odd number of j's, is broken by the external field. This new term requires a tadpole counter term. The $S_F(x,y,A^{ext}_\mu)$ only involves Dirac theory. The high energy part is calculated in first order of A^{ext}_μ and has the form:

$$\delta E_N\,|_{h.e.} = ie\int\int d^3pd^3p'\bar{u}(\mathbf{p}')\Gamma_\mu(\mathbf{p},E_N;\mathbf{p}'E_N)u(\mathbf{p})A^{ext}_\mu(\mathbf{p}-\mathbf{p}')$$

The Γ_μ is almost the previous form factor Γ except for the fact that the hydrogen wave functions push it slightly off shell. One replaces this by the on shell Γ. The infrared dependence of the latter causes a problem. One solution (Itzykson-Zuber) consists in converting the unphysical photon mass μ into an infrared photon energy cutoff K via use of the soft bremsstrahlung. K is than used to define an upper integrarion limit in $\delta E\mid_{l.e.}$ whereas $\delta E\mid_{h.e.}$ is calculated from the previous formula with μ in the formfactor Γ_μ being replaced in favour of K. The result for the s-p splitting in hydrogen is:

$$\delta E_{2s} - \delta E_{2p_{\frac{1}{2}}} = \frac{\alpha^5 m}{6\pi}\left\{\ln\frac{\Delta E_{2p}}{\alpha^2 \Delta E_{2s}} + \frac{19}{30} + \frac{1}{8}\right\} \tag{4.45}$$

Here the $\Delta E's$ are suitably avaraged energies of the hydrogen atom (only numerically accessable). This result corresponds to the famous value 1052,19 MHZ. instead of a photon energy cutoff one may also base the division on the decomposition for the photon propagator:

$$\frac{1}{k^2 + i\varepsilon} = \frac{1}{k^2 - \mu^2 + i\varepsilon} + \left(\frac{1}{k^2 + i\varepsilon} - \frac{1}{k^2 - \mu^2 + i\varepsilon}\right)$$

The first part leads to the Γ contribution whereas the second faster decreasing part enters the atomic physics calculation. Again the infrared singular terms cancel. This somewhat more attractive calculation (invariant cutoff) can be found in the first volum of Weinberg's book. Other physical problems related to the formfactor are the radiative corrections to the Coulomb scattering i.e. the second order correction to the Mott-formula and the bremsstrahlung correction to the Mott formula. Both are seperatly infrared divergent for $\mu \to 0$, but their joint cross section (for fixed photon infrared resolution) approaches a finite limit. This is a special case of good infrared behaviour of photon inclusive cross sections which are the principle observables of QFT's involving photons.

In this section we met two slightly different reasons for renormalization. One is entirely physical: if we describe matrix elements between particle states (i.e.on shell quantities) we must use the W_{adiab} as our interaction, independent of whether the counter terms have infinite coefficients or not. There is no other operator description for such quantities then the one in a Fockspase with the correct mass. In the next section we will study off shell quantities which do not require W_{adiab}. Any auxilary mass Fockspace may be used for their perturbative evaluation. It will be shown later that scattering theory reconciles the description of on and off shell quantities. The Hilbertspace for scattering theory requires a reparametrization from the off shell auxilary mass to the physical mass as well as a multiplicative adjustment. In momentum space the difference between these two type of quantities looks deceivingly simple: one just sends certain p-variables to the physical mass shell. In x-space the distinction looks more dramatic: it is the difference between global (particles involve asymptotic limits) and local quantities. In algebraic QFT it corresponds to the local equivalence of algebras which belong globally to inequivalent (different charges) representations(see later sections).

We end this section with some formal remarks on how to use the above time-ordered formalism to obtain perturbative correlation functions. As a prototype theory which is free of infrared problems, tensor-and spinor-indices etc., we take the model with $W(x)=:A_0(x)^4:$.Previously we have seen that the Gell-Mann Low representation for time ordered n-point functions has the following form:

$$\langle TA(x_1)....A(x_n)\rangle = \lim_{g(x)\to g} \left\langle TA_0(x_1)....A_0(x_n)e^{i\int g(x)W(x)d^4x}\right\rangle_0^{v.r.} \qquad (4.46)$$

The subscript 0 on the right hand side is a reminder that the free field expressions are to be evaluated in the A_0 Fockspace and the superscript v.c. indicates that "vacuum.bubbles" in the Wick-ordering must be omitted. We also mentioned the extension method of distributions which succeeds to give an iterative definition of the expanded right hand side. More popular with physicists (but not necessarily more physical) are the various regularization methods which we will dicuss in the next section. Let us consider the purely formal aspects of the A^4 model. This time we introduce counterterms W_c solely for the elimination of the divergencies which arise from the removal of the unphysical regularizations:

$$\begin{aligned} \hat{W} &= g:A_0^4:+W_c \\ W_c &= \delta m^2 Z:A_0^2:+Z(:\partial_\mu A_0\partial^\mu A_0 - m^2A_0^2:)+g(Z_g-1):A^4: \end{aligned} \qquad (4.47)$$

The claim (proven partially later) of renormalization theory is that δm^2,Z and Z_g can be chosen such that the correlation function stay finite in the limit of removed regularization. This time the mass appearing in Fockspace does not have to be the physical one and the normalization of A is not required to be standard. However in order to have a simple form for scattering formulas it is convenient to implement the physical parametrization and the standard field normalization already in every order of perturbation theory.

4.4 Invariant Parametrizations, Regularization

The x-or p-space integrations of perturbation theory extend over noncompact regions and are difficult to perform in their original form. An efficient formalism which also allows to maintain the manifest Lorentz covariance in the presence of regularizations and cutoffs is based on Schwinger's α-parametrization (another one is due to Feynman):

$$\frac{1}{p^2-m^2+i\epsilon} = \int_0^\infty e^{ia(p^2-m^2+i\epsilon)}da \qquad (4.48)$$

$$\frac{1}{\not p - m+i\epsilon} = \frac{\not p+m}{p^2-m^2+i\epsilon} = (\not p+m)\int_0^\infty e^{ia(p^2-m^2+i\epsilon)}da$$

where the $i\epsilon$ provides a damping factor for the upper integration limit. Applying this representation to the second order vacuum polarization we obtain:

$$\pi_{\mu\nu}(k) = -e^2\int\frac{d^4p}{(2\pi)^4}\left\{\frac{Tr\gamma_\mu(\not p+m)\gamma_\nu(\not p-\not k+m)}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\right\}$$

$$
\begin{aligned}
&= -4e^2 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{p_\mu (p-k)_\nu + \{\mu \Leftrightarrow \nu\} - g_{\mu\nu}(p^2 - p\cdot k - m^2)}{(p^2 - m^2 + i\varepsilon)((p-k)^2 - m^2 + i\varepsilon)} \right\} \\
&= -4e^2 \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu} + \{\mu \Leftrightarrow \nu\} - g_{\mu\nu}\left(\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial y} + m^2\right) \right\} \\
&\quad \times \int_0^\infty \int d\alpha_1 d\alpha_2 \exp i \left\{ \begin{array}{c} \alpha_1(p^2 - m^2 + i\varepsilon) + \alpha_2((p-k)^2 - m^2 + i\varepsilon) \\ + x\cdot p + y\cdot(p-k) \end{array} \right\}_{x=0=y} \qquad (4.49)
\end{aligned}
$$

where in the last step we used the α-parametrization and eliminated the polynomial in the numerator by differentiation and setting the auxilary variables zero at the end. In this form the p-integration involves easy to do oscillatory Gaussian integrals and the original divergence has been shifted into the α-integrals as divergencies at α=0:

$$
\begin{aligned}
\pi_{\mu\nu}(k) &= \frac{i\alpha}{\pi} \int_0^\infty \int \frac{d\alpha_1 d\alpha_2 \alpha_1\alpha_2}{(\alpha_1+\alpha_2)^4} \left\{ \begin{array}{c} 2(k_\mu k_\nu - g_{\mu\nu}k^2) \\ -g_{\mu\nu}\left(k^2 + \left[m^2 - \frac{i}{\alpha_1+\alpha_2}\right]\frac{(\alpha_1+\alpha_2)^2}{\alpha_1\alpha_2}\right) \end{array} \right\} \qquad (4.50) \\
&\quad \times \exp i \left\{ \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}k^2 - (\alpha_1+\alpha_2)m^2 \right\}
\end{aligned}
$$

Here we split the polarization into a transversal and longitudinal part. Note that the transversality property:

$$
\begin{aligned}
k^\mu \pi_{\mu\nu}(k) &= -e^2 \int \frac{d^4p}{(2\pi)^4} Tr\left(\not{k}\frac{1}{\not{p} - m + i\varepsilon}\gamma_\nu \frac{1}{\not{p} - \not{k} - m + i\varepsilon}\right) \qquad (4.51) \\
&= -e^2 \int \frac{d^4p}{(2\pi)^4} Tr\gamma_\nu \left(\frac{1}{\not{p} - \not{k} - m + i\varepsilon} - \frac{1}{\not{p} - m + i\varepsilon}\right) \stackrel{?}{=} 0
\end{aligned}
$$

does not follow because the translation of integration variables is not allowed. Instead of enforcing the transversality condition by "brute force" (vanishing of the longitudinal term) we may also use regularizations which maintain transversality (gauge invariance). There are two gauge invariant methods: the Pauli-Villars method of auxilary fields and the more recent dimensional regularization method. The P-V method adds fictitious spinor fields with masses $m_i = \lambda_i m$ and strength C_i :

$$
\pi^{PV}_{\mu\nu}(k, m_1, m_2 \ldots) = \pi_{\mu\nu}(k, m) + \sum_{i=1}^{n} C_i \pi_{\mu\nu}(k, m_i) \qquad (4.52)
$$

The power counting of the integrand indicates convergence for:

$$
1 + \sum_{i=1}^{n} C_i = 0 \qquad (4.53)
$$

For sufficiently convergent integrals one can shift integration variables and obtain the transversality of $\pi^{PV}_{\mu\nu}$. The tranversal $\pi^{PV}_{\mu\nu}$ has the following α-representation:

$$
\pi^{PV}_{\mu\nu}(k) = -i(g_{\mu\nu}k^2 - k_\mu k_\nu)\pi(k) \qquad (4.54)
$$

$$\pi(k) = \frac{2\alpha}{\pi}\int\int_0^\infty \frac{d\alpha_1 d\alpha_2 \alpha_1\alpha_2}{(\alpha_1+\alpha_2)^4}\sum_{i=0}^n C_i \exp i\left\{k^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - (m_i^2 - i\epsilon)(\alpha_1+\alpha_2)\right\}$$

$$= \frac{2\alpha}{\pi}\int\int_0^\infty d\alpha_1 d\alpha_2 \alpha_1\alpha_2\delta(1-\alpha_1-\alpha_2)\int_0^\infty \frac{d\rho}{\rho}\sum_{i=0}^n C_i \exp i\rho\left\{k^2\alpha_1\alpha_2 - m_i^2 + i\epsilon\right\}$$

where in the last line the identity $\int d\rho\delta(\rho-\alpha_1-\alpha_2)=1$ was used to introduce a radial variable ρ. By appropriate choice of the C_i one improves the small ρ behaviour. For our purpose it is sufficient that the above relation $\sum_0^n C_i = 1+\sum_1^n C_i = 0$ leads to the vanishing of the sum in the integrand. For this n=1 suffices. Higher order zeros could be obtained by requiring higher moments to vanish as well i.e. $\sum_i m_i^j C_i = 0$. The ρ integration gives (considering only leading contributions for large λ_i):

$$\lim_{r\to 0}\int_r^\infty \frac{d\rho}{\rho}\sum_{i=0}^n C_i .... = \lim_{r\to 0}\sum_0^n C_i(-e^{i\sigma}\ln\sigma\,|_{\sigma=r(k^2\alpha_1\alpha_2-m_i^2)} \qquad (4.55)$$

$$+\int_0^\infty d\sigma e^{i\sigma(1+i\epsilon)}\ln\sigma)$$

$$= -\left\{\ln(1-\frac{\alpha_1\alpha_2 k^2}{m^2}) - \ln\frac{\Lambda^2}{m^2}\right\}, \qquad (4.56)$$

$$\text{with}\ \sum_1^n C_i \ln\lambda_i^2 = -\ln\frac{\Lambda^2}{m^2} \qquad (4.57)$$

This yields the Pauli-Villars regularized vacuum polarization:

$$\pi^{PV}(k) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left\{\ln(1-x(1-x)\frac{k^2}{m^2}) - \ln\frac{\Lambda^2}{m^2}\right\} \qquad (4.58)$$

$$= -\frac{\alpha}{3\pi}\left\{2(1+\frac{2m^2}{k^2})[y\,arccoty - 1] + \frac{1}{3} - \ln\frac{\Lambda^2}{m^2}\right\}$$

$$\text{with}\ y = (\frac{4m^2}{k^2}-1)^{\frac{1}{2}}$$

Another more recent regularization scheme which also maintains gauge invariance of $\pi_{\mu\nu}$ is the dimensional regularization. This method only works in the euclidean formulation of perturbation theory which will be discussed in a later section.The regularized expressions for $i\Sigma$ and Γ_μ are also conveniently derived in the α-parametrization. since the integration over the loop momentum is always a simple (oscillatory) Gaussian, we write directly:

$$\Sigma(p) = \frac{\alpha}{2\pi}\int\int\frac{d\alpha_1 d\alpha_2}{(\alpha_1+\alpha_2)^2}(2m - \frac{\alpha_1}{\alpha_1+\alpha_2}\not{p})\exp\left\{i(\frac{\alpha_1}{\alpha_1+\alpha_2}p^2 - \alpha_1\mu^2 - \alpha_2 m^2\right\} \qquad (4.59)$$

$$= \frac{\alpha}{2\pi}\int_0^\infty\frac{d\rho}{\rho}\int\int d\alpha_1 d\alpha_2\delta(1-\alpha_1-\alpha_2)(2m-\alpha_1\not{p})e^{i\rho(\alpha_1\alpha_2 p^2 - \alpha_1\mu^2 - \alpha_2 m^2)}$$

As in the previous case we add a PV regularization term but this time through an auxilary "heavy photon field" of mass Λ^2. Again only retaining the leading term, we obtain the answer by the substitution($C_0 = 1, C_1 = -1$):

$$\int \frac{d\rho}{\rho} e^{i\rho(\ above\ )} \rightarrow \int \frac{d\rho}{\rho}(e^{i\rho(\ above\ )} - e^{-i\rho\alpha_1\Lambda^2}) \tag{4.60}$$

The $\rho-$integration and one of the $\alpha-$integrations say α_2 can be done and we are left with the following integral representation:

$$\Sigma(p,\Lambda) = \frac{\alpha}{2\pi}\int_0^1 dx(2m - x\ \not{p})\ln\frac{x\Lambda^2}{(1-x)m^2 - x(1-x)p^2 + x\mu^2 - i\varepsilon} \tag{4.61}$$

If we stay within $p^2 < m^2$ we may set $\mu = 0$ and obtain the explicite results

$$\Sigma = \frac{\alpha}{2\pi}\left\{\begin{array}{c} \ln\frac{\Lambda^2}{m^2}(2m - \frac{1}{2}\not{p}) + 2m(1 + \frac{m^2-p^2}{p^2}\ln(1 - \frac{p^2}{m^2}) \\ -\frac{1}{2}\not{p}[\frac{m^4-(p^2)^2}{(p^2)^2}\ln(1-\frac{p^2}{m^2}) + \frac{m^2}{p^2} + \frac{3}{2}] \end{array}\right\} \tag{4.62}$$

This is a matrix-valued analytic function in the cut p^2-plane which has a diverging derivative on the mass shell $p = m^2$ as a reminder of the infrared problem. By keeping μ finite, the mass shell limit has finite derivatives.

The regularization of the one particle irreducible second order contribution to the vertex function 4.31 is more involved since as a result of the presence of three propagators one has to introduce 3 $\alpha's$. The α-representation for the three denominators reads as:

$$\int \frac{d^4q}{(2\pi)^4}\frac{e^{iqx}}{(q^2-\mu^2+i\varepsilon)(q^2 - 2p'\cdot q + i\varepsilon)(q^2 - 2p\cdot q + i\varepsilon)} \tag{4.63}$$
$$= \frac{1}{(4\pi)^2}\int\int\int \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^2}\exp -i\left\{\alpha_1\mu^2 + \frac{(\frac{x}{2} - \alpha_2 p' - \alpha_3 p)^2}{\alpha_1+\alpha_2+\alpha_3}\right\}$$

where the exponential was added for the same reasons as in the previous case of $\pi_{\mu\nu}$ namly to convert the polynomial q-dependent numerator into a differentiation acting on the variable x. The (PV type) regularization can be again implemented through the photon propagator. Since there is no essential new idea involved but (even if one passes from the off shell vertex to the on shell formfactor) only some lengthy calculations, we skip the details in the derivation of formula ??The divergent Λ-dependent parts are evidently local (polynomial in the external momentum variables) ant therefore can be compensated by counter terms of the following kind:

$$W_{c.t.} = -\frac{1}{4}(Z_3-1)F^2 + (Z_2-1)(\frac{i}{2}\bar{\psi}\overleftrightarrow{\not\partial}\psi - m\bar{\psi}\psi) + Z_2\delta m\bar{\psi}\psi - e(Z_1-1)\bar{\psi}\ \not{A}\psi \tag{4.64}$$

Finite parts in counterterms would remain unspecified unless one imposes normalization conditions. Natural normalization conditions are the conditions which result from the adiabatic requirement (a must for on shell quantities)

augmented by the physical charge paramertization of the formfactor. The regularized formulas of the previous section 4.35have been written in such a way that the natural normalization means omission of the δm and Z-terms. It is easily seen that the remaining second order π, Σ and Γ terms have the correct zeros required by the adiabatic principle resp. physical charge parametrization. The proof of n^{th} order renormalizability, i.e. the statement that the old local counterterms iterated together with the original W in the S(g) expansion lead to higher order correlation functions which in turn may be liberated from their infinities by new higher order counterterms of the same local structure, requires a significant extension of the regularization formalism. In an appendix we will present a proof in the spirit of the Bogoliubov-Shirkov and Epstein-Glaser method.

We already stressed the fact that the close connection between particles and fields is an artifact of perturbation theory and not a result of the use of the choice of a particular Fockspace for the definition of local operator algebras. In some sense the infrared singularities of Maxwell like (gauge) theories can be interpreted as a perturbative indication that the theory is not compatible with the zero order particle content.

We have previously shown that the description of e.g. $(m = 0, h = 1)$ representation in terms of vector potentials leads to a significant deviation from the principles of local quantum physics. Whereas in classical Maxwell-like theory the use of gauge fields in the form of vector potentials is a harmless and useful and even natural step, which in addition gives rise to pretty mathematics (fibre bundles, differential geometry...), the use of covariant vector potentials for the $(m = 0, h = 1)$ representation leads outside quantum theory to unphysical degrees of freedom in indefinite metric spaces (ghosts). One needs these objects not for conceptual reasons but only in order to define *the perturbatively correct* long range coupling in a scheme which requires covariance and locality for unphysical quantities for the sake of easier calculations. The problem would not be so serious if the prescription for the elimination of these undesired companions (the prize for using covariant vector potentials) would be a kinematical procedure independent of the interaction, but unfortunately this turned out to be wishfull thinking. In the case of QED the prescription of Gupta and Bleuler defined a positive semidefinite subspace, but in the more complicated nonabelian gauge theories this did not work. The formal geometric setting of path integrals suggested (Faddeev-Poppov 1967) that in addition to the unphysical (scalar and longitudinal vectormesons) degree of freedoms one should introduce s=$\frac{1}{2}$ ghosts with the wrong relation between spin and statistics. Whereas the Gupta-Bleuler approach at least stays in the vicinity of the Wigner representation theory, the relation of the Faddeev -Poppov scheme to representation theory is less clear. It was shown later that these formal ghost objects carry a new formal symmetry (Becchi-Rouet-Stora 1975) which allows to organize the perturbation theory and extract the ghost free quantum physics (modulo infrared proplems) in an efficient way. We will formulate this procedure directly within the S(g) interaction operator approach without using differential geometric properties.

There exists a widespread misconception that a Lagrangian quantization

viewpoint is important for the intrinsic physical understanding of interactions in QFT. From such a point of view the ultraviolet divergencies appear as serious flaw of perturbation theory. The history of renormalization (see Laurie M. Brown: "Renormalization", From Lorentz to Landau (and Beyond)) seems to favour such thinking. After all, Renormalization arose from some remarks of Kramers who suggested to use similar distinctions between bare and physical masses (and other parameters) in QED as they were used by Lorentz and Poincaré in classical field theory at the beginning of the century. selfenergy. Additional support came from the observation that for a few models (e.g.A^4 in d=1+1) which were "well-behaving" in the perturbative treatment, it was possile to prove their mathematical existence by extending the perturbative method. Additional support in those cases comes from the functional integral method (based on the euclidean Feynman-Kac representation) which furnishes a rather direct relation between QFT and classical physics, more tight than canonical quantization. However the recent progress in a nonperturbative understanding of interactions from a different starting point (sometimes called the "bootstrap approach") has cast grave doubts on the universal correctness (apart from those few exceptions) of such quantization approach to interactions. We will return to this important point at a more appropriate place.

It also should be stressed that renormalized perturbation theory does not lend credibility to the idea of a physical cutoff i.e. a Λ which cannot be absorbed into the renormalization constants but rather enters the physics. As we will see more clearly in a general critical review at the end of the monography, a physically interpretable nonlocal theory with an elementary length does presently not exist.

4.5 Specialities of Perturbative Gauge Theories

Gauge quantum field theories are (ever since their nonabelian generalization was shown to be reconcilable with perturbative renormalization at the beginning of the 70^{ies}) mostly presented as the quantized counterparts of classical field theories in the differential geometric setting setting of fibre bundles. This standard approach can be criticized on two points. On the one hand, as we learned previously, the peculiarity of quantum zero mass phenomena lies in the special structure of the the m=0 Wigner little group (i.e. the nonfaithful representation theory of the noncompact $\tilde{E}(2)$). One hardly can claim that the differential geometric gauge principle of Lagrangian field theory follows from this quantum observation. This would not be so bad however, if the use of this principle would have led to a conceptually clear and computationally successful new theory. But (if such historical comparisons in physics are allowed) the present state of gauge theory versus the largly nonunderstood physical reality behind the standard model in relation to a yet unknown future theory may turn out to be similar to the Bohr atomic model and its relation to QM. This lack of understanding is also visible on a pure theoretical level: e.g. it is completely obscure how the physical (gauge invariant) correlation functions reveal

that they come from the implementation of the gauge principle or whether there is an intrinsic meaning of gauge invariance. In the words of one of my collegues: "success (the phantastic precision of QED effects) and desaster (the mentioned problems) may lie close together". Such situations in physics are usually the germ of future progress and I have written this section with this hope in mind.

Following the line of arguments in section 5 of the preveous chapter, one should start with $S(g) = e^{iW}, W(g) = \int g(x)j^{\mu}(x)A_{\mu}(x)d^4x + ...$where all the fields are free fields in Fock space and the dots denote possible "nonabelian" contributions containing the back-reaction of A_{μ} on the source. Hier A_{μ} is a nonlocal potential of the form 3.158. As a result of the nonlocal gauge term under L-transformations 3.159 there is an appearant clash of the localiy of the *supp.g* interaction $W(g)$ and partial action of Lorentz-transformation inside the support region. In other words the surface term in $W(g)$ resulting from the transformatiom 3.159 seems to be nonlocal. We will not persue the problems of this indefinite metric free formulation because it does not yet exist in an elaborated form.

Instead we will follow the more standard approach of working in an extended indefinite metric auxilary space. The treatment will however not be identical to the standard geometrical functional integration quantization of "gauge theories" since we do not care about of differential geometric elegance which shovels important physical concepts under the rug[2]. This is done because we believe that the "bumpy" method is less prejudicial as far as a discovery of a future more physical elegance is concerned (i.e. we are not blind against elegance but we expect it at a different place from differetial geometry and fibre bundles. I am convinced that a different more intrinsic way of introducing s=1 long range interaction through modular localization (which generalizes the peculiar duality structure mentioned in the section on the m=0 s=1 Wigner representation) will be found in due time. Therefore we will emphasize the statement that in perturbation theory *the consistency of vector fields with the principles of local quantum physics alone (and not any "gauge principle")* determines the structure of the interactions, but we will do it with all the awareness that there is an unsolved deep nonperturbative conceptual problem hidden behind the gauge issue.

Leaving this for the future, we remind ourselves that the use of a point-like *covariant* vector potential $A_{\mu}(x)$, as one needs it to describe the characteristic long range coupling (minimal e.m. coupling), requires to forget the positivity principles of quantum physics (a step which is not yet necessary if one only treats matter in external potentials) in intermediate calculations and work in an indefinite metric (Pseudo)Fock space. An equivalent method is to use an auxilary Hilbertspace and to work with η-adjoint fields and pseudounitary representation theory. The fact that there is no (nonperturbative) controllable mathematical framework for "η-star algebras" makes the connection of the perturbative prescriptions with an existing QFT even more flimsy than it is already outside of gauge theories. In addition the indefinite metric coming with the extension of

[2]One may think of Einsteins aphorism: "Eleganz ist etwas fuer die Scheider".

the (m=0,h=1) Wigner representation theory to vector potentials (via a Weyl "pseudo-functor"?) and the identification of a positive semidefinite subspace turns out to be insufficient for nonabelian gauge fields. Although historically a cure for this last additional deficiency has first been seen in the geometric functional setting (Faddeev-Poppov), we will present it here in the perturbative operator formalism. Last but not least we discard matters of esthetical beauty as gauge theory in the spirit of classical fibre bundles because they suffer severe damage from the renormalization procedure and donot reveal the physical content. In addition from the conceptual physical point of view the really ugly feature of the formalism which will be presented below is not the absence of estetical gauge principles based on differential geometry, but rather the permission of indefinite metric resp. pseudo-unitarity. As the elementary interaction we take SU(n) free fields coupled in first order as:

$$W = i\frac{g}{2} : \bar{\psi}_\alpha \lambda^i_{\alpha\beta} \gamma_\mu \psi_\beta : A^\mu_i + igf_{ijk}(\frac{1}{2} : A_{\mu i} A_{\nu j} F^{\nu\mu}_k : + Gh) \tag{4.65}$$

The reason for the nonlinear term is that we would like to include the possibility that not only the matter field ψ but also the (interaction)mediating field A_μ participates in the source (the equality of the coupling constants is already anticipated). With the linear relation between A_μ and $F_{\mu\nu}$, this is the first order (linear part) of the Yang-Mills interaction, apart from an auxilary term Gh which we need in a moment in order to implement a quantum gauge principle which is necessary in order to recover a positve semidefinite subspace (from which one can form a Hilbertspace by equivalence classes) at least in the perturbative sense. If one uses the functional quantization approach of Faddeev and Poppov, the Gh term would be (in the Feynman gauge):

$$Gh = -A_{\mu i} : \xi_j \partial^\mu \tilde{\xi}_k : \tag{4.66}$$

where $\xi_i, \tilde{\xi}_i$ are n-component auxilary massless free scalar field with anti-commutation relation (an object which violates the quantum structure even more severely than the A_μ):

$$\xi_i(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{2\omega} (c_{i,2}(p) e^{-ipx} + c^*_{i,1}(p) e^{ipx}) \tag{4.67}$$

$$\tilde{\xi}_i(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{2\omega} (-c^*_{i,1}(p) e^{-ipx} + c_{i,2}(p) e^{ipx})$$

$$\{c_{i,\alpha}(p), c^*_{j,\beta}(p')\} = \delta_{ij}\delta_{\alpha\beta} 2\omega\delta(\mathbf{p} - \mathbf{p}') \tag{4.68}$$

$$A_{\mu,i}(x) = \begin{cases} \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2\omega} (e^{-ipx} a_\mu(k) + e^{ipx} a^*_\mu(k)), & \mu = 1,2,3 \\ \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{2\omega} (e^{-ipx} a_0(k) - e^{ipx} a^*_0(k)), & \mu = 0 \end{cases} \tag{4.69}$$

$$[a_\mu(k), a^*_\nu(k')] = \delta_{\mu\nu} 2\omega\delta(\mathbf{k} - \mathbf{k}'), \quad i.e. a^*_{i,0} \neq a_{i,0}, c^*_{i,\alpha} \neq c_{i,\alpha}$$

$$A^\dagger : = \eta A^* \eta, \quad A^\dagger_{i,\mu} = A_{i,\mu}(x), \xi^\dagger_i(x) = \xi_i(x) \quad with:$$

$$\eta a^*_\mu(k)\Omega = \begin{cases} a^*_\mu(k)\Omega, & \mu = 1,2,3 \\ -a^*_0(k)\Omega, & \mu = 0 \end{cases}, \eta c_i(p)\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c_i(p)\Omega$$

The guiding idea is now to look for a "gauge principle" in the form:

$$[Q, T_n(x_1....x_n)] = i\sum_{k=1}^{n}\partial_\mu^k T_{n,k}(x_1....x_n), \quad Q: \text{conserved charge} \tag{4.70}$$

$$S(g) = 1+\sum\frac{1}{n!}\int d^4x_1..d^4x_n T_n(x_1..x_n)g(x_1)..g(x_n)$$

where the T_n are the iteratively renormalized T-products of the interaction density W. The candidate for Q is:

$$Q = \int d^3x\chi_i(x)\overleftrightarrow{\partial}_0\xi_i(x), \quad \chi_i(x) := \partial^\mu A_{\mu i} \tag{4.71}$$

The form of the gauge generator Q is suggested by analogy with the Gupta-Bleuler treatment of QED in which $Q(\Lambda) = \int d^3x\chi(x)\Lambda(x)$ with Λ a c-number solution of the wave equation generates the classical gauge transformations. Let us first argue that the above commutation relation arise and insure the existence of a semidefinite metric subspace The difference to the classical $\Lambda(x)$ gauge transformation is that ξ_i is required to be a pseudo-quantum field in order to obtain in first order:

$$\begin{aligned}[Q, T_1] &= i\partial_\nu T^\nu_{1,1} \\ T^\nu_{1,1} &= :\left\{igf_{ijk}(A_{i,\mu}F_k^{\nu\mu}\xi_j - \frac{1}{2}\xi_i\xi_j\partial^\nu\tilde{\xi}_k) + igj_i^\nu\xi_i\right\}:\end{aligned} \tag{4.72}$$

Note that equality of the coupling as well as the nessecity to add the "ghost-term" to the interaction density is a consequence of the first order gauge principle. The necessity to add the quadratic term g^2AAAA as in standard geometrically motivated gauge theory would show up in the next order. In the same vein the nilpotency of Q i.e. $Q^2 = 0$ is required by nonabelian consistency. In fact Q is the Fockspace analoge of the BRST-charge. Before we go into technicalities it may be helpful to present the physically relevant results.

- The perturbative gauge principle is sufficient (I believe also necessary) in order to reconcile local couplings involving massless vector fields with the principles of local quantum physics.

- If the vector mesons are massive, the same consistency requirements lead to the additional perturbative presence of another (bosonic) scalar field and one obtains the standard perturbative Schwinger-Higgs picture of broken symmetries through screened charges. As expected from the algebraic viewpoin, an intrinsic physical meaning to the notion of "spontaneously broken local gauge invariance" unlike in the spontaneous Nambu-Goldstone breaking does no show up. There are however nonperturbative arguments (B.Schroer "Motivations and Physical Aims of Algebraic QFT" to appear in Ann.of Phys. March 1997) which suggest that there may be an intrinsic understanding of the Schwinger-Higgs mechanism in terms of screened semiinfinite string-like charges which could reappear for short distances.

The only exception is the case of abelian vectorpotentials (no contribution to the source) in which case the perturbative existence of a symmetry-breaking Higgs field is not a necessary consequence of a massive A_μ (Remark: in the massive abelian theory without Higgs scalars the fermion field is nonrenormalizable and only the current- and the vectormeson- correlation functions only suffer logarithmic corrections to their canonical behaviour).

Before we go on, let us give a (somewhat formal) argument why these commutation relations suffice to secure the existence of a physical subspace. Besides the gauge charge Q we may also define a "ghost-charge"(summation convention always used):

$$\begin{aligned} G &= \int d^3x : \tilde{\xi}_i \partial_0 \xi_i :, \quad G^2 = 0 \\ &= \int \frac{d^3k}{2\omega} c^*_{i,\alpha}(k) c_{i\alpha}(k) \end{aligned} \tag{4.73}$$

Different from the pseudo-selfadjoint Q, G is a positive definite (pseudo)particle counting operator. Q and G commute, so formally (forgetting that Q is only pseudo-selfadjoint) the physical subspace $\mathcal{H}_{phys}$ can be represented as an intersection of nullspaces:

$$\mathcal{H}_{phys} = \{\psi \mid Q\psi = 0 \wedge G\psi = 0\} \tag{4.74}$$

and since the G-nullspace is a subspace of the Q-nullspace:

$$KerQ = H_{phys} \oplus H^{\perp}_{phys} \tag{4.75}$$

where the complement is taken inside $KerQ$. Using now $Q^2 = 0$ and $\left\{Q, \tilde{Q}\right\} = G$ with:

$$\tilde{Q} := \int \frac{d^3k}{2\omega} (c^*_{i,1}(k) c_{i,2}(k) + c^*_{i,2}(k) c_{i,1}(k)) \tag{4.76}$$

We may now derive the following equation between projectors:

$$\begin{aligned} P(H^{\perp}_{phys}) &= GG^{-1} P(H^{\perp}_{phys}) = (Q\tilde{Q} + \tilde{Q}Q) P(H^{\perp}_{phys}) \\ &\curvearrowright \quad P(H^{\perp}_{phys}) \leq P(ranQ) \curvearrowright P(H^{\perp}_{phys}) = P(ranQ) \end{aligned} \tag{4.77}$$

where the last step uses the nilpotency $Q^2 = 0$. This characterization of the physical subspace in terms of projectors suffices in order to derive the gauge relation 4.70Before we go into these technicalities we should explain why this relation insures positivity. Formally this is so because the $S(g)$ in the adiabatic limit (if it exists !) $g \to 1$ commutes with the gauge charge or in other words, S_{adia} commutes with $P(KerQ)$ and $P(H^{\perp}_{phys})$. For massive gauge theories (Schwinger-Higgs theories) the adiabatic limit exists perturbatively. In the massless case the infrared divergencies force one to shift the extraction of a physical theory to the construction of sufficiently many local objects with the commuting property. The gauge relation of $S(g)$ is then expected to be an important ingredience in a complete proof.

Presently this formalism has not been elaborated to the same degree as the standard gauge formalism. But the second order calculation shows that there is no need for a gauge principle; the interactions are fixed just by the physical consistency of theories in which vector particles contribute to their own source! The main purpose of this "dirty" presentation was to counteract the geometric mysticism which has developped around gauge theories with its deceiving elegance..

4.6 Perturbative Thermo-QFT

Justification for "thermal" Feynman-rules, the Bros & Buchholz framework of relativistic KMS, instability of supersymmetry against temperature.

Left out, because of lack of time.

4.7 Functional techniques

Formal perturbative Feynman-Kac representation of perturbative Schwinger functions. Feynman-Kac representations in Schroedinger theory. Basic facts about φ^4_2. Mathematical and structural limitations of F.-K. representability. A particularly useful application: the F-K representation of d=1+1 order-disorder fields and solitons (example the φ^4-kink). Limitations of F-K representability: no such representation for chiral conformal theories and more generally for fields with braid group commutation relations.

Again deferred because of lack of time and beauty of Brazilean beaches.

4.8 Interactions with External Fields, CST-Problems

Interactions of quantum fields with external (classical) fields played an important role in the development of full QFT. The simplest situation of this kind one meets in case the quantum fields are free. In fact free Dirac or Schrödinger fields interacting with external electromagnetic fields preceeded QED, and led (with some hindsight concerning interpretations, see the introduction in Weinberg's book) to many correct formulas. If we look at these external field problems from the point of view of Poicaré-invariant QFT we notice a conceptual problem. Since the vacuum and also the particle states are defined in terms of P-covariance properties, it is not immediately clear how one should define such reference states if P-covariance is broken. In an elegant formalism like Schwinger's, there is not much chance to ask such a question since the formalism itself takes care of it (see his treatment of the astrophysically relevant $\mu\bar{\mu}-$pair creation in e.m.fields). A closer look reveals that his reference state is the "adiabatic vacuum" which in a more mundane formalism corresponds to the approximation of the actual external interaction by a sequence of softly switched on and off external interactions. Wheras this is eminently reasonable for e.m.interactions, this is generically speaking unreasonable for problems in

curved space time (CST) i.e. with external gravitational fields. Since the Hawking radiation effect belongs to this class of problems, these structural questions are not without physical interest and relevance.

In the following we will briefly scetch some ideas (Radziskowski, Brunetti, Fredenhagen,Köhler..) which not only led to an answer for the correct reference states, but gave a framework for the renormalized perturbation theory of *interacting* quantum matter in CST. Even if, as in the case of the present author, one is not an actively working specialist in this area, one should take notice for the following reason. General QFT as it stands, is not quite that perfect quantum counterpart of the classical Faraday-Maxwell theory with its "Nahewirkungsprinzip". Wheras the algebraic part (the net theory) is local, the energy positivity and the vacuum homogeneity are very nonlocal stability requirements. This is the cause of the above mentioned difficulty. Therefore if QFT in CST requires to think about a substitute, this may be very beneficial for Minkowski space QFT. Since all of the renormalization schemes use either euclidean space or momentum space, one also is forced to rethink the renormalization formalism. Looking at the literature on notices two very strange facts: almost all the papers (before the above work) on the subject are about euclidean QFT, and nobody ever tried to define a Wick-polynomial (needed for the energy-momentum tensor etc.)

Since the answer to both questions requires the use of somewhat unfamiliar (including to the present author), let me only make some qualitative comments on the microlocal (or Fourier integral operator) analysis (developped by the mathematicians Hörmander and Duistermaat around 1971) which is then used in the formulation of a "microlocal spectrum condition" in QFT.

The basic idea is to refine the local analysis of singularity structure (the singsuppu is the complement of the largest open smoothness region of u) Schwartz distribution u and more generally of distribution densities on a manifold, by shifting it from the base space to the cotangent bundle. In brief, one first zooms in on a u-singularity and then the uses a directional Fourier "telescope". If ϕ is a localyzing test function, one studies:

$$\widetilde{\phi u}(\xi) = \left\langle u, e^{-i(\cdot,\xi)}\phi \right\rangle \tag{4.78}$$

where $\langle\cdot,\cdot\rangle$ denotes dual pairing. This is a fast decreasing function in ξ as long as suppϕ does not touch the singularity points. If suppu on the other hand does extend into the singular region of u, the singularity may be directional dependent and in certain ξ−directions one may still encounter a fast decrease. Therefore one uses the following definition (V denotes the euclidean base space)

Definition 3 *The wave front set, $WF(u)$, of u is the complement in $V \times \mathbf{R}^n\setminus\{0\}$ of the points (x_0,ξ_0) in cotangent space $V \times \mathbf{R}^n\setminus\{0\}$ s.t. for each ϕ, $\exists$a nbhd. $U \times \Gamma$ with Γ conic (directional) nbhd. of ξ_0 and an $N \geq 0$ with:*

$$\left|\left\langle u, e^{-i(\cdot,\xi)}\phi \right\rangle\right| \leq C_{\phi,N}\,(1+|\xi|)^{-N}, \quad \forall \xi \in \Gamma \tag{4.79}$$

Returning to physics, we recall that as the result of the positive energy property, (unordered) correlation functions belong to a class of distributions which can be freely multiplied. For example the product of two two-point functions $w_i(x,y) = W_i(x-y)$ is again a well-defined distribution in the same class because the convolution of their Fourier transforms with the $V^\dagger$spectral support amounts to an interal over a finite (phase space) region. Using the Kallén-Lehmann spectral representation:

$$W_i(\xi) = \int_0^\infty i\Delta^+(\xi,\kappa)\rho_i(\kappa)d\kappa, \quad i=1,2$$

the convolution of the $\rho_i's$ extends over a finite mass region. This property does not hold for time-ordered or retarded distributions since they do not posses a spectral support in momentum space.

The main property of the wave front sets of distributions is that they allow a simple criterion for the existence of the product: the conic nbhds. Γ_i must add up to a resulting conic nbhd. in $V \times \mathbf{R}^n \setminus \{0\}$. The coordinate free adaption to densities (distribution valued differential forms) on manifolds M is easy. The wave front sets are now cones in T^*M and they behave additively under multiplication in the following sense:

$$WF(u_2u_1) \subseteq WF(u_2) \oplus WF(u_1) \cup WF(u_2) \cup WF(u_1) \qquad (4.80)$$

The product exists if the zero section in T^*M does not intersect $WF(u_2u_1)$.

Let us now test this idea for free QFT in CST. We start with the structure of the algebra generated by a Klein-Gordon field ϕ in a globally hyperbolic space time:

$$\begin{aligned} (g^{\mu\nu}\partial_\mu\partial_\nu - m^2)\,\phi(x) &= 0, \quad [\phi(f),\phi(g)] = E(f\otimes g) \qquad (4.81)\\ \forall f,g &\in C_0^\infty(M), \quad E(x,y) = \Delta^{av}(x,y) - \Delta^{ret}(x,y) \end{aligned}$$

It is defined uniquely in terms of the manifold data i.e. the ret (av) functions are uniquely determined by the geometry whereas the onordered and time ordered functions are determined by the states. One now defines a wave front set for the (yet unknown) two-point functions ω_2:

$$WF(\omega_2) = \{(x,k;x',-k') \in T^*M^2 \setminus \{0\} \mid (x,k) \sim (x',k'),\ k \in \bar{V}_+\} \qquad (4.82)$$

where the equivalence relation $\sim$ means that there exists a light-like geodesic from x to x' s.t. k is coparallel to the tangent vector to the geodesic and k' is its parallel transport from x to x'. This physically appealing local requirement for the selection of physical states was shown by Radzikowski to be mathematically equivalent to the more global Hadamard condition (an older recipe to obtain physical states). Free field structure, i.e. the Wick combinatorics means that the higher point functions are products of ω_2 i.e. that the states ω on the algebra 4.81 are so called quasi-free states on a CCR algebra. Using a theorem of Hörmander about the product structure of distributions[3] with a known wave

[3]Distributions allow a pointwise multiplication provided the convex combination of their wave front sets do not meet elements of the zero section.

front one then proves the existence of the n-point functions and a formula for their WF set. In a similar vein one shows the existence of Wick-products as e.g. $:\phi^n:$. As in the Minkowski case, the time-orderd propagator:

$$iE_F(x,y) = \omega_2(x,y) + E_{ret}(x,y) \tag{4.83}$$

does not have the one-sided spectral structure in order to allow for pointwise multiplication. Its wave front set is:

$$\begin{aligned} WF(E_F) &= \{\text{same, but } x \neq x', k \in \bar{V}_\pm \text{ if } x \in \mathcal{T}_\pm(x')\} \\ &\quad \cup \{(x,k;x,-k), x \in M, k \in T_x^*M \setminus \{0\}\} \end{aligned} \tag{4.84}$$

where $\mathcal{T}_\pm(x')$ are the future/past of x'.

Actually the formula 4.82 turns out to be not general enough in order to incorporate theories beyond free fields. The more general formula which does not contain the restriction to light like geodesics and its stable (under multiplication of n-point functions) generalization to ω_n which is most conveniently expressed in terms of graphs with vertices x_i and directed geodetic edges between them) can be found in the work of Brunetti et.al.[?].

We still have to understand why the microlocal Spectrum Condition (μSC) is not capable of unique selection of a state and what kind of family it selects. A local spectral condition is naturally not able to single out states with global symmetry. With one particular state in this family all other states in the same folium (vector or density matrix in the same GNS Hilbert space) turn out to share the same WF set. In fact the states with coalescing wave front sets form exactly one folium of the set of all states on the C^*-algebra $\mathcal{A}$. That folium contains of course states with different superselection charges, a situation which is vaguely reminiscend of the "no hair" property of black holes. There are two physical questions which enter ones mind. One is whether physical properties (corrections to electro-weak effects as anomalous moments, Lambshift etc.) change significantly in one folium. For this one has to onderstand renormalization theory and the implementation of physical normalization conditions (the adiabatic separation or interaction discussed at the beginning of this chapter. The other is whether the μSC can perceive interactions. An affirmative answer to this question would be extremely interesting even for Minkowski QFT (since the global vacuum condition is not capable of such a distinction. Both questions are presently open??.

The last issue in this section is how to do renormalization theory for interacting QFT in CST (i.e. how to avoid euclidian- and momentum-space). A framework which stays in x-space is that of Epstein and Glaser. The main problem in its adaption to the present case is how to avoid translational invariance (on which EG rely heavily). Let us look at their starting formula for the coefficients of the Bogoliubov Shirkov operator S(g) (see chapter 3) after Wick-ordering:

$$\begin{aligned} T_n^{k_1,\ldots,k_n}(x_1,\ldots,x_n) &= \sum t^{l_1,\ldots,l_n}(x_1,\ldots,x_n) \times \\ &\quad : \phi^{k_1-l_1}(x_1)\ldots\phi^{k_n-l_n}(x_n) : \end{aligned} \tag{4.85}$$

In the EG approach it is crucial that Wick-products (i.e. operator-valued distributions) can be multiplied with translational invariant numerical distributions. The CST substitute is:

$$WF(t_n) \in \Gamma_n^{to}, \quad to: \text{ time ordered} \tag{4.86}$$

where Γ_n^{to} is a subset of T^*M^n with a certain graphical characterization. The construction of the time ordered operators $T_n^{k_1 \cdots k_n}$ is achieved by induction in two steps. First one shows that T_n for $(x_1 \ldots . x_n) \notin \Delta_n$ (the total diagonal) can be patched together from all lower n T's. Let us call this T_n on $M^n \backslash \Delta_n$ T_n^0. the second step (the more difficult one) is the extension the diagonal. For a detailed presentation we refer to the recent literature.

The CST renormalization theory contains of course the proof for the renormalizability of the standard theory as a special case. If ones main interest is the explicite calculation of individual Feynman diagrams in the standard theory and not so much a structural inductive understanding of renormalization, the approach in the textbooks (Itzykson-Zuber, Weinberg,..) is still the most appropriate.

It would have been too nice if QFT in CST could furnish a gateway into "Quantum Gravity". After all, QFT in external e.m.fields was an essential step towards QED. But unfortunately this does not seem to be the case and Quantum Gravity continues te exist as only an enigmatic nice sounding word, which presently lacks physical content.

Chapter 5

The General Framework of QFT

5.1 Model-independent Properties of pointlike Fields

The conceptual situation of QFT after the discovery of renormalized perturbation theory was at first somewhat confused. Despite the impressive agreement of low order radiative corrections, the precise relations between particles and fields as well as the mathematical consistency of QFT beyond perturbation theory were ill understood. Most of the post renormalization progress was in the area of structural understanding about the particle-field dichotomy and an compilation of mathematically well posed and physically necessary properties. These developments are often named with the names of the principle protagonists of those problems: Lehmann,Symanzik and Zimmermann (LSZ) in the first case and Wightman in the second. The strong return of perturbative methods in the 70^{ies} via the Standard Model and QCD only led to a temporary lull in the ongoing research on general structural properties of QFT, especially in view of the fact that those nonabelian gauge theories after some initial success (notably in the area of small distance behavior off mass shell) run into tough problems which appear unsolvable in the standard approach.

We already have explained the relation between free fields and the net of local algebras generated by them in terms of an analogy to differential geometry: the fields are like coordinates and the net corresponds to the coordinate-free (intrinsic) approach to QFT. Many of the properties of fields appear in a clearer physical light, if one thinks about them in terms of local generators. Therefore let us list some properties (the main properties of the Haag-Kastler net theory) of nets before we write down the (model-independent) postulates for fields.

- (i) There is a map of double cones $\mathcal{O}$ in Minkowski space into von Neumann operator algebras $\mathcal{A}(\mathcal{O})$ which are subalgebras of all operators $\mathcal{B}(\mathcal{H})$

in some Hilbertspace $\mathcal{H}$:

$$\mathcal{A} : \mathcal{O} \to \mathcal{A}(\mathcal{O})$$

The C*-completion of this family yields the global C*−algebra $\mathcal{A}_{quasi}$:

$$\mathcal{A}_{quasi} = \bigcup_{\mathcal{O}\in\mathcal{M}} \mathcal{A}(\mathcal{O})$$

- (ii) The family $\mathcal{A}$ forms a "net" i.e. a coherent (isotonic) family of local algebras:

$$\text{if } \mathcal{O}_1 \subset \mathcal{O}_2 \text{ then } \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$$

In case the local algebras represent observables one requires another physically motivated coherence property namely Einstein causality:

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$$

- (iii) covariance with respect to the Poincare group. For observable nets:

$$\alpha_{(a,\Lambda)}(\mathcal{A}) = \mathcal{A}(\Lambda\mathcal{O} + a)$$

The subsequent properties of fields and their physical interpretation is facilitated by thinking about them as coordinatizations of generators for local nets. The main difference of the field approach versus the net approach is that properties of charge carrying fields are put in and not derived from those of neutral fields. In the net approach charges (and their field carriers) are constructed via the superselection theory. The latter approach is more fundamental and more suitable in situations which are far away from quantization prescriptions and Lagrangians (e.g. low dim.QFT with braidgroup statistics). In the following we explain the properties of fields in the setting of Wightman. Here and in the following we use the symbol A as a generic notation for collection of generating fields but the standard situation underlying illustrations and proofs is mostly that of one generating scalar field.

Properties of Fields:

- A $\mathcal{H}$-space and $\mathcal{P}$-group

1. Unitary representation $U(a,\alpha)$ of $\tilde{\mathcal{P}}$ in $\mathcal{H}$, $\tilde{\mathcal{P}}$:covering of $\tilde{\mathcal{P}}$
2. Uniqueness of the vacuum Ω, $U(a,\Lambda)\Omega = \Omega$
3. Spectrum condition: $specU \in \bar{V}_+$,the forward light cone.

- B Fields

1. operator-valued distributions: $A(f)=\int A(x)f(x)d^4x$, $f \in \mathcal{S}$(the Schwartz space of "tempered" testfunctions) is an unbounded operator with a dense domain $\mathcal{D}$ such that the function $\langle\psi_2 |A(x)|\psi_1\rangle$ exists as a sesquilinear form.for $\psi_i \subset \mathcal{D}$

2. Hermiticity

With A also A^* belongs to the family of fields and the affiliated sesquilinear forms are as follows related: $\langle\psi_2 |A^*(x)|\psi_1\rangle = \langle\psi_1 |A(x)|\psi_2\rangle$

- C. $\tilde{\mathcal{P}}$-covariance of fields: $U(a,\alpha)A(x)U^*(a,\alpha) = D(\alpha^{-1})A(\Lambda(\alpha)x+a)$

For observable fields only integer spin representations (i.e.representations of $\mathcal{P}$) occur.

- D. Locality: $[A^{\#}(f), A^{\#}(g)]_{\mp} = 0$ for suppf×suppg (supp are spacelike separated).

- E. Stability of local algebras under causal completion: Alg(A,$\mathcal{O}$)=Alg(A,$\mathcal{O}''$), where (for $\mathcal{O}$ convex) the causal complement is the smallest double cone which contains $\mathcal{O}$.

A weaker form of this requirement is the so called "time-slice"property.

Comments:

The domain requirements on (unbounded) smeared-out fields $A(f)$ are reminiscent of properties which are required of generators of noncompact groups. Their motivation here is entirely pragmatic; they insure that the standard calculational methods of physicists are applicable. The more technical domain requirements will be absent in the net approach. The latter only knows the very fundamental (and physical) domain properties of the Tomita-Takesaki modular theory. The existence of the sesquilinear forms for pointlike fields is the substitute for the classical notion of field strength. The $\tilde{\mathcal{P}}$-transformation property of the hermitian adjoint field is that of the complex conjugate transformation which is isomorphic to the antiparticle field: $A^{(\lambda)}(x) = CA^{(\lambda)*}(x)$, C=charge conjugation matrix.

The causality requirement strictly speaking applies to observable fields only (example: electromagnetic field strength and currents but not to vectorpotentials and charged matter fields). The restriction to local fields, which by definition obey the $\mp$ commutation relations, is too strong in $d \leq 2+1$ (see later). In d=3+1 all charge carrying fields are equivalent (by Klein-transformations) to local fields. The strong causal completion property is the substitute for an hyperbolic equation of motion (which,due to ill-defined nonlinear terms is apriori meaningless in QFT). Its formulation and exploration is more natural in the algebraic setting where it simply means that $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'')$ where the double prime denotes the causal complement of a region (the causal future and past shadows as well as the side-caps) which for convex $\mathcal{O}$ is a double cone.

Another physically important property which has been omitted here (but makes its appearance in the net theory later on) is the nuclearity or compactness property which is the QFT counterpart of the statement that a finite cell in phase space can only accomodate a finite (in QFT a nuclear set of vectors) number of degrees of freedom.

The most useful objects which one can form in such a Wightman setting of fields are the vacuum expectation values or (terminology of condensed matter physics) correlation functions:

$$w_n^{(\lambda_1,\ldots,\lambda_n)}(x_1,\ldots,x_n) = \left\langle 0 \left| A^{(\lambda_1)}(x_1)\ldots A^{(\lambda_n)}(x_n) \right| 0 \right\rangle \qquad (5.1)$$

$$
\begin{aligned}
&= W_n^{(\lambda_1 \ldots \lambda_n)}(\xi_1, \ldots \xi_{n-1}) \quad \xi_i = x_i - x_{i+1} \\
&= \left\langle 0 \left| A^{(\lambda_1)}(0) e^{-iP\xi_1} \ldots A^{(\lambda_{n-1})}(0) e^{-iP\xi_{n-1}} A^{(\lambda_n)}(0) \right| 0 \right\rangle \\
&= \int \cdots \int \tilde{W}_n^{(\lambda_1 \ldots \lambda_n)}(q_1 \ldots q_{n-1}) e^{-i \sum_k q_k \xi_k} d^4 q_1 \ldots d^4 q_{n-1}
\end{aligned}
$$

The spectrum property: $\mathrm{spec}(\mathbf{P}) \subset \bar{V}_+$ evidently implies that $\mathrm{supp}\tilde{W}^{(\cdots)}(q_1 .. q_{n-1}) \subset \otimes^n \bar{V}_+$ and, as a property of a Fourier-Laplace transform of a cone supported distribution the "tube analyticity": W is boundary value of a function $W^{(\cdots)}(z_1 \ldots z_{n-1})$ analytic in the tube $T^{(n-1)}$ $z_i = \xi_i - i\eta_i$ with $\eta_i \in \bar{V}_+$ fulfilling the "tempered" bound (assuring the temperedness of the singular boundary values):

$$
\left| W_n^{(\cdots)}(z_1 \ldots z_{n-1}) \right| \leq C \frac{\left(1 + \sum_i^{n-1} |z_i|^2\right)^k}{\left(\min_j (\eta_j^2)\right)^l}, \quad |z|^2 := \sum_\mu |z_\mu|^2 \tag{5.2}
$$

This tube analyticity together with the Lorentz-invariance of the W's (a consequence of the invariance of the vacuum and the covariant transformation properties of the fields) yields the invariance under the complex Lorentz-group L^c:

$$
\begin{aligned}
L^c &= \{A, B\} \quad A, B \in SL(2, C) \\
\underline{z} &= \sigma_\mu z^\mu \rightarrow A \underline{z} B^*, \quad z^\mu z_\mu = inv.
\end{aligned} \tag{5.3}
$$

This complex extension is a rather direct consequence of the previous analyticity and the fact that the finite dimensional representations $D^{(A,\dot{B})}$ permit an extension to a transformation in which the undotted and dotted spinors transform independently. the details can be looked up in the literature. L^c has different from L only two (instead of four) connected components: det=±. It takes some additional calculations to prove that the extended tube $T_{ext}^{(n-1)} = L^c T^{(n-1)}$. this is a natural analyticity region and the W are univalued . It is remarkable that T_{ext} contains real points. It is easy to see that the convex real set: $\xi_k^2 < 0 \quad (\sum_k \lambda_k \xi_k)^2 < 0 \quad \lambda_k \geq 0 \quad \sum_k \lambda_k > 0$ (the so called Jost points) is contained in T_{ext}. The locality binds all the n! different $\mathrm{w}^{(n)}(\mathrm{x}_{i_1} \ldots \mathrm{x}_{i_n})$ together to one (anty-)symmetric holomorphic "master function" with T_{ext}^{perm} the extended permuted tube being the enlarged analyticity region :

$$
w_n^{(\lambda_{i_1} \ldots \lambda_{i_n})}(z_{i_1} \ldots z_{i_n}) = \begin{cases} w_n^{(\lambda_1 \ldots \lambda_n)}(z_1 \ldots z_n) \\ sign P\, w_n^{(\lambda_1 \ldots \lambda_n)}(z_1 \ldots z_n) \end{cases} \tag{5.4}
$$

The mathematical structure behind this extension is the so called "edge of the wedge" theorem which generalizes the well known Schwartz reflection principle from one to several complex variables. The resulting "permuted extended tube" is not a natural holomorphy domain but its holomorphic completion is difficult (and fortunately physically not as relevant as it appeared during the 60^{ies}. For a discussion of this and related matters we refer to the literature. One physically relevant fact is the univaluedness of the masterfunction in d=3+1

theories. In d=1+1 the possibility of richer spacelike commutation relations (e.g. braid group statistics) which have multivalued master functions. Naturally its restriction to the real analytic (Jost) points is always univalued (the branching happens in the euclidean region) since otherwise the Hilbertspace setting of quantum physics would get lost.

The crucial question is now wether the family of w's with those properties following from the operator postulates suffice in order to reconstruct uniquely (up to isomorphism) the quantum field theory. From our experience with the GNS construction we would expect a positive answer. However the "field algebras" are not C^*-algebras of bounded operators and therefore a special construction which is more adapted to this problem is necessary. One uses the polynomial algebra $\mathcal{P}(M)$:

$$\left\{ f_0 \underline{1} + \sum_{n=1}^{N} \int \cdots \int f_n(x_1 .... x_n) A(x_1) .... A(x_n) \mid f_n \in \mathcal{S}(\mathrm{R}^{dn}), \forall N \right\} \tag{5.5}$$

Here we have again surpressed all Lorentz- and charge-indices i.e. used our standard neutral scalar illustration. As in the case of CCR and CAR we can interprete the expectation values $w^{(n)}$ as affiliating a positive linear functional on a *- algebra of test functions:

$$f = \{f_0 .... f_N\} \in \bigoplus_n \mathcal{S}(R^{dn}) \equiv T\mathcal{S} \tag{5.6}$$

$$(f \cdot g)_n(x_1 .... x_n) = \sum_k f_k(x_1 .. x_k) g_{n-k}(x_{k+1} .. x_n) \tag{5.7}$$

$$(f^*)_n(x_1 .... x_n) = \overline{f_n(x_n .... x_1)} \tag{5.8}$$

Note that different from the CCR and CAR case this is not a Hilbertspace of "one particle" functions but a tensor algebra $T(M)$ on sequences of functions. Here M indicates that the testfunction space consists of functions on Minkowski-space. The localized polynomial algabra $\mathcal{P}(\mathcal{O})$ is a subalgebra of $\mathcal{P}(\mathcal{M})$. The vacuum expectation values w_n just define a state (positive definite normalized functional) on T :

$$w(f) = \sum_n W_n(f_n), \quad w(f^* f) \geq 0 \tag{5.9}$$

In the operator way of writing this is just the positivity of the norm squared:

$$\left\| \left(f_0 \underline{1} + \sum_{n=1}^{N} \int \cdots \int f_n^{(\lambda_1 .... \lambda_n)}(x_1 .... x_n) A^{(\lambda_1)}(x_1) .... A^{(\lambda_n)}(x_n) \right) |0\rangle \right\|^2 \geq 0 \tag{5.10}$$

With an appropriately defined action α of $\mathcal{P}$ on the tensoralgebra, W is covariant:

$$w(\alpha_{a,\Lambda}(f)) = w(f) \tag{5.11}$$

The reconstruction is completely analogous to the GNS situation. One obtains a triple (H, π, Ω) i.e. a * representation of the Borchers-Uhlmann tensor-algebra

which is covariant with positive spectrum and a unique vacuum vector Ω. Certain properties as the time-slice property and its local version have no known equivalent in terms of correlation function; they need the reconstructed operator theory for their formulation.

5.2 Simple Structural Properties

1. The Cluster decomposition property. Its weak form is defined:

$$\begin{aligned} \lim_{\lambda\to\infty} W(f\alpha_{\lambda x}(g)) &= \lim_{\lambda\to\infty} \langle 0|A_f^* U(\lambda x) B_g|0\rangle = W(f)W(g) \qquad (5.12) \\ A_f, B_g &\in \mathcal{P}(M) \end{aligned}$$

and results from the fact that only the discrete part of the energy momentum spectrum (i.e.the unique vacuum contribution) survives, whereas the continuum oscillates to zero (Riemann-Lesbegue Lemma). The strong form is coveniently formulated in terms of the connected correlation functions:

$$\begin{aligned} &\langle 0|A_{f_1}(x_1)....A_{f_n}(x_n)|0\rangle_{con} \overset{clustering}{\to} 0 \qquad (5.13) \\ A_f(x) &= U(x)A_f U^*(x), \quad A_f \in \mathcal{P}(M) \end{aligned}$$

It uses locality (in order to disentangle overlapping clusters) and needs more mathematical efford for its derivation from the postulates. Note that a vacuum degeneracy would show up in form of a very specific violation (containing information about the dimension of the vacuum projector) of the cluster property.

2.The Reeh-Schlieder Theorem. The localized polynomial algebra $\mathcal{P}(\mathcal{O})$ is cyclic and separating on Ω, i.e.

$$\overline{[\mathcal{P}(\mathcal{O})\Omega]} = H, \quad A\Omega = 0 \curvearrowright A = 0, \quad A \in \mathcal{P}(\mathcal{O}) \qquad (5.14)$$

For the cyclicity assume that $\psi \in [P(O)\Omega]^{\perp}$, $\psi \neq 0$. Then for $A_i \in \mathcal{P}(\tilde{O})$, $\tilde{O} \ll O$ (no boundary touching) define:

$$\begin{aligned} F(x_1....x_n) &= \langle \psi|\alpha_{x_1}(A_1)....\alpha_{x_n}(A_n)|0\rangle \quad \psi \in \mathcal{D} \qquad (5.15) \\ &= 0 \quad \text{on } \left\{(x_1....x_n) \mid x_i \in V, \tilde{O}+V \in O\right\} \end{aligned}$$

The Fourier transform $\tilde{F}(p_1....p_n)$ vanishes outside the support: $\cap_l \{\sum_l^n p_i \in \bar{V}^+\}$ as follows from the spectrum condition. Therefore also the matrixelement F enjoys tube analyticity in $z_1....z_n$ (instead of n-1 z's as the W's). They agree with the (obviously holomorphic) zero function in the above real neighbourhood. The already mentioned mulidimensional generalization of the Schwartz reflection principle termed "edge of the wedge theorem" will then lead to the identical vanishing which contradicts the assumption of nontriviality of ψ. The proof of separability of Ω with respect to $\mathcal{P}(\mathcal{O})$ can be reduced to cyclicity by using locality. We have:

$$AA'\Omega = A'A\Omega, \quad A \in \mathcal{P}(\mathcal{O}), \quad A' \in \mathcal{P}(\mathcal{O}') \qquad (5.16)$$

Since $\mathcal{O}'$ is nonvoid, $\mathcal{P}(\mathcal{O}')$ acts cyclically on Ω and therefore $A'\Omega = 0 \curvearrowright A' = 0$ on the dense set $\mathcal{P}(\mathcal{O})\Omega$ and hence $A \equiv 0$.

With the Reeh-Schlieder theorem we have met the first characteristic property of local quantum physics. It has no counterpart in Schrödinger theory and general quantum theory. Indeed the idea that one can emulate vacuum exitations "behind the moon" by operating with hardware localized on an earthly laboratory with increasing accuracy,souds somewhat exotic. It has led to many misunderstandings especially with respect to causality (One of the more spectacular conceptional mistakes even cast doubt on Fermi's conclusion that Einstein's causality statements about classical relativistic field theory are also valid in QFT). On the positive side this property led to deeper thoughts about long range correlation and the proper operational formulation of causality and phase space localization of degrees of freedom (nuclearity).

3. Irreducibility of $\mathcal{P}(M)$ Starting from the time-development automorphism which (according to the positive energy assumption) is implemented by a positive hamiltonian:

$$\alpha_t(A) = e^{iHt}Ae^{-iHt}, \quad H \geq 0, \quad A \in \mathcal{P}(\mathcal{M}) \tag{5.17}$$

we study the analytic properties of matrixelements of time translated operators from the commutant:

$$\begin{aligned} f(t) \quad &: \quad = \langle A_1\Omega \,|e^{-iHt}A'e^{iHt}|\, A_2\Omega\rangle = \langle \Omega \,|A_1^*\alpha_{-t}(A')A_2|\,\Omega\rangle \qquad (5.18)\\ A_i \quad &\in \quad \mathcal{P}(\mathcal{M}), \quad A' \in \mathcal{P}(\mathcal{M})', \quad \text{where:}\\ \mathcal{P}(\mathcal{M})' \quad &: \quad = \{C \mid \langle A^*\phi, C\psi\rangle = \langle \phi, CA\psi\rangle \ \forall A \in \mathcal{P}(\mathcal{M}),\, \phi,\psi \in \mathcal{D}\} \end{aligned}$$

One computes:

$$\begin{aligned} f(t) \quad &= \quad \langle A'^*\Omega, \alpha_t(A_1^*A_2)\Omega\rangle = \langle A'^*\Omega, e^{itH}A_1^*A_2\Omega\rangle \qquad (5.19)\\ &= \quad \langle \alpha_t(A_2^*A_1)\Omega, A'\Omega\rangle = \langle A_2^*A_1\Omega, e^{-itH}A'\Omega\rangle \end{aligned}$$

The first line represents f(t) as a matrixelement of e^{itH}, and the second of e^{-itH}. Therefore f is a bounded function which is simultaneously analytic in the upper and lower halfplane. According to a theorem of Liouville this forces f to be a constant i.e.

$$\begin{aligned} f(t) \quad &= \quad \langle A'^*\Omega, E_0A_1^*A_2\Omega\rangle = \langle \Omega, A'\Omega\rangle \,\langle A_1\Omega, A_2\Omega\rangle\,, \; E_0 = \text{proj on vac.}\\ &\curvearrowright \quad A' = \langle \Omega, A'\Omega\rangle \cdot \mathbf{1} \end{aligned}$$

4. TCP Symmetry. We first remind ourselves of the TCP-transformation property of free fields:

$$\begin{aligned} \Theta\,|p,\lambda,i\rangle \quad &= \quad \sum_{\lambda'} |p,\lambda',\bar{i}\rangle\, D_{\lambda',\lambda}(i\sigma_2) \quad \text{in } H_1, \text{ antilinear} \qquad (5.20)\\ \bar{i} \quad &: \quad \text{antiparticles of type i, } \Theta^2 = (-1)^{2s}\mathbf{1} \end{aligned}$$

$$\Theta\Phi^{[A,\dot{B}]}(x)\Theta^{-1} = (-i)^F(-1)^{|\dot{B}|}\Phi^{[A,\dot{B}]*}(-x) \equiv \Phi^{\theta}(-x) \quad (5.21)$$

F # of fermions, $\left|\dot{B}\right|$ = # of dotted spinor indices

We will show that this formula holds in general (for local interacting fields). If $\Phi = A$ =scalar field, the proof startes from first rewriting the content of TCP symmetry in terms of correlation functions:

$$\begin{aligned} & \langle \Theta A(x_m)....A(x_1)\Omega, \Theta A(x_{m+1})....A(x_n)\Omega\rangle \\ = \; & \langle A(x_{m+1})....A(x_n)\Omega, A(x_m)....A(x_1)\Omega\rangle \end{aligned} \quad (5.22)$$

$$\Leftrightarrow w(-x_1, ... - x_n) = w(x_n....x_1)$$

where in the last line we used the above Θ-action. We now take notice of the fact that by combining the symmetry relation from locality in T_{ext}^{perm} with the $L_+(C)$ invariance (which included the total reflection) we have:

$$w(x_n....x_1) = w(x_1....x_n) \overset{L_+(C)}{=} \quad (5.23)$$

$$w(-x_1.... - x_n) \quad (5.24)$$

This means that we obtain the above relation in T and hence on the physical boundary (the boundary $i\varepsilon$-prescription in the above relation is the same on both sides) which is the desired relation for the operators.

5. Spin & Statistics. If we require the wrong local commutation relations for $\Phi^{[A,\dot{B}]}$:

$$\begin{aligned} \{\Phi(x)\Phi(y)\} &= 0, \quad (x-y)^2 < 0, \quad A + \dot{B} = \frac{n}{2}, \; n \text{ odd} \\ [\Phi(x)\Phi(y)] &= 0 \quad (x-y)^2 < 0, \quad n \text{ even} \end{aligned} \quad (5.25)$$

then $\curvearrowright \Phi \equiv 0$. With other words within the framework of local fields, the standard relation between spin and statistics is a consequence of the postulates. The proof (again for neutral scalar fields) only employs the two-point function W(z) which is analytic in T_{ext} and fulfills (as a consequence of $L_+(\mathbf{C})$-invariance) W(z)=W(-z):

$$\begin{aligned} \langle 0\,|\{\Phi(x), \Phi(y)\}\,|\,0\rangle &= W(\xi) + W(-\xi) = 0, \; (x-y)^2 < 0 \\ &\curvearrowright \; 2W(z) = 0, \quad \curvearrowright \Phi(x)\Omega = 0 \end{aligned} \quad (5.26)$$

Finally the Reeh-Schlieder theorem gives $\Phi \equiv 0$. The general case with dotted and undotted spinors is left to the reader.

The TCP and Spin&Statistics theorem are considered to represent the deeper parts of structural QFT. They even gave the title for the first monography on the subject. Later we will see that they continue that role in algebraic QFT with an additional gain in profoundness (in particular with low dimensional QFT)..

6. Normal C.R. and Klein transformations. The previous theorem left open the commutation relations between different Lorentz-multiplets. One

defines as "normal" the spacelike commutativity of two local fields of which at least one is bosonic, as well as the spacelike anticommutativity between two fermionic (A+B=halfinteger) fields. As a preparatory step towards proving that normal commutation relation can always be achieved, let us prove that commutation relations remain stable under transition to the hermitian adjoint field:

$$\begin{aligned} [\Phi_1(x), \Phi_2^*(y)]_\pm &= 0 \quad (x-y)^2 < 0 \\ &\curvearrowright \quad [\Phi_1(x), \Phi_2(y)]_\pm = 0 \quad (x-y)^2 < 0 \end{aligned} \tag{5.27}$$

The proof uses the cluster decomposition property (i.e. the uniqueness of the vacuum):

$$\begin{aligned} & \langle \Omega, \Phi_1^*(f)\Phi_2^*(g)\Phi_2(g)\Phi_1(f)\Omega \rangle \\ = \; & \|\Phi_2(g)\Phi_1(f)\Omega\|^2 \geq 0 \\ \overset{loc.}{=} \; & \sigma \langle \Omega, \Phi_1^*(f)\Phi_1(f)\Phi_2^*(g)\Phi_2(g)\Omega \rangle \\ \overset{cluster}{\Rightarrow} \; & \sigma \|\Phi_1(f)\Omega\|^2 \|\Phi_2(g)\Omega\|^2 \end{aligned} \tag{5.28}$$

Here $\sigma = \pm$. Consistency requires that σ agrees with the ± 1 in the original c.r. between Φ_1 and Φ_2^*. Now we are prepared to construct the Klein transformation which carries anomalous into normal c.r. For the typical anomalous situation assume that:

$$\begin{aligned} [\varphi(x), \psi(y)] &= 0, \quad (x-y)^2 < 0 \\ \varphi &: \text{ bosonic}, \quad \psi : \text{fermionic} \end{aligned} \tag{5.29}$$

Define:

$$\begin{aligned} \varphi'(x)H_{even} &= \varphi(x)H_{even} \\ \varphi'(x)H_{odd} &= -\varphi(x)H_{odd} \\ \psi'(x)H_{even} &= \psi(x)H_{even} \\ \psi'(x)H_{odd} &= \psi(x)H_{odd} \end{aligned} \tag{5.30}$$

or briefly: $\varphi'(x) = (-1)^F \varphi(x), \quad \psi'(x) = \psi(x)$ with

$$\{\varphi'(x), \psi'(y)\} = 0, \quad (x-y)^2 < 0 \tag{5.31}$$

The general situation is analogous.

7. Characterizations of free fields. Assume first that the two-point function agrees with that of a free field, i.e.

$$\begin{aligned} \langle \Omega, \varphi(x)\varphi(y)\Omega \rangle &= i\Delta^{(+)}(x-y) \curvearrowright \\ \varphi(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int (e^{-ipx} a(p) + h.c.) \frac{d^3p}{2\omega} \end{aligned} \tag{5.32}$$

in case of a neutral scalar field. The first step in the proof consists in deriving the Klein-Gordon equation $(\partial^\mu\partial_\mu + m^2)\varphi(x) = 0$. From the two-point function one obtains

$$\begin{aligned} \langle\Omega, j(x)j(y)\Omega\rangle &= 0, \quad \text{where } j(x) := (\partial^\mu\partial_\mu + m^2)\varphi(x) \\ &\curvearrowright \; j(x)\Omega = 0 \end{aligned} \tag{5.33}$$

The analytic properties in the tube T of the mixed $j - \varphi$ correlation functions together with the relative spacelike commutativity bring about an edge of the wedge situation with:

$$\begin{aligned} &\langle\Omega, \varphi(x_1)....j(x_i)\varphi(x_{i+1})....\Omega\rangle \\ = \; &\langle\Omega, \varphi(x_1)....\varphi(x_{i+1})....j(x_i)\Omega\rangle = 0 \end{aligned} \tag{5.34}$$

on an open set of the boundary and hence the vanishing of all matrix elements of j on the dense domain $\mathcal{D}$ i.e. j(x)$\equiv$ 0. Therefore φ indeed fulfills the free field equation and hence permits a frequency decomposition:

$$\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x), \quad \varphi^{(-)}(x)\Omega = 0 \tag{5.35}$$

A characterizing property of free fields is their c-number (anti)commutator, in our case:

$$[\varphi(x), \varphi(y)] = i\Delta(x - y)1 \tag{5.36}$$

But this follows by again using analyticity properties. First we use the spectrum condition to obtain:

$$\varphi^{(-)}(x)\varphi^{(+)}(y)\Omega = i\Delta^{(+)}(x - y)\Omega \tag{5.37}$$

since $\varphi^{(\pm)}$ transfers momentum on the forward (backward) mass shell and hence the spectral transfer of the product is spacelike +zero i.e. the intersection with the physical spectrum consists of just p=0 corresponding to the vacuum vector Ω. This is a much stronger statement than the assumed two-point function structure. For the commutator applied to Ω we now have:

$$[\varphi(x), \varphi(y)]\,\Omega = i\Delta(x - y)\Omega + \left[\varphi^{(+)}(x), \varphi^{(+)}(y)\right]\Omega \tag{5.38}$$

Let $\psi \in \mathcal{D}$ and consider the analytic properties of:

$$F(x, y) := \left\langle \psi, \left[\varphi^{(+)}(x), \varphi^{(+)}(y)\right]\Omega \right\rangle \tag{5.39}$$

Since the momentum transfer of each $\varphi^{(+)}$ is on the forward mass shell, this distribution is the boundary value of a funtion $F(z_1, z_2)$ analytic in $z_i = x_i - iy_i$, $y_i \in V^+$. Since this function vanishes in a neighborhood of the real boundary, $\curvearrowright F \equiv 0$ .But:

$$B(x, y) := [\varphi(x), \varphi(y)] - i\Delta(x - y)1 \tag{5.40}$$

is a bilocal operator-valued distribution ($B(f,g) \in \mathcal{P}(\mathcal{O})$) for which Ω is a separating vector, i.e.

$$B(x,y)\Omega = 0 \curvearrowright B(x,y) \equiv 0 \quad q.e.d. \tag{5.41}$$

This property has no analogon in quantum mechanics i.e. the interaction cannot be seen by considering two-point funtions (related to the absentselfinteraction).

Remembering that a free field has vanishing connected n-point correlation functions for n>2, the question arises whether this property is typical. The affirmative answer is:

$$\begin{aligned} \text{if } \exists n \;>\; 2 \quad & w_n(x_1,....x_n)^{conn} = 0 \\ \curvearrowright \quad & \varphi(x) \text{ is a generalized free field} \end{aligned} \tag{5.42}$$

A generalized free field shares with the free field the property of having a c-number commutator. But this commutator is a (continuous) superposition of free field commutators:

$$[\varphi(x),\varphi(y)] = \int d\rho(\kappa^2) i\Delta(x-y,\kappa^2) \tag{5.43}$$

We will not prove the above statement here. Another characterization of (generalized) free fields not presented here is in terms of gaps in the spacelike momentum transfer of fields. Reductions to generalized free fields are reductions to free fields in view of the time slice property and in particular the phase space nuclearity property presented in a later section.

8. Shape of Energy Momentum Spectrum. The asymptotic factorization or clustering of correlation functions suggests that the energy-momentum spectrum **specP** is an additive set i.e. with $\mathbf{p}_1, \mathbf{p}_2 \in$ **specP** $\curvearrowright \mathbf{p}_1+\mathbf{p}_2 \in$ **specP**. To see this consider the vector:

$$\psi_{21}(a) = U(a)A_2U^*(a)A_1\Omega, \quad A_i \in \mathcal{P}(\mathcal{M}) \tag{5.44}$$

Assume that the energy-momentum transfer is limited to regions $\Delta_i \in$ **specP**. Then the Fourier-transform of ψ has its support in $\text{supp}\tilde{\psi} \in \Delta_1 + \Delta_2$. The clustering:

$$\begin{aligned} \lim \|\psi_{21}(a)\|^2 \;&=\; \langle\Omega, A_2^*(a)A_2(a)\Omega\rangle \, \langle\Omega, A_1^*A_1\Omega\rangle \\ &=\; \|\psi_2\|^2 \, \|\psi_1\|^2, \quad \psi_i = A_i\Omega \end{aligned} \tag{5.45}$$

serves to show that $\|\psi_{21}(a)\| \neq 0$ i.e. does not vanish identically thus assuring the nontriviality of the vector carrying the sum of the momenta.

Classically the hyperbolic causal propagation in classical field theory is inexorably linked with Lorentz-covariance. By analogy one would expect that causality, even if it does not extend translational covariance to full Poincaré covariance, at least forces the energy-momentum spectrum to have a Lorentz-invariant shape. Indeed, the implementer of the translation can always be chosen in such a way:

$$\begin{aligned} \exists U(a) s.t. \alpha_a(A) \;&=\; U(a)AU^*(a) \\ U(a) \;&=\; e^{iPa}, \quad \textit{spec}\mathbf{P} \text{ inv. shape} \end{aligned} \tag{5.46}$$

This theorem is easier to prove in an algebraic setting and hence will be deferred.

5.3 Euclidean Fields

Analytic continuations through Wick-rotation have been useful in perturbation theory because certain regularization techniques only work if noncompact L-invariance can be replaced by compact euclidean invariance. Therefore it is interesting to know if a euclidean formulation is also possible outside perturbation theory and whether it is useful. Schwinger and later Symanzik were the first to realize that a euclidean framework opens a useful connection with statistical mechanics.

The starting point for a nonperturbative euclidean approach is the analyticity and univaluedness of the analytic extension of correlation functions into the extended permuted tube T^{ext}_{perm}. It is obvious that the noncoinciding $(\hat{x}_i \neq \hat{x}_j \forall i,j)$ euclidean points are inside this domain. The Wick-rotation $(\vec{x}, x^0) \to (\overrightarrow{\hat{x}} = \vec{x}, \hat{x}^0 = ix_4)$ relates the Minkowski inner product with the Euclidean one: $x^\mu x_\mu = -\sum_{i=1}^4 \hat{x}_i^2$ and the group $O_+(4)$ with a subgroup of $L_+(C)$. Here and in most of what follows we present the euclidean formulation for integer spin fields, the adaption to halfinteger spin will be commented on later. The restriction of the analytically continued correlation functions to the euclidean points $(\vec{x}, x_4) \in \mathbf{E} = R^d$ are called Schwinger functions:

$$\begin{aligned} s(x_1 .... x_n) &= w(\hat{x}_1 .... \hat{x}_2) \\ &= S(\xi_1 .... \xi_{n-1}) \end{aligned} \tag{5.47}$$

where we used translation invariance in the last line. As for time-ordered functions there is no spectrum condition which assures that they are distributions on the Schwartz-space $\mathcal{S}$, rather their natural domain of definition are those test-functions which vanish at coinciding points of sufficiently high order. If the dimension of the fields is canonical i. e. for scalar fields $\dim\varphi = \frac{d}{2} - 1 = \dim$(free field) then S is naturally (i.e. without Hahn-Banach extension) integrable and hence a $\mathcal{S}(\mathbf{E}^n)$ distribution. We now collect those properties of the Schwinger functions which allow to reconstruct a local Poincaré-invariant QFT. These properties are called the Osterwalder-Schrader axioms. In the following we present these axioms for the illustrative case of scalar neutral fields.

- **S1** The Schwinger functions are translation invariant real analytic function for noncoinciding euclidean variables. They are distributions in $\mathcal{S}'(\mathcal{E}_-^{n-1})$ with $\mathcal{E}_-^{n-1} = \{\xi \in \mathbf{E}^{n-1} \mid \xi_1^4 < \xi_2^4 .... < \xi_n^4\}$ where $\mathcal{S}(\mathcal{E}_-^{n-1})$ is given a weaker topology which is defined by the following system of seminorms:

$$\begin{aligned} \|f_<\|_{l,m} &= \left\|\tilde{f}\right\|_{l,m} \quad f_< \in \mathcal{S}(\mathcal{E}_-^{n-1}) \\ \tilde{f}(q_1 .... q_{n-1}) &= \int .... \int e^{\sum_1^{n-1}(\xi_j^4 q_j^0 + i\, ; q_j)} f_<(\xi_1 .... \xi_{n-1}) d^d\xi_1 .... d^d\xi_{n-1} \end{aligned} \tag{5.48}$$

Here we used the property of the Laplace-Fourier transforms of mapping continuously $\mathcal{S}(\mathcal{E}_-^{n-1})$ onto a dense set in $\mathcal{S}(\mathcal{M}_+^{n-1})$ which are Minkowski-space test functions $\tilde{f}$ with

$$supp\tilde{f} \in \{q \in \mathrm{M}^{n-1} \mid q_i^0 \geq 0 \ \forall i\} \equiv \mathcal{M}_+^{n-1} \tag{5.49}$$

The above topology is the one which $\mathcal{S}(\mathcal{E}_-^{n-1})$ inherits from $\mathcal{S}(\mathcal{M}_+^{n-1})$ through this map. The Schwinger distributions are just the continuous linearfunctionals on $\mathcal{S}(\mathcal{E}_-^{n-1})$ in this topology. It is the analoge of the growth condition on the holomorphically extended correlation functions W which insured the temperedness of their distributional boundary values and often called the Osterwalder-Schrader growth condition.

- **S2.** Hermiticity. For the Schwinger functions of a scalar neutral field:

$$\begin{aligned} s(x_1....x_n) &= \overline{s(Tx_1....Tx_n)} \\ Tx &= (\vec{x}, -x_4) \quad \text{euclidean time reversal} \end{aligned} \tag{5.50}$$

- S3. Reflection-Positivity:

$$\sum_{n,m} \int s(Tx_m, ..., Tx_1, y_1, ...y_n)\overline{f_m(x_1, ...x_m)}f_n(y_1, ...y_n)d^dx_1...d^dy_n \geq 0 \tag{5.51}$$

where the sum only involves a finite sequence of test functions $(f_0, f_1...f_n...f_N)$ with their support on the time wedges $\mathrm{E}_<^n = \{x \in E^n \mid 0 < x_1^4 < ... < x_n^4\}$. Clearly this property is the analogon of the Wightman-positivity for the W's. In fact it results from the positivity of "euclidean states":

$$\psi(x_1....x_2) = \varphi(\vec{x}_1, ix_1^4)....\varphi(\vec{x}_n, ix_n^4)\Omega, \quad x \in \mathrm{E}_<^n \tag{5.52}$$

Note that the spectrum condition allows to interpret the analytic continuation as a smearing with a an exponential damping factor(fast decreasing test function in time).

- **S4.** Euclidean covariance:

$$s(Rx_1....Rx_n) = s(x_1....x_n) \tag{5.53}$$

- **S5.** Permutation symmetry:

$$s(x_{P(1)}, ...x_{P(n)}) = s(x_1....x_n) \tag{5.54}$$

- **S6.** Cluster property:

$$\begin{aligned} &\lim_{a\to\infty} \int s_n(x_1, ...x_m, x_{m+1} + a, ..., x_n + a)f(x_1, ...x_m)g(x_{m+1}, ...x_n) \\ = &\int s_m(x_1, ...x_m)f(x_1, ...x_m) \times \int s_{n-m}(x_{m+1}, ...x_n)g(x_{m+1}, ...x_n) \end{aligned} \tag{5.55}$$

The generalization to charged fields with arbitrary finite spin is obvious: the covariance law involves the representations of the $SU(2) \times SU(2)$ covering of $O(4)$ and the permutation symmetry carries an additional $sign(P)$

It is fairly obvious that a theory in terms of correlation functions fulfilling positivity, hermiticity, P-covariance and locality leads to Schwinger functions fulfilling S1-S6. One just defines euclidean vectors $\psi(x_1, ...x_n)$ as above. The reflection positivity allows to equip the linear vector space:

$$\left\{ \sum_{n=1}^{N} \int .. \int f_n(x_1, ...x_n)\psi(x_1, ...x_n) \mid f_n \in \mathcal{S}(\mathbf{E}^n_<) \right\} \qquad (5.56)$$
$$\mathcal{S}(\mathbf{E}^n_<) \text{ with } \|f\|_{l,m} - \text{topology}$$

with a positive semidefinite inner product. Factoring out the null-space and forming the closure one obtains a euclidean Hilbertspace which thanks to the Reeh-Schlieder theorem is equal to the GNS space of the real time correlation functions. The short-distance growth condition of the W's in the tube (controlling the temperedness of the distributional boundary values) are equivalent to the $\|\cdot\|_{l,m}$ topology of the Schwinger functions:

$$s(x....x) = \langle \Omega, \psi(x....x) \rangle \qquad (5.57)$$

The permutation symmetry of s is a result of that symmetry for the analytic w's (from locality). Actually already the $\psi's$ are symmetric as real analytic functions in the euclidean domain for $x_i \neq x_j, i \neq j, x_j^4 > 0$, as can be shown by he application of the edge of the wedge theorem. Note that the Osterwalder-Schrader (euclidean) reflection positivity S3 cannot be interpreted as a state on a *-algebra (the Borchers-Uhlmann tensor algebra of functions) but only serves to define a scalar product.on a linear space (finite sequences of test functions $f_n \in S(E^n_<)$. The reconstruction of the real time theory can then be carried out in two different ways. Either one uses fuctional analysis (contractive properties of space-time semigroups) or the analytic properties of the previous Laplace-Fourier transforms in S1 which relate the Schwinger distributions $\in S(\mathcal{E}^{n-1}_-)$ to the spectral supported correlation functions $\tilde{W} \in \mathcal{S}(\mathcal{M}^{n-1}_+)$ and carries the reflection positivity into the W positivity. The latter method is more appropriate in the present context whereas the first method also works in situations without space-time analyticity e.g. the derivation of the transfer matrix formalism in clsssical statistival mechanics on a lattice (see a later section). We collect the result:

Theorem 12 ***(Osterwalder-Schrader) Every set of Schwinger functions with S1-S6 comes from a real time QFT by analytic continuation and restriction to the euclidean points.***

The euclidean framework described here is primarily a structural reformulation, it does not really solve any problem of the real time theory. In fact

even from a mathematical viewpoint it looks somewhat mocked up, since the topology we used on $S(E^n_<)$ is not natural. Only under very special circumstances it becomes a powerful constructive tool of QFT. This happens e.g. if the Schwinger functions allow an interpretation in terms of a continuous classical mechanics. Mathematically this amounts to the Feynman-Kac representability of the Schwinger functions in terms of a (infinite dimensional) functional measure theory e,g, (the φ^4-theory):

$$\begin{aligned} s(x_1,....x_n) &= \frac{1}{Z}\int [d\varphi]\, e^{-A[\varphi]}\varphi(x_1)....\varphi(x_n) \\ A[\varphi] &= \frac{1}{2}(\partial\varphi\partial\varphi + m^2\varphi^2) + g\varphi^4 \end{aligned} \tag{5.58}$$

A physically fruitful formal interpretation is in terms of a continuous version of a Gibbs formula for classical statistical mechanics on a lattice:

$$\begin{aligned} \langle\varphi(x_1)....\varphi(x_n)\rangle_{Gibbs} &= \lim_{\Lambda\to\infty}\frac{1}{Z_\Lambda}\sum_{conf,\Lambda} e^{-\beta H_\Lambda[\varphi]}\varphi(x_1)....\varphi(x_n) \\ Z_\Lambda &= \sum_{conf,\Lambda} e^{-\beta H_\Lambda[\varphi]} \end{aligned} \tag{5.59}$$

Here the dynamical variables φ over each lattice point take on either values in a discrete (e.g. Z_n) or continuous manifold (e.g. C, SU(2) etc.) in which case the sum over configurations represents an integral over the field values at each lattice point within the volume Λ. There are two questions to be asked: (i) can one work out a measure theory for stochastic variables such that the above functional integrals have mathematical meaning? , (ii) can one controll "critical limits" (second order phase transitions) of classical statistical mechanics precisely enough in order to obtain possibly existing local QFT? Deferring the second problem to a later section, we comment here only on the first one, namely the relation between a Nelson-Symanzik stochastic euclidean theory and realtime QFT. Euclidean fields are continuous linear maps ϕ from test function spaces $\mathcal{S}(E^d)$ into random variables over a probability space (Q,Σ,μ) with μ a normalized measure on Q and Σ the μ-measurable subsets. Let us define a generating functional W for the euclidean correlation functions of ϕ in a reference state (the euclidean "vacuum") which has the following properties:

$$S(f) = \int_Q e^{i\phi(f)}d\mu, \quad i.e. S(0)=1,\ S(f)=\overline{S(-f)}, \tag{5.60}$$

$S(f)$ is of positive type and invariant under euclidean transf. and "time" reflections

Here we may any axis declare to be the time axis. According to a famous theorem of Minlos this measure-theoretical setting is equivalent to the following (Nelson-Symanzik) positivity and covariance properties of the functional S(f)

(the functional fourier-transform of μ):

$$\sum_{i,j=1}^{n} \bar{c}_i c_j S(f_i - f_j) \geq 0, \quad S(0) = 1 \tag{5.61}$$

$$S(f) = S(\vartheta f), \quad S(f) = S(\alpha_{a,R} f)$$

the last line expressing the time reflection ϑ (the choice of the time axis is arbitrary) and euclidean invariance. In addition S(f) is continuous on $\mathcal{S}$ in the Schwartz topology.

This euclidean field setting is obviously appropriate for the Feynman-Kac representation which assumes that the measure μ on the space of field configurations is given by an invariant statistical mechanics-like local "hamiltonian" which consists of a quadratic free and a polynomial interacting part. We already know that the validity of the reflection positivity is a prerequisite for obtaining real time local quantum physics. It is not difficult to prove that such a stochastic euclidean theory with reflection positivity is equivalent to a special class of real time QFT namely the so called stochastic positive QFT.

Definition 4 *A QFT is said to fulfill stochastic positivity if its associated von Neumann algebra $\mathcal{A}$ contains an abelian subalgebra $\mathcal{B}$ ("fields at one time") and an automorphism α_t("time translation") such that: $\overline{\bigcup_t \alpha_t(\mathcal{B})} = \mathcal{A}$*

Theorem 13 *(Klein-Landau) A reflection positive stochastic euclidean theory is equivalent to a stochastic positive real time QFT.*

Hence the equivalence requires the stochastic theory to have an additional QFT positivity property (reflection positivity) and the QFT to pocess an additional stochastic (Nelson-Symanzik) positivity. We will not prove this theorem since our main motivation here is to counteract the erronous but widespread belief that QFT can be always be defined in terms of measure theory or Feynman-Kac Formulas. Only theories which "stay close" to the d=1+1 ϕ^4-theory (the standard relativistic illustrative example of the above theorem) allow for a Feynman-Kac representation. Whatever the intuitive appeal of Lagrangian quantization and functional integrals may be worth, one of its conceptual and mathematical limitation is set by the above theorem.

Note that we are here not concerned with mathematically fine points caused by renormalization (e.g. ϕ^4 in d=1+2 or d=1+3) wrecking the canonical (equal time) structure. Rather we mean that certain theories are structurally incompatible with Feynman-Kac representations. Examples are chiral conformal theories and d=1+2 theories with braid group statistics (Chern-Simons actions). They are easily shown to fail on the stochastic positivity property. The reason is the nonexistence of an abelian subalgebra with the required density property. There are indications that combinatorical or topological field theories result from imposing singular states on Weyl like algebras. This is again looking like a promising clash between the geometric approach via quantization and functional formalism and the algebraic approach relating directly to physical principles. To

be clear on this point, nobody is forbidden to write down functional integrals and study their properties also for those cases. The misunderstandings only start with the claim (generally made by physicist under the spell of differential geometry) that this can be taken (apart from mathematically fine points of regularizing such ∞-dim. integrals) as the *definition of quantum physics*. The interesting aspect of the above theorem is that it limits the myth of a general functional integral "quantization" as a parallelism of quantum with classical physics (for more details we refer to B.Schroer, Rev. in Math Phys.,Vol 7, No.4 (1995), page 669). Although there are formal-intuitive arguments that d=2+1 anyons and plektons are in some way related to the quantized Chern-Simon structure nobody has been able to extract a practically useful description of "free" fields describing such objects which could match the clear formalism of free Bosons and Fermions. For this reason we prefer to study this problem by starting from the point of view of Wigner's analysis of d=2+1 irreducible representations of the Poincaré-group with abelian spin $\neq$ (semi)integer together with the powerful formalism of algebraic scattering theory (see chapter 7, last section).

A closely related, conceptionally more robust constructive idea is to try to define QFT as scaling limits of mathematically controllable lattice systems. The guiding priciple going back to Kadanoff, Wilson and others was to use the possible existence of second order phase transitions ("criticality") to loose the memory of the lattice and recover $\mathcal{P}$-covariance and locality. This approach always has a "light" start since the mathematical controll of lattice systems is rather simple. But in the last step, the investigation of criticality and the execution of the scaling limit, one has to pay heavily for the easy life at the beginning. The mere control of existence is not enough, the last step requires a deep structural understanding of the lattice theory. Wheras it is true that most of the QFT concepts as conserved charges, particles, multiparticle scattering, antiparticles etc. can be transferred to the lattice (albeit with much more sweat, since the helpful causality structure is absent), a sufficiently detailed structural controll is only possible under special circumstances as integrability (meaning the Yang-Baxter structure for 2-dim. lattice systems). This kind of temporary practical restriction is quite different from the above restriction through Feynman-Kac representability. In particular there is no limitation on the short-distance behaviour: the operator short-distance dimensions of e.g. the Ising, RSOS etc. models is too far away from canonicity as required by the euclidean F-K approach. Real time short distance singularities which go significantly byond canonical behaviour do not threaten the existence of real time QFT but only limit certain methods as quantizations by funtional integrals. Although we do not really advocate a lattice approach, the reader can find detais on this subject in a later section. Our main constructive contribution (presented only after the chapter on algebraic QFT) will be based on the net approach.

5.4 Constructive use of Euclidean Fields

The $P\phi_2$-models. Feynman-Kac representations for d=1+1 (dis)order fields

5.5 Scattering Theory

Whereas scattering theory in e.g. Schrödinger QM is very important for the comparision of theory with experiments but less so for the formulation and construction of quantum mechanical models, the S-operator takes on a more fundamental significance in local quantum physics. The reason is threefold: in addition to its standard role of permitting experimental verification of the theory, S is an invariant of the net (i.e. S is attached to a Borchers class and should not be affiliated with individual fields) and finally S is related to the modular reflection J for the wedge algebra and the TCP-operator θ by $S = JJ_0 = \theta\theta_0$ where the subscript zero refers to the incoming fields (considered as a free theory). In this section we will present the scattering content and the class invarince property of S. In the perturbative approach we already met the S-matrix as the adiabatic limit of S(j). But we also realized that from a conceptual point of view such limits should be avoided since that formalism is good for the local net properties, but becomes unnatural for the calculation of "on shell" quantities, in particular for the scattering operator. The conceptually most satisfying method is to first calculate the approximations for the correlation function and then to use the scattering theory for on shell quantities. Similar to the nonrelativistic theory, the main objective is to use the time dependent formulation because of its physical clarity but convert its content into analytically simple stationary formulas.

This aim is accomplished in the Lehmann-Symanzik-Zimmermann (LSZ) approach. As quantum mechanical time dependent scattering theory relates interacting wave functions for t$\rightarrow \infty$ with those of a free system, scattering theory in QFT should relate interacting (Heisenberg) fields with free fields. By checking with stationary external source models as well as with renormalized perturbation theory these authors proposed the following asymptotic condition (for the standard scalar situation):

$$\begin{aligned} lim_{t\to\pm\infty} \langle \phi | A_f(t) | \psi \rangle &= \langle \phi | A_f^{ex} | \psi \rangle \quad ex = out, in \qquad (5.62) \\ A_f(t) &= \int_{x^0=t} f(x) \overleftrightarrow{\partial}_0 A(x) d^3x \end{aligned}$$

Here f(x) is a solution of the Klein-Gordon equation, A_f^{ex} is defined by the same formula with A replaced by the free incoming or outgoing field (and therefore time-independent) and the state vectors ϕ, ψ are taken from a dense set of in states (with nonoverlapping wave functions in velocity space, as we know nowdays). Later the Haag-Ruelle formulation which is based on strong convergence was derived from the locality and spectral principles of QFT, from which 5.62 follows, was derived. But before we discuss these refinements, we will derive the useful LSZ reduction formulas.

Let us start with the reduction of an incoming particle in the following matrix-element:

$$\begin{aligned} & {}^{out}\langle f_{n+1}....f_{n+m}\,|A(x)|\,f_1....f_n\rangle^{in} \\ = \;& lim^{out}_{t\to-\infty}\,\langle f_{n+1}....f_{n+m}\,|A(x)A_f(t)|\,f_2....f_n\rangle^{in} \\ = \;& \left\langle f_{n+1}....f_{n+m}\left|\int K_y TA(x)A(y)f(y)d^4y\right|f_2....f_n\right\rangle^{in} + c.t. \end{aligned} \tag{5.63}$$

where T denotes the time ordering, K is the KLein-Gordon operator and c.t. (contraction terms) is the generic notation for terms in which f's in the in or out states have been contracted with resulting $(f_i, f_j)\times$lower terms (example: the annihilation part of A^{in}_f may contract with f_i in the in state if the overlapp is nonvanishing). In the third term the time ordering occurs since we want the outgoing boundary contribution in: $lim_{t\to-\infty}\{A_f(t) - A_f(-t)\}$ = volum term to appear on the left hand side of the local operators whose matrix elements we are reducing (then its contribution just produces outgoing contraction terms). The same statements apply verbatim to the reduction of outgoing states. The iterative application of this procedure clearly leads to to the following reduction formula:

$$\begin{aligned} & {}^{out}\langle f_{n+1}....f_{n+m}\,|A(x)|\,f_1....f_n\rangle^{in} \\ = \;& \int....\int \bar{f}_{n+1}(y_{n+1})...\bar{f}_{n+m}(y_{n+m})f_1(x_1)...f_n(y_n)\times \\ & K_{n+1}...K_{n+m}K_1...K_n\,\langle 0\,|TA(x)A(y_1)....A(y_{n+m})|\,0\rangle \end{aligned} \tag{5.64}$$

Instead of $A(x)$ we could have also started with any multilocal product of local fields. In the special case of A$\to$ **1** we obtain the reduction formula for the S-matrix:

$$\begin{aligned} & {}^{out}\langle f_{n+1}....f_{n+m}\,|\,f_1....f_n\rangle^{in} \\ & \int....\int \bar{f}_{n+1}(y_{n+1})...\bar{f}_{n+m}(y_{n+m})f_1(x_1)...f_n(y_n)\times \\ & K_{n+1}...K_{n+m}K_1...K_n\,\langle 0\,|TA(y_1)....A(y_{n+m})|\,0\rangle + c.t. \end{aligned} \tag{5.65}$$

By going to the limit of plane waves one obtains for theconnected part of the momentum space kernel of the S-matrix:

$$\begin{aligned} S(p_{n+1}..p_{n+m};p_1..p_n)^{conn.} \;=\; & lim_{p_i^2\to m^2}\prod_i(p_i^2-m^2)\times \\ & \tau(-p_{n+1}..-p_{n+m},p_1..p_n) \end{aligned} \tag{5.66}$$

i.e. the residua on mass shell of the Fourier transforms of the time ordered function τ. These reduction formulas are very suggestive of the so called crossing symmetry:

incoming particle p $\to$ outgoing antiparticle -p

Such a "symmetry" in order to be physically meaningful must however be interpretable as a relation between different boundary values of an on shell meromorphic master function. Although in renormalized perturbation theory this was true in each checked case, a proof of the necessary analyticity derived from the principles of QFT is only known in special cases i.e. the reduction formula is only suggestive but does not establish the crossing symmetry. A related question is the existence of time-ordered functions outside perturbation theory. According to the best of my knowledge, this has not been demonstrated in the general setting of QFT. A closer look at the derivation of the reduction formula reveals that a pointlike covariant time ordering is not needed; any asymptotic ordering will lead to the same on-shell values i.e. the residua on mass shell are independent on the precise ordering prescription for finite space time separations. In a later section we will see that time ordered fields is not a natural concept in non-perturbative QFT. The more natural objects turn out to be certain sesquilinear forms of the fields, the so called "generalized formfactors".

In the following we will derive the Haag-Ruelle scattering theory in the general setting of QFT and then comment in the derivation of the LSZ theory.

In n-particle Schrödinger theory, the physical input for the existence of scattering state vectors as large time limits of suitably chosen time dependent vectors is the strong fall-off property of the two body potential. Although one can somewhat relax those properties, potentials as the Coulomb potential fall off too weakly in order to belong to the standard situation (the large time wave functions oscillate with a logarithmic factor which does not contribute to the probabilities). In QFT the corresponding property is the strong cluster property of correlation functions in spacelike directions. a sufficient condition for this property is the existence of a spectral gap in the mass operator.

An operator from the polynomial $\mathcal{P}$ algebra (see section 2 of this chapter):

$$\begin{aligned} Q &= \sum_n \int f_n(x_1,....,x_n)A(x_n)....A(x_n)d^dx_1....d^dx_n \\ f_n &\in \mathcal{S}^{4d} \end{aligned}$$

will be called "almost local" (if $suppf_n \in O$, "local"). We will be interested in the behaviour of correlation functions of $Q(x) := U(x)QU(x)^{-1}$. The relevant theorem is

Theorem 14 *(Ruelle,1962) In a local QFT with a spectral mass gap (isolated one-particle mass shells) the quasilocal operators fulfill the strong cluster property:*

$$\forall N \in \mathbf{N}, \exists C \; s.t. \; \langle Q_1(x_1)....Q_n(x_n)\rangle_{con} < C_N R^{-N}$$

Here R denotes the maximal space like distance:

$$R = \max_{i,k} -(x_i - x_k)^2$$

We will not prove the theorem but rather try to understand how it can be used in order to understand the convergence for large times and the structure

of the incoming and outgoing multi-particle states. We first pick $Q_i s$ which applied to the vacuum create one-particle states with given wave function $\tilde{\varphi}(p)$. By choice of $f_n \in \mathcal{S}^{nd}$ with appropiate energy-momentum support this is always possible. Then we form the operators:

$$Q_i(h_i;t) := i \int_{x_0=t} Q_i(x) \overleftrightarrow{\partial_{x_0}} h_i(x) d^3x$$

where h_i is a positive energy solution of the Klein-Gordon equation and the derivative act with a minus sign to the left. Clearly:

$$Q_i(h_i;t)\Omega = |\psi_i\rangle\,, \quad \tilde{\psi}_i(p) = \tilde{\varphi}(p)\tilde{h}_i(p)$$

i.e. one obtaines time independent one-particle states. On the other hand the multiple application (at least two) of these operators leads to time dependent states whose large time behaviour is controlled by the following theorem:

Theorem 15 *(Haag 1958)*

(i) The sequence of state vectors

$$\Psi(t) = \prod_i^n Q_i(h_i;t)\Omega \tag{5.67}$$

converge strongly for t$\to \pm\infty$. The limiting states have the physical interpretation of incoming and outgoing multiparticle scattering states:

$$\Psi^{in} = \lim_{t\to-\infty} \Psi(t) = |\psi_1, \ldots \psi_n\rangle^{in}$$

$$\Psi^{out} = \lim_{t\to+\infty} \Psi(t) = |\psi_1, \ldots \psi_n\rangle^{out}$$

(ii) The scalar product of these scattering states has the Fock space structure:

$${}^{in}_{out}\langle \psi'_1 \ldots \psi'_n \mid \psi_1 \ldots \psi_m \rangle^{in}_{out} = \delta_{nm} \sum_{P\in S_n} \left\{ \begin{array}{c} + \\ sign(P) \end{array} \right\} \prod_k \left\langle \psi'_{P(k)} \mid \psi_k \right\rangle$$

according to bosonic or fermionic spacelike behaviour of the Heisenberg fields $A(x)$. One should add that the Poincaré transformation act naturally on the asymptotic Fock space structure i.e. the in and out states do not remember the special frame in which the time direction was defined.

The idea of the proof consists in showing that $\Psi(t)$ is a Cauchy sequence i.e. that $\left\|\frac{d}{dt}\Psi(t)\right\| < Ct^{-\frac{3}{2}}$. What one needs in addition to the cluster properties of the $Q(x)$–correlations is a refined asymptotic estimate on the single particle wave functions which goes beyond the result of the well-known stationary phase method:

$$\begin{aligned} h(\mathbf{x},t) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \tilde{h}(\mathbf{p}) e^{-i(\omega(\mathbf{p})t - \mathbf{px})} \underset{t\to\infty}{\to} \\ &= const.t^{-\frac{3}{2}} \exp(-im\gamma^{-1}t)(\gamma^{\frac{3}{2}}\tilde{h}(m\gamma\mathbf{v}) + O(t^{-1})) \\ \gamma &= \frac{1}{\sqrt{1-\mathbf{v}^2}}, \quad \mathbf{v} = \frac{\mathbf{x}}{t} \end{aligned} \tag{5.68}$$

The refined version determines the "essential" x-space support of h in terms of the velocity support in momentum space $\Sigma = \left\{ \mathbf{v} = \frac{\mathbf{p}}{\omega} \mid p \in supp\tilde{h} \right\}$. one has:

Theorem 16 *(Ruelle 62) Let h be a positive energy solution of the Klein-Gordon equation and Σ its velocity support. With $\mathcal{U}$ an open set containing Σ we have:*

(i) for $v \in \mathcal{U} : |h(\mathbf{v}t, t)| < C\,|t|^{-\frac{3}{2}}$ *as*5.68

(ii) for $v \notin U : |h(\mathbf{v}t, t)| < C_N(1 + |\mathbf{v}|)^{-N}\,|t|^{-N}$

If we now choose one-particle wave functions h' with nonoverlapping velocity supports relative to the unprimed h then

$$\langle \Omega \,|Q_1(h_1';t)^* \ldots . Q_m(h_m';t)^* Q(h_n;t) \ldots . Q(h_1;t)|\rangle$$

$$\underset{t\to\infty}{\to} \delta_{nm} \left\{ \sum_{P\in S_n} \left\{ \begin{array}{c} + \\ sign(P) \end{array} \right\} \prod_{k=1}^{n} \left\langle \psi'_{P(k)} \mid \psi_k \right\rangle \right\}, \quad \begin{array}{l} bosonic \\ fermionic \end{array}$$

The connected part, upon integration with the dissipating wave packets, does not contribute at all to the limit, as follows from the elementary geometrical (essential) support pictures in Minkowski space.. The same holds for any cluster with more than two operators Q_i. this fixes the structure of the in/out scalar products. The fall-off of $\left\| \frac{d}{dt}\Psi(t) \right\|^2$ is even simpler, because each term which contributes to this norm square for large t contains one two-poit function where one operator is a time derivative of $Q(h;,t)$ which vanishes upon acting on the vacuum.

The restriction to nonoverlapping wave packets has a physical origin: parallel flying particles lead to a weaker convergence. The best strategy is to prove formulas for the nonoverlapping situation and only at the end take the plane wave limit. The formalism does not only allow to derive the LSZ theory and the reduction formulas, but also gives higher order t corrections to LSZ (Haag, chapterII section 4).

The above scattering formalism needs to be modified in an essential and interesting way, if the fields have a spacelike commutation structure which leads to braid group statistics. In the physically interesting case of d=2+1 dimensions, these "plektonic" fields have really a string-like spatial extension i.e. they are not fields in the sense of Lagrangian QFT. Their construction falls into the realm of general or algebraic QFT. One still can prove the asymptotic convergence, but the asymptotic state vectors loose their tensor product structure and the cut between kinematics (in/out structure) and dynamics (genuine interactions) has to be essentially modified. The fact that such theories are outside the Lagrangian framework and even outside quantization ideas, does not make them any lass physical or unsusceptible to explicit and perurbative constructions, but the perturbation around free "plektons" is expected to have more in common with ideas on perturbing around chiral conformal theories than with Feynman perturbation theory arond bosonic/fermionic free fields.

The scattering treatment of plektons is still missing here.

Chapter 6

Modular Localization and Bootstrap-FormfactorProgram

6.1 Introductory comments

Presently QFT presents itself as being formed of several parts which seem to drift apart into different directions. On the one hand there is the standard approach which is centered around renormalized perturbation theory and the various quantization methods (canonical, functional). Enriched with geometrical ideas it has led to recent (mainly) mathematical advances via string theory and the Seiberg-Witten duality structure. On the other hand there is the more algebra-based low dimensional approach which has led to the construction of rich families of chiral conformal and integrable QFT. The latter approach, although being somewhat conservative in its use of physical principles, has nevertheless led to many startling results concerning e.g. fusion of antiparticles from particles, confined objects and solitons as being two sides of the same coin, and other extended (and somewhat surprising) manifestations of the principle of "nuclear democracy". Thirdly there is a very small group of theoreticians who find it profitable to continue the structural investigations of algebraic QFT.

In fact the most interesting message of the low dimensional constructive bootstrap-formfactor program seems to be that the emphasis on the scattering matrix advocated way back by Heisenberg and later by Chew, Stapp and others, was well founded. What went wrong in those early attempts was related to the enforced and artificial separation from local QFT and the (cyclically recurrent) ideologically motivated working hypothesis of a theory of everything (in this case: everything minus gravity). The main theme of this article is the realization that the S-matrix in algebraic QFT acquires a new hitherto unknown pivotal role in the construction of local nets (whose generators are local fields). It belongs

to the foundation of the local field theory (in its role as *the* net invariant which carries local modular information) as well as to its roof (in its role as describing scattering observables), a truly vexing "bootstrap" situation. The fact that in d=1+1 factorizable theories Chew's bootstrap ideas for the S-matrix work without fields (but with the help of "fusion" and "Yang-Baxter") is not due to the correctness of the underlying philosophy but rather to undeserved luck: the physical rapidity scattering variable is at the same time the uniformization parameter of the analytic properties [1]. In higher dimensions or without the factorization, Chew's program would fail without the use of local fields (and it did fail). In that case an iterative procedure which corrects the S-matrix together with a locality improvement of states and fields may have a constructive chance, a situation which could be vaguely reminiscent of the Hartree-Fock iteration in Schrödinger theory..

In this note we propose a new concept [1] of "modular localization" which, as will be shown, is capable of reconquering the lost unity of QFT. In particular, we will learn a new and very interesting lesson from the d=1+1 formfactor program. Far from being a special "exotic" construction, remote from any "real" QFT, this approach, if analyzed with general and deep concepts related to the TCP theorem and the S-matrix (interpreted as an invariant of a local net), reveals a surprising new and powerful nonperturbative construction principle which, so we hope, may turn out to be the basis of a future new iterative constructive approach. Locality of observables and localization of states (always relative to the vacuum or some other distinguished reference state) in QFT is a conditio sine qua non for the physical interpretation (without any outside impositions). Global topology as in the so called "topological field (?) theories" or the vacuum structure assigned to certain effective potentials which does not result from the local structure of real time QFT remain part of mathematics.

The fastest way to get a glimpse at the "modular localization" is to look at the relation of the Wigner representation theory [2] for positive energy representations of the Poincaré group and free fields. Whereas in Wigner's theory these irreducible representations in $H_{Wig}^{(m,s)}$ are uniquely specified by their mass and their spin (resp. helicity), this uniqueness is lost if one passes to free fields in the Fock-space $H_F^{(m,s)}$. There are infinitely many free fields in Fock-space and they constitute the linear part (in creation and annihilation operators) of a huge local equivalence class of fields, the so called Borchers class $B(m,s)$ [3][8] Any cyclic (with respect to the vacuum) representative field from this class generates the same net of local von Neumann algebras in H_F:

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \tag{6.1}$$

In fact the emerging picture of pointlike fields, that behave similar to coordinates in differential geometry, was the prime motivation for formulating algebraic QFT in terms of nets of algebras [3]. In our illustrative example[1] we regained the lost Wigner unicity on the level of nets. For a detailed presentation of the

[1] Even in d=1+1 the situation is very far removed from the desired uniqueness of Chew's S-matrix approach.

physical motivations and aims of this algebraic QFT we refer to a forthcoming article in Annals of Physics [4].

For the following it is important to understand the *direct* construction of this net in terms of the "modular localization" principle. For that one uses the operators which Wigner's theory affiliates with a reference wedge for which we take the x-t wedge:

$$\begin{aligned} W_r &: |t| < x \\ \delta^{it} &\equiv U(\Lambda_r(2\pi t)), \quad j \equiv U(r_r) \end{aligned} \tag{6.2}$$

Here $\Lambda_r(\chi)$ and r_r are the x-t Lorentz-boost and the x-t reflection. The latter is represented by a antiunitary operator j related (via a π-rotation around the x-axis) to the TCP transformation ϑ. For a charged particle this requires doubling of the Wigner space $H_{Wig}^{(m,s)}$. The reflection commutes with the L-boost. It has been shown elsewhere [5][1] that the unbounded antilinear involution:

$$s = j\delta^{\frac{1}{2}} \tag{6.3}$$

from which j and $\delta^{\frac{1}{2}}$ can be recovered by polar decomposition, can be used in order to define a real subspace H_R :

$$H_R = \{\psi \in H_{Wig} \mid s\psi = \psi\} \tag{6.4}$$

The momentum space wave functions in H_R are in the domain of $\delta^{\frac{1}{2}}$ and hence have analytic properties in the rapidity variable θ associated to the standard wedge W: $p_r = (|p|\,cosh\theta, |p|\,sinh\theta, p_2, p_3)$, $|p| = (m^2 + p_2^2 + p_3^2)^{\frac{1}{2}}$. For selfconjugate particles H_R consists of analytic functions in the strip $0 < \theta < \pi$ for which the two boundary values are related by a generalized reality condition:

$$D^{(s)}(i\sigma_2) \underset{\chi \to i\pi}{a.c.} \left[D^{(s)}(R_W(\Lambda_r(\chi), p_r)\right] \psi(\theta + i\pi, p_2, p_3) = \overline{\psi(\theta, p_2, p_3)}$$

where ψ is the 2s+1 component Wigner wave function, $R_W(\Lambda_r, p)$ the Wigner rotation associated with the boost Λ on which the analytic continuation *a.c.* acts, and $D^{(s)}(i\sigma_2)$ the charge conjugation matrix. In the non-selfconjugate case (particles $\neq$ antiparticles) the reflection j involves a flip in a doubled Wigner space. H_R has a property which is called "standard" i.e.:

$$H_R \cap iH_R = \{0\}, \quad H_R + iH_R \; dense \; in \; H_{Wig} \tag{6.5}$$

and j transforms H_R into its symplectic complement H_R' (the symplectic form is the imaginary part of the scalar product in H_{Wig}) which in the case of integer spin representations (the modifications for halfinteger spin are explained in [1]) is the same as the geometric opposite wedge space:

$$\begin{aligned} H_R' &\equiv H_R(W_r)' = H_R(W_r') \\ W_r' &\equiv W_r^{opposite}, \quad H_R(W) \cap H_R(W') = \{0\} \end{aligned} \tag{6.6}$$

$\mathcal{P}$-covariance generates from W_r a family of wedges $W = gW_r$ and associated $s(W), j(W), \delta(W)$ and $H_R(W)$ with the net isotony property:

$$H_R(W_1) \subset H_R(W_2) \quad W_1 \subset W_2 \tag{6.7}$$

where the properness of the inclusion is a consequence of the positivity of the energy (in fact equivalent to it, as it turns out) [5]. Wedge localization in the Wigner theory (not to be confused with the Newton-Wigner localization) [1] is the statement that the dense subspace (the precise domain of s) $H_R + iH_R$ describes localization inside the reference wedge. From QFT it is known that if one applies smeared local fields ϕ with test functions which are supported in W_r to the vacuum, one obtains a dense set of W-localized state vectors. The relation between the field algebras restricted to wedges and the Tomita-Takesaki modular theory was first observed by Bisognano and Wichmann [6] and later used by Sewell [7] in order to obtain a structural understanding of the Hawking-Unruh effect. What is new is that these spaces allow (in the present case for free systems) for a very neat characterization in terms of closed real subspaces whose position within the total space contains the full information about localization regions. Localization inside compact regions viz. double cones C (which are inaccessible by direct geometric modular theory) may be defined in terms of (dense if nontrivial) intersections:

$$H_R(C) = \bigcap_{W \supset C} H_R(W) \tag{6.8}$$

As already mentioned, the double cone localization is fulfilled in all positive energy representations with halfinteger spin, but not in d=3+1 m=0 "continuous spin" [1] representation and also not in massive d<3+1 representations with "anyonic"spin.

The important last step in the construction of a localized net of von Neumann algebras is the application of the CCR (Weyl) and the CAR functor which maps the net of real Hillbert subspaces into the net of algebras [5][1]

$$\begin{aligned} H_{Wig} &\rightarrow B(H_F) = alg\,\{Weyl(f) \mid f \in \mathcal{S}(M_4)\} \\ H_R(W) &\rightarrow \mathcal{A}(W) = alg\,\{Weyl(f) \mid f \in \mathcal{D}(W)\} \\ \check{S} &= J\Delta^{\frac{1}{2}}, \quad J = e^{i}, \quad \Delta = e^{\delta} \end{aligned} \tag{6.9}$$

These operators $\check{S}, J$ and Δ^{it} are the Tomita-Takesaki operators of the T-T modular theory of the von Neumann algebra $\mathcal{A}(W)$ in "standard position". The inverse hat $\vee$ on S helps to distinguish the Tomita involution from the later appearing scattering S-operator, usually referred to as the S-matrix.

For the somewhat subtle point that the obstruction against the equality of the geometric opposite with the modular opposite localization in the case of halfinteger spin requires the introduction of a "Klein twist" and the CAR functor we refer to the mentioned literature [1].

6.2 Modular Localization and Interaction

In order to obtain a clue of how to incorporate an intrinsic notion of interactions into this modular localization setting, we remind ourselves that if we do use pointlike fields, the modular localization for free fields agrees with what we get by applying the polynomial in the localization region supported smeared fields. In contrast to the conventional characterization of localization in terms of x-space pointlike fields, the modular characterization solely works in the *momentum-(Fock)space* of the (incoming) *free particles*. It attributes a physical significance to the precise position of the Reeh-Schlieder [3] dense set of localized vectors and the change of this position resulting from the change of localization region. In order to formulate the modular localization principle in the case of interactions, one must take note of the fact that the scattering matrix S of local QFT is the product of the interacting TCP Θ with the free (incoming) TCP Θ_0 and (since the rotation by which the Tomita reflection J differs from Θ is interaction-independent as all connected Poicaré transformations are interaction-independent) we have:

$$S = \Theta \cdot \Theta_0, \quad S = J \cdot J_0 \tag{6.10}$$

and as a result we obtain for the Tomita involution:

$$\check{S} = J\Delta^{\frac{1}{2}} = SJ_0\Delta^{\frac{1}{2}} = S\check{S}_0 \tag{6.11}$$

Again we may use covariance in order to obtain $\check{S}(W)$ and the localization domain of $\check{S}(W)$ as $\mathcal{D}(\check{S}(W)) = H_R(W) + iH_R(W)$ i.e. in terms of a net of closed real subspaces $H_R(W)$ of the incoming Fock space. However now the construction of an associated von Neumann algebra is not clear since an "interacting" functor from subspaces of the Fock space to von Neumann algebras is not known. We will make some remarks (still short of a solution of this important problem) in the concluding section and continue here with some more helpful comments on modular localization of interacting state vectors.

As in the free case, the modular wedge localization does not use full Einstein causality but only the so-called "weak locality", which is just a reformulation of the TCP invariance [8] Weakly local fields form an equivalence class which is much bigger than the local Borchers class but they are still associated to the same S-matrix (or rather the same TCP operator). Actually the S in local quantum physics has two different interpretations: S in its role to provide modular localization in interacting theories, and S with the standard scattering interpretation in terms of (nonlocal!) large time limits. There is *no parallel outside local quantum physics* to this state of affairs. Whereas all concepts and properties which have been used hitherto in standard QFT (perturbation theory, canonical formalism and path integrals) as e.g. time ordering [2] and interaction picture formalism, are shared by nonrelativistic theories, modular localization is a new

[2] There is a conspicuous absence of the time-ordering operation in the bootstrap construction of factorizable field models. Instead the basic objects are generalized formfactors i.e. sesquilinear forms on a dense set of state vectors.

structural element in local quantum physics [3] and a characteristic property for Einstein causal quantum physics. The simplest kind of interacting theories are those in which the particle number is at least asymptotically ("on shell") conserved i.e. $[\tilde{S}, \mathrm{N}_{in}] = 0$.

In the next section we will briefly review the d=1+1 bootstrap-formfactor program in a manner which facilitates the later application of modular localization.

6.3 The Bootstrap-Formfactor Program

In this section we will meet a constructive approach for "integrable" d=1+1 QFT. Our first task is to obtain an intrinsic QFT understanding of integrability in a way which avoids classical notions as e.g. complete sets of conservation laws etc. For this purpose we note an important d=1+1 peculiarity.. Our generic expectation is that large spatial separation of the center of wave packet of two particles in the elastic two-particle scattering matrix leads to the weakening of scattering, or in momentum space:

$$\left\langle p_1' p_2' \left| S \right| p_1 p_2 \right\rangle = \left\langle p_1' p_2' | p_1 p_2 \right\rangle + \delta(p_1 + p_2 - p_1' - p_2') T(p_1 p_2 p_1' p_2') \qquad (6.12)$$

where the identity contribution is more singular (has more δ-factors) than the T-term and therefore the second term drops out in x-space clustering. This argument fails precisely in d=1+1 and therefore the cluster property of the S-matrix is not suitable in order to obtain an intrinsic understanding of interaction. The two-particle S-matrix looses its higher particle threshold structure, but it remains nontrivial (in distinction to d=3+1). However for all higher particle scattering processes the behavior for d=1+1 is qualitatively the same as in higher dimensions: the decreasing threshold singularities (which decrease with increasing particle number) are responsible for the spatial decrease. Therefore any d=1+1 QFT is expected to have a limiting $\mathrm{S}_{\mathrm{lim}}$-matrix which is purely elastic and solely determined by the elastic two-particle $S^{(2)}$-matrix. The Yang-Baxter relation results as a consistency relation for the elastic 3→3 particle $S^{(3)}_{\mathrm{lim}}$-matrix. If this limiting S-matrix would again correspond to a localizable QFT, we would have a new class division of QFT, this time based on a long distance limit (which in some sense is opposite to the scale invariant short distance limit). It is this *(long distance) class property* [4] *which makes these factorizing models so fascinating*, as much as the fascination of chiral conformal QFT results from their role of representing short distance universality classes. In d=3+1 $\mathrm{S}_{\mathrm{lim}} = 1$ and therefore the limiting theory is expected to maintain the same superselection rules but in the "interaction freeest" possible way (literally free theories as we will argue later on). Hence in d=1+1 we are invited to

[3] This characteristic modular structure lifts local quantum physics to a new realm by its own which cannot be obtained by specialization from general quantum theory.

[4] Although I know of no article in which this has been spelled out, its pervasive presence behind the scene is is recognizable in some publications.

speculate on the validity of the following commutative diagram:

$$\mathcal{F} \swarrow \begin{matrix} \mathcal{F}_{ld} \\ \downarrow \\ \mathcal{F}_{sd} \end{matrix} \tag{6.13}$$

Here $ld(sd)$ labels the long (short) distance limits. There are also arguments [9] that with the help of a perturbative idea one may ascend from $\mathcal{F}_{sd}$ to $\mathcal{F}_{ld}$. It is however presenly not clear how one can use the known properties of the ld theories (i.e. integrable models) in order to formulate a constructive program for the nonintegrable members of the ld equivalence class. We hope that our modular localization principle (which is not restricted to factorizable models) may turn out to be helpful for this purpose.

The constructive approach based on the bootstrap idea proceeds in two steps. One first classifies unitary, crossing symmetric solutions of the Yang-Baxter equations which fulfill certain minimal (or maximal, depending on the viewpoint) requirements. Afterwards we use these factorizing S-matrices together with the Watson equations (a notion from scattering theory relating formfactors with the S-matrix) and analytic properties for formfactors in order to compute the latter. One obtains the complete set of multi-particle matrix elements of "would be" local fields, i.e. one constructs the fields as sesquilinear forms. It is characteristic of this method that one does not use the "axiomatic" properties of the beginning of this section but rather less rigorously known momentum-space analytic properties which, although certainly related to causality and spectral properties, are more part of the LSZ+dispersion theoretic folklore than of rigorous QFT. As long as one demonstrates at the end that the so obtained fields fulfill local commutativity this is a legitimate procedure. It leaves open the question whether there exists a more direct conceptual link between the S-matrix and the local fields or rather the field independent local nets. That this is indeed the case will be shown after the presentation of the formfactor program.

6.3.1 Properties of Factorizing S-Matrices

Consider first the analytic structure of an elastic S-matrix for a scalar neutral particle. In terms of the rapidity variable θ:

$$|p_1,p_2\rangle^{out} = S\,|p_1,p_2\rangle^{in} = S_{el}(p_1,p_2)\,|p_1,p_2\rangle^{in} \tag{6.14}$$

$$\begin{aligned} S_{el}(p_1,p_2) &= : S(\theta), \quad p_i = m(cosh\theta_i, sinh\theta_i), \quad \theta := |\theta_1 - \theta_2| \\ {}^{in}\langle p_1',p_2'\,|S|\,p_1,p_2\rangle^{in} &= S(\theta)^{in}\langle p_1',p_2' \mid p_1,p_2\rangle^{in} \end{aligned} \tag{6.15}$$

Usually the elastic S-matrix is written in terms of the invariant energy $s = (p_1+p_2)^2 = 2m^2(1+cosh\theta)$ and the momentum transfer (not independent in d=1+1) $t = (p_1-p_2)^2 = .2m^2(1-cosh\theta)$. As a result of undeserved fortune, the rapidity θ turns out to be a uniformization variable for the real analytic S i.e. the complex s-plane with the elastic cut in $s \geq 4m^2$ is dumped into the

strip $0 \leq Im\theta \leq \pi$ and the S-matrix becomes a meromorphic function $S(\theta)$ with $S(-\theta) = S^*(\theta) = S^{-1}(\theta)$. (unitarity). Hence the strip $-\pi \leq \theta \leq \pi$ is the physical strip for $S(\theta)$. Crossing symmetry in our special (neutral) case means a symmetry on the boundary of the strip: $\theta \rightarrow i\pi - \theta$. Note that the presence of inelastic thresholds would destroy the uniformization.

The factorization implies the operator relation:

$$\begin{aligned} & S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3) \\ = \; & S_{23}(p_2,p_3)S_{13}(p_1,p_3)S_{12}(p_1,p_2) \end{aligned} \tag{6.16}$$

According to Liouville's theorem, the only minimal solution (minimal number of poles,smallest increase at ∞) for this scalar diagonal case is $S = \pm 1$. More general solutions are obtained by placing bound-state poles into the minimal solution. In order to maintain unitarity, the pole factor must be of the form:

$$P(\theta) = \frac{sinh\theta + isin\lambda}{sinh\theta - isin\lambda} \tag{6.17}$$

Transforming back this pole at $\theta = i\lambda$ into the original individual particle variables, we obtain the following parametrization in terms of a center of mass and relative rapidity:

$$\begin{aligned} p_1 &= m\left(cosh(\chi + \frac{i\lambda}{2}), sinh(\chi + \frac{i\lambda}{2})\right) \\ p_2 &= m\left(cosh(\chi - \frac{i\lambda}{2}), sinh(\chi - \frac{i\lambda}{2})\right) \end{aligned} \tag{6.18}$$

Clearly the two-particle bound state has the momentum:

$$\begin{aligned} p_{1,2} &= (p_1 + p_2)_{at\ bd.state} = 2mcos\frac{\lambda}{2}(cosh\chi, sinh\chi) \\ p_{1,2}^2 &= m_2^2, \quad m_2 = \frac{m}{2sin\frac{\lambda}{2}}sin\lambda \end{aligned} \tag{6.19}$$

The "fusion" of particles may be extended. For a 3-particle bound state we would look at the 3-particle S-matrix which, as a result of factorization has the form:

$$S^{(3)}(p_1,p_2,p_3) = S(\theta_{12})S(\theta_{13})S(\theta_{23}) \tag{6.20}$$

We first fuse 1with 2 and simultaneously 2 with 3 as before. The center of mass + relative rapidity parametrization yields:

$$\begin{aligned} p_1 &= m(cosh(\chi + i\lambda), sinh(\chi + i\lambda)) \\ p_2 &= m(cosh\chi, sinh\chi) \\ p_3 &= m(cosh(\chi - i\lambda), sinh(\chi - i\lambda)) \end{aligned} \tag{6.21}$$

Again we get the mass of the 3-particle bound state by adding the zero components in the $\chi = 0$ frame:

$$m_3 = (p_1 + p_2 + p_3)_0 = m_2 cos\frac{\lambda}{2} + mcos\lambda = 2\frac{m}{2sin\frac{\lambda}{2}}sin\frac{3\lambda}{2} \tag{6.22}$$

Induction then gives the general fusion mass formula:

$$m_n = 2\mu sin\frac{n\lambda}{2}, \quad \mu = \frac{m}{2sin\frac{\lambda}{2}} \tag{6.23}$$

We will meet such trigonometric fusion formulas later in algebraic QFT where they are related to the statistical dimensions of fused charge sectors. They were first known through the Dashen-Hasslacher-Neveu quasiclassical approach. The above fusion calculation was done as far back as 1976 [10] and consisted in a synthesis of the quasiclassical work of DHN with some ideas of Sushko using the factorization principle, but still without the ideas of Yang and Baxter (which are not needed for this scalar case). The decisive step towards a general factorizable bootstrap program was taken two years later [11][12].

The consistency of these particles as incoming and outgoing objects leads to additional structures. Consider the scattering of the mass m_2 bound state with a third m-particle. This S-matrix for the scattering of these two different particles is obtained from $S^{(3)}$ by:

$$S_{b.e.}(p_1+p_2,p_3)\,|_{(p_1+p_2)^2=m_2^2} = \frac{1}{R}\underset{(p_1+p_2)^2\to m_2^2}{Res} S_{12}S_{13}S_{23} \tag{6.24}$$

where the projector P_{12} together with a numerical residue value R is defined by:

$$\underset{(p_1+p_2)^2\to m_2^2}{Res} S(p_1,p_2) = RP_{12} \tag{6.25}$$

and we used the word elementary e.and bound b. as labels on the new two-particle $S_{b.e.}$. The factorization insures that:

$$P_{12}S_{13}S_{23} = S_{23}S_{13}P_{12} \tag{6.26}$$

A prominent family of scalar S-matrices with N -1 bound state fulfilling all these requirements are the Z_N models [13]. Consistency requires that the bound state of N-1 m-particles is again a m-particle. For N=2 this family contains the Ising field theory with $S^{(2)}_{Ising} = -1$ which we already met in the section on (dis)order variables. Instead of elaborating this scalar factorization situation, we pass immediatly to the matrix case where we meet a new and interesting phenomenon. We assume that the particle from which we start has an internal "charge" which can take on a finite number of values i.e.

$$|p,\alpha\rangle \in H_1 \otimes V, \quad dimV < \infty \tag{6.27}$$

The two-particle S-matrix is then written as a matrix acting on $V \otimes V$ whose entries are operator-valued (represented as in the previous case by momentum-space kernels):

$$S\,|p_1,....,p_n\rangle^{in}_{\alpha_1....\alpha_n} = |p_1,....,p_n\rangle^{in}_{\alpha'_1....\alpha'_n}\, S^{\alpha'_1....\alpha'_n}_{\alpha_1....\alpha_n}(p_1,....p_n) \tag{6.28}$$

$$S^{(n)}(p_1,....,p_n) = \prod_{i<j} S^{(2)}(p_i,p_j) \tag{6.29}$$

The factorization requires a specific order of the product of matrices. Consistency requires the validity of a Artin (braid-group) like relation:

$$\begin{aligned} & S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3) \\ = \; & S_{23}(p_2,p_3)S_{13}(p_1,p_3)S_{12}(p_1,p_2) \end{aligned} \tag{6.30}$$

The notation should be obvious: the subscript on S indicates on which of the tensor factors in the 3-fold tensor product of one-particle spaces the object acts. The relation with the Artin relations becomes clear if one ignores the p-dependence and rewrites the Y-B relation in terms of $\tilde{S}. = PS$, where P is the permutation of two tensor factors.

This is the famous Yang-Baxter relation, since at the time of the discovery of the S-matrix bootstrap it became clear that such a mathematical structure had appeared before outside QFT in a quite different setting. Here this identity permits to change the temporal order of individual rescatterings so that the n-particle scattering $S^{(n)}$ is independent of those (graphically: invariance under parallel shifts of 2-momenta in graphical illustrations of scattering processes). The problem of finding the natural parametrization (e.g. Baxter's elliptic parametrization) for these Yang-Baxter relations does not arise in QFT; the *uniformizing rapidity θ is already the natural Yang-Baxter variable:*

$$S_{12}(\theta)S_{13}(\theta+\theta')S_{23}(\theta') = S_{23}(\theta')S_{13}(\theta+\theta')S_{12}(\theta) \tag{6.31}$$

If fermion-antifermion pairs can go into boson-antiboson pairs, the object which fulfills the Yang-Baxter relation is not S but σS where $\sigma = \pm 1$ with + for bosons. As the braid group relation, this is an overdetermined system of equations. For the former one found a powerful mathematical framework within V.Jones subfactor theory [14]. Although the attempts to get an equally powerful mathematical framework for the latter was less than successful (the "Baxterization" of the subfactor representations of Artin braids) one was able to find many interesting families of nontrivial solutions of which some even allowed a comparision with Lagrangian perturbation theory.

The S-matrix bootstrap idea originated in the early 60^{ies} from dispersion theory. Its revival in connection with d=1+1 factorization in the late 70^{ies} showed that its premises were physically reasonable, except the idea that it could be seen as a "theory of everything" (TOE) which was wrong and even obsert (for the more recent TOE's one would be hard pressed to say friendly words about their physical content).

The basic new message [15][16] is that one should use these factorizing S-matrices as computational tools for the construction of local fields and local nets as explained in the following subsection

6.3.2 Generalized Formfactors

Now we will probe the idea that these S-matrices belong to localizable fields. Let A be any local field which belongs to a Borchers equivalence class of local

fields. We write the generalized formfactor of A(x) as:

$$ {}^{out}\langle p_1,....,p_m\,|A(0)|\,p_{m+1},....,p_n\rangle^{in}_{\alpha_{m+1}....\alpha_n} \qquad (6.32) $$

We are interested in its analytic p-space properties. "On shell" p-space analytic properties are more elusive than x-space analytic properties. For the latter the spectral support properties play the important role, whereas p-space analyticity relies heavily on caucality. The above matrix element still contains energy-momentum δ-functions resulting from contracting incoming p's with outgoing. These are removed by taking the connected parts of the formfactors. Only for the distinguished formfactor:

$$ \langle 0\,|A(0)|\,p_1,....p_n\rangle^{in}_{\alpha_1....\alpha_n} = \langle 0\,|A(0)|\,p_1,....p_n\rangle^{in,con}_{\alpha_1....\alpha_n} \qquad (6.33) $$

we have coalescence with its connected part. Similar to x-space analyticity, one expects the existence of one analytic master-function whose different boundary values correspond to the different n-particle formfactors:

$$ \begin{aligned} & {}^{out}\langle p_1,....,p_m\,|A(0)|\,p_{m+1},....,p_n\rangle^{in,con}_{\alpha_{m+1}....\alpha_n} \qquad (6.34)\\ &= F^A_{\underline{\alpha}}(s_{ij}+i\varepsilon, t_{rs}-i\varepsilon, s_{kl}+i\varepsilon), \quad i<j\le m<k<l\le n \\ t_{rs} &= (p_r-p_s)^2, \quad r\le m<s\le n \end{aligned} $$

There are Watson relations between the S-matrix and the formfactors. In the d=1+3 dispersion theory setting it is well known that the cuts below the inelastic threshold of $\langle 0\,|A(0)|\,p_1,p_2\rangle$ is related to the partial wave phase shifts in that elastic region. In a factorizing d=1+1 theory these Watson relations can be written down in general:

$$ \begin{aligned} F^A_{\alpha_1....\alpha_n}(s_{ij}+i\varepsilon) &= \langle 0\,|A(0)|\,p_1,....p_n\rangle^{in}_{\alpha_1....\alpha_n} \qquad (6.35)\\ &= \sum_{out}\langle 0\,|A(0)|\,out\rangle\,\langle out\,|\,p_1,....,p_n\rangle^{in}_{\alpha_1....\alpha_n} \end{aligned} $$

$$ \curvearrowright F^A_{\alpha_1....\alpha_n}(s_{ij}+i\varepsilon) = F^A_{\alpha'_1....\alpha'_n}(s_{ij}-i\varepsilon)S^{\alpha'_1....\alpha'_n}_{\alpha_1....\alpha_n}(s_{ij}) \qquad (6.36) $$

and for the mixed formfactors6.32:

$$ \begin{aligned} & F^A_{\underline{\alpha}}(s_{ij}+i\varepsilon, t_{rs}-i\varepsilon, s_{kl}+i\varepsilon) \qquad (6.37)\\ = \; & S^{\alpha_m....\alpha_1}_{\alpha'_1....\alpha'_m}(s_{ij})F^A_{\underline{\alpha}'}(s_{ij}-i\varepsilon, t_{rs}+i\varepsilon, s_{kl}-i\varepsilon)S^{\alpha'_n....\alpha'_{m+1}}_{\alpha_{m+1}....\alpha_n}(s_{kl}) \end{aligned} $$

Using the uniformazing $\theta's$ this is like a generalized quasiperiodicity property on θ-strips for the F's (instead of the periodicity of S). The first who considered formfactors beyond two-particles [18] and presented a system of axioms for their calculation was Smirnov [15] Following a recent presentation by Babujian, Fring and Karowski [16] in a more standard field theoretic setting (LSZ+ dispersion theory), the formfactor program for the construction of d=1+1 QFT is as follows. Introduce the orderd formfactors:

$$ f^A_{\underline{\alpha}}(\theta_1,....,\theta_n) := \langle 0\,|A(0)|\,p_1,....,p_n\rangle^{in}_{\underline{\alpha}}, \quad \theta_1>...>\theta_n \qquad (6.38) $$

and define the value for reordered $\theta's$ by analytic continuation (starting with this ordering in the physical region). Demand that the f's fulfill the following properties:

- (i) $f^A_{...ij...}(...,\theta_i,\theta_j,...) = f^A_{...ji...}(...,\theta_j,\theta_i,...)S_{ij}(\theta_i-\theta_j) \quad \forall\ \theta's$
- (ii) $f^A_{12...n}(\theta_1+i\pi,\theta_2,...,\theta_n) = f^A_{2...n1}(\theta_2,...,\theta_n,\theta_1-i\pi)$
- (iii) $f^A_{1...n}(\theta_1,...\theta_n) \underset{\theta_1\to\theta_2+i\pi}{\approx} \frac{2i}{\theta_1-\theta_2-i\pi}C_{12}f^A_{3...n}(\theta_3,...,\theta_n)(1-S_{2n}...S_{23})$

where $C_{\alpha\beta}=\delta_{\bar{\alpha}\beta}$ is the charge conjugation matrix.

Here we have not mentioned the poles from bound states (states which appear by the previous fusion) since they are automatically entering the formfactors via the S-matrix. The word "axiom" in the context of this paper has the significance of working hypothesis i.e. an assumption which receives its legitimation through its constructive success. Physical principles on the other hand, as the spectral and causality properties of general QFT, will not be called axioms. Our main aim is to show how one can reduce the above axioms of the bootstrap-formfactor approach to the principles of QFT and thereby recuperate the unity of this nonperturbative approach with the rest of QFT.

The conceptually somewhat unusual property is the "symmetry" property (i). Here one should bear in mind that from the point of view of the LSZ formulation f is an auxiliary object to which the statistics property under particle exchange does not apply (it would apply to the original matrix-element). The above exchange property for f is a statement about analytic continuation. The statistics of incoming particle is only used in order to get the charges (i.e. the tensor factors) into the same $j-i$ order as the analytically interchanged $\theta's$. Following BFK [16] let us first remind ourselves of the standard argument for (i) in somewhat detail. For the special case $\langle 0|A(0)|\theta_1\theta_2\rangle^{ex}$ $ex = in, out$ it is evident that:

$$lim_{\epsilon\to 0}F(s_{12}\pm i\epsilon) = \begin{cases} \langle 0|A(0)|\theta_1\theta_2\rangle^{in} \\ \langle 0|A(0)|\theta_1\theta_2\rangle^{out} \end{cases} \tag{6.39}$$

i.e. there is one analytic masterfunction $f(z)$ (assuming identical particles) with different boundary values on the $s\geq 4m^2$ cut having the in, out interpretation. Assuming Bose statistics, the physical matrix elements on the right hand side are symmetric under the interchange of the $\theta's$. In terms of the uniformization variable θ_{12} in F the transition from $in \to out$ means a change of sign via analytic continuation i.e. without changing the charge quantum numbers α i.e. the position of the tensor factors. After accomplishing this last step by the bose commutation relation the negative θ_{12} formfactor $F(\theta_{21})$ can according to the definition 6.38 be identified with $f_{21}(\theta_2,\theta_1)$ and the relation

$$\langle 0|A(0)|\theta_2\theta_1\rangle^{out}\, S(|\theta_1-\theta_2|) = \langle 0|A(0)|\theta_1\theta_2\rangle^{in} \tag{6.40}$$

agrees with 6.38 The generalization to ${}^{out}\langle\theta_3...\theta_n|A(0)|\theta_1\theta_2\rangle^{ex}_{con}$ has a problem because replacing in by out means passing from time-ordering to anti-time-ordering but the LSZ scattering theory produces boundary terms contributing

to the connected part. Although they are absent for theories in which the number of in-particles are conserved, it is unclear what property of general QFT is bringing about (i) through specialization to factorizing d=1+1 models.

On the other hand (ii) and (iii) are consequences of the following standard crossing formula [16] which relate the connected part of the generalized formfactors to the analytic master function f:

$$\begin{aligned} & \bar{\mathrm{I}}\,\langle 0|A(0)|p_2,...,p_n\rangle^{in}_{2...n} \qquad (6.41) \\ = & \sum_{j=2}^{n} \bar{\mathrm{I}}\,\langle p_1 \mid p_j\rangle_j\, f^A_{2..\hat{j}..n} S_{2j}...S_{j-1j} + f^A_{12...n}(\theta_1 + i\pi_-,...,\theta_n) \\ = & \sum_{j=2}^{n} \bar{\mathrm{I}}\,\langle p_1 \mid p_j\rangle_j\, f^A_{2..\hat{j}..n} S_{jn}...S_{jj+1} + f^A_{2...n1}(...,\theta_n\theta_1 - i\pi_-) \end{aligned}$$

The fastes way to understand this is to draw the corresponding graphs and remember that a positive energy particle crosses into a negative energy antiparticle. Successive application leads to a formula which expresses the formfactors in terms of the analytic auxiliary function f .The analytic part of this relation gives (ii) whereas the δ-function part is responsible for (iii). A proof that these properties do not only insure TCP-invariance (weak locality) but also Einstein causality can be given by using JLD spectral representations [19] However the direct derivation of the bootstrap-formfactor axioms from the principles of QFT was hitherto not achieved. It is part of the complicated and incomplete momentum space analyticity problem. Even the derivation of forward dispersion relations in particle physics took several years, not to mention the derivation of the analytic aspects [5] of crossing symmetry which remained utterly incomplete. It is precisely at this point where our modular localization approach shows its strength. To anticipate one result, it shows that the crossing symmetry is a kind of strengthened TCP-property and that the cyclicity it leads to is identical to the KMS-temperature (≡Hawking -Unruh temperature in this special case) characterization of the (Rindler-)wedge based Hawking-Unruh effect. From our point of view the most valuable result is that it opens for the first time the way to a new constructive iterative (but not perturbative) approach to non-quantization non-Lagrangean based QFT. My confidence that this may amount to more than just another fashion rests on the observation that the tool of modular localization comes from a refinement of TCP which, as anybody will immediately admit, *the central structure* of local QFT.

6.3.3 Modular Theory and the Formfactor Program

In this section we will analyze the formfactor program from the viewpoint of modular localization. To avoid complications we start with theories which have a diagonal S-matrix. A prototype is the Ising field theory with $S^{(2)} = -1, S = K$

[5] Only together with the (mass shell) analytic properties the crossing symmetry aquires a physical content.

where K is the Z_2 Klein twist of the previous section. We look first for a weakly local (not necessarily local) field in the TCP class associated to this simple S-matrix. For this purpose we modify the Fermi-creation and annihilation operators $a^{\#}_{in}$ of the free Majorana field associated to the Ising field theory in order to obtain the bosonic $\mathbf{Z_2}$ commutation relations. With some experience [20] one immediately writes the following expressions:

$$\begin{aligned} b(\theta) &= :a_{in}(\theta)e^{-i\pi\int_{-\infty}^{\theta}a^{*}_{in}(\theta)a_{in}(\theta)d\theta}: \\ c^{*}(\theta) &= :a^{*}_{in}(\theta)e^{i\pi\int_{\theta}^{\infty}a^{*}_{in}(\theta)a_{in}(\theta)d\theta}: \end{aligned} \tag{6.42}$$

$$\begin{aligned} B(x) &:= \frac{1}{\sqrt{2\pi}}\int\left\{e^{-ipx}b(\theta)+e^{ipx}c^{*}(\theta)\right\}d\theta \\ &\curvearrowright \quad \Theta B(x)\Theta = B^{*}(-x), \quad \Theta = S_{Ising}\cdot\Theta_0, \quad \Theta_0 \equiv \Theta_{in} \end{aligned} \tag{6.43}$$

This field creates wedge localized vectors in the n-particle projections of $\mathcal{D}(W) = H_R + iH_R$:

$$B^{\#}(x_1)....B^{\#}(x_n)\Omega, \quad x_i \in W \tag{6.44}$$

These $b^{\#}s$ produce the S_{Ising} S-matrix:

$$\begin{aligned} b^{*}(\theta_1)....b^{*}(\theta_n)\Omega &= S_{Ising}a^{*}(\theta_1)....a^{*}(\theta_n)\Omega \\ b^{\#}(\theta) &= S_{Ising}a^{\#}(\theta)S^{*}_{Ising} \end{aligned}$$

and fulfill the Zamolodchikov algebra:

$$\begin{aligned} b(\theta)b^{*}(\theta') &= S^{(2)-1}(\theta-\theta')b^{*}(\theta')b(\theta)+\delta(\theta-\theta') \\ b(\theta)b(\theta') &= S^{(2)}(\theta-\theta')b(\theta')b(\theta) \end{aligned} \tag{6.45}$$

Operators of this kind have been used a long time ago in order to exemplify the fact that massive d=1+1 theories describe "statistical schizons" [20] in distinction to conformal field theory, where the statistics (in form of field commutation relations) are inexorably linked with the fusion law of charges[6]. Recently Fring [21] and Lashkevich [22] constructed these operators using the realization of anyionic deformations [20] of Fock space d=1+1 creation- and annihilation operators $a^{\#}(p)$ as a guiding principle. It was through discussions I had with Lashkevich that I later recognized that there may be an interesting relation between the above Zamolodchikov algebra and certain auxiliary operators in my modular localization approach.

[6]For this reason the widespread use of the terminology "bosonization" in conformal QFT is unfortunate , but appropriate in d=1+1 massive theories. Whereas "bosonization" (or "fermionization") in conformal QFT means that the current algebra admits fermionic superselection sectors, in the massive case the *same sector* allows different "pseudo"statistical descriptions since schizons only have intrinsic charges but not intrinsic statistics. A periodic table of elements in a d=1+1 world does not require fermions.

The above expressions may be generalized to factorizable S-matrices which are diagonal.. The corresponding b-operators are then of the form:

$$b(\theta) = a_{in}(\theta)e^{i\int_{-\infty}^{\theta}\delta_{sc}(\theta-\theta')a^*_{in}(\theta')a_{in}(\theta')d\theta'} \tag{6.46}$$

where $\delta_{sc}(\theta)$ is the scattering phase shift. Again the TCP invariant operators and the wedge localization through the $\tilde{S}$-domain is analogous to the constant case. From those wedge localized n-particle states one reads off the formfactors of the would be local field. The solution is unique.

The main observations which links the bootstrap-formfactor axiomatics (i) (ii) and (iii) with modular localization are contained in the following statements:

- The TCP-covariant (not necessarily local, but weakly local) fields $B(x)$ with $x = r(sinh\chi, cosh\chi) \in W$ which generate a right wedge algebra $\mathcal{A}(W)$ equipped with the global vacuum state (i.e. that state which is the vacuum with respect to the global algebra $\mathcal{A}$) result in a KMS-temperature state with the Hawking-Unruh temperature $\beta = 2\pi$. The KMS boundary condition for a correlation function of $B's$ together with local fields $A(x), x \in W$:

$$\begin{aligned}&\langle 0\,|B(r_1,\chi_1)B(r_2,\chi_2)...A(x)...B(r_n,\chi_n)|\,0\rangle \\ &= \langle 0\,|B(r_2,\chi_2)..A(x)...B(r_n,\chi_n)B(r_1,\chi_1+2\pi i)|\,0\rangle\end{aligned} \tag{6.47}$$

  is equivalent to the cyclicity property (ii) (the $f^{\mathcal{A}}$is obtained from the connected part of the B-correlation function). The 2π strip-analyticity in the $\chi's$, which is provided by the KMS-theory, translates into momentum space analyticity for the rapidity variables θ_i of $f^{\mathcal{A}}$. In particular the analyticity and the crossing symmetry of the S-matrix is a consequence of this temperature structure 6.47for $A = 1$ e.g.:

$$\begin{aligned}&\langle 0\,|B(r_1,\chi_1+i\pi)B(r_2,\chi_2)B^*(r_3,\chi_3)B^*(r_4,\chi_4)|\,0\rangle \\ &= \langle 0\,|B(r_2,\chi_2)B^*(r_3,\chi_3)B^*(r_4,\chi_4)B(r_1,\chi_1-i\pi)|\,0\rangle \\ &\qquad\qquad \Leftrightarrow\ S(\theta) = S(i\pi-\theta)\end{aligned} \tag{6.48}$$

- The (improper) state vectors $B(r_1,\chi_1)....B(r_n,\chi_n)\Omega$ are boundary values of "half-strip" analytic vectors:

$$B(r_1,z_1)....B(r_n,z_n)\Omega, \quad \pi > Imz_n > ....Imz_1 > 0$$

  Upon taking the n-particle component this property translates into the half strip analyticity of the vectors:

$$b^*(z_1)....b^*(z_n)\Omega, \quad z_i = \theta_i + i\vartheta_i$$

  As a result of the Zamolodchikov commutation relations of the $b^{\#'s}$, we find that there is one half strip-analytic "master"state vector $\psi(z_1,....z_n)$,

whose different boundary values correspond to different operator orderings:

$$\lim_{\pi>Im z_{P(1)}>...>Im z_{P(n)}>0 \atop Im z_i \to 0} \psi(z_1...z_n) = b^*(\theta_{P(1)})...b^*(\theta_{P(n)})\Omega \qquad (6.49)$$

The analytic structure in rapidity space has an intriguing similarity with the analytic x-space structure known from Wightman's formulation of QFT. The reason is of course the close connection between the L-boost variable χ and the rapidity θ.

- The above state vectors generate the n-particle component of the modular localization subspace:

$$P_n \check{S} B(x_1)...B(x_n)\Omega = P_n B^*(x_n)...B^*(x_1)\Omega \qquad (6.50)$$

$$\Leftrightarrow \check{S} c^*(\theta_1)...c^*(\theta_n)\Omega = b^*(\theta_n)...b^*(\theta_1)\Omega \qquad (6.51)$$

$$\curvearrowright c^*(\theta_1)...c^*(\theta_n)\Omega + b^*(\theta_n)...b^*(\theta_1)\Omega \in H_R^{(n)}(W) \qquad (6.52)$$

Here we used the relation 6.45 i.e. the factorizability of the theory. As in the free case, the closed real subspace $H_R(W)$ represents the encoding of the complex dense modular localization space associated with W. Note that this last discussion used the factorization structure.

The generalization to charged particles and to halfinteger Lorentz spin is straightforward but the case of nondiagonal S-matrices gives rise to additional problems. The modular localization equation for the real subspace $H_R^{(n)}(W)$ is now:

$$\begin{aligned} & \check{S} \int \psi_{\alpha_1...\alpha_n}(\theta_1,...\theta_n)\,|\theta_1,...\theta_n\rangle_{in}^{\alpha_1...\alpha_n} \qquad (6.53) \\ = & \int (\bar{\psi}_{\alpha_1'...\alpha_n'} S_{\alpha_1...\alpha_n}^{\alpha_1'...\alpha_n'})(\theta_1 + i\pi...\theta_n + i\pi)\,|\theta_1,...\theta_n\rangle_{in}^{\alpha_1...\alpha_n} \\ = & \int \psi_{\alpha_1...\alpha_n}(\theta_1,...\theta_n)\,|\theta_1,...\theta_n\rangle_{in}^{\alpha_1...\alpha_n} \end{aligned}$$

$$\curvearrowright (\bar{\psi}_{\alpha_1'...\alpha_n'} S_{\alpha_1...\alpha_n}^{\alpha_1'...\alpha_n'})(\theta_1 + i\pi...\theta_n + i\pi) = \psi_{\alpha_1...\alpha_n}(\theta_1,...\theta_n) \qquad (6.54)$$

This "S-reality" equation seems to be a new mathematical structure as far as the mathematical physics literature is concerned. In the formulation of crossing symmetry the charge multiplicity indices α must be replaced by their conjugate values:

$$S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta_1 - \theta_2) = S_{\alpha_2\beta_2}^{\bar{\alpha}_1\beta_2}(\theta_2 - (\theta_1 - i\pi)) \qquad (6.55)$$

In the case of nondiagonal S one does not have operators $B(x)$ at ones disposal. It turns out that analogous to the Bethe Ansatz inspired solution of this problem [16]in the formfactor approach, the modular wedge localization (i.e. the S-reality) entails a natural Bethe Ansatz structure which permits an explicit

description of the space $H_R^{(n)}(W)$. The latter is indispensable for a realization of the Zamolodchikov algebra in the incoming Fock space and the construction of local fields within the formfactor program. These matters will be taken up in a separate paper [19]. There we will also present the new rich structures which result from intersections of wedge spaces in order to describe the localization spaces e.g. of the compact double cone regions. Although the modular structure of wedge localized von Neumann algebras in QFT as we used it here may be found in the literature, we do not expect most of our readers to know it. Therefore we decided to present the modular material in a broader context ??.

The coordinate-free point of view of the net theory suggests that the formfactor program may not be the most efficient and natural way to relate an S-matrix with local fields i.e. the inverse problem of the net field theory: given the modular invariant S of the net, reconstruct the net. Since an S-matrix is not associated to a particular field but rather is an invariant of a local equivalence class or a net, a direct construction (in the spirit of the functorial construction for free systems in the introduction) of the net instead of individual fields may be simpler than the rather cumbersome formfactor program. Such an algebraic approach would then consist of two parts, the conceptual problems related to the modular wedge localization of state vectors in Fock space i.e. TCP, antiparticles, crossing symmetry etc. and the ascend to local nets of subspaces and associated nets of von Neumann algebras.

Apart from the new analytic modular structure, the wedge localization equation is reminiscent of Yang's use of the Bethe Ansatz idea [17]. Our modular localization method therefore suggests that a suitably generalized Bethe idea may be a valuable tool in a new constructive approach to algebraic QFT (and not just for factorizable models). Related to this is the hope that our approach may lead to an explicit Fock space representation of the Zamolodchikov algebra (beyond the above diagonal cases) also in nondiagonal factorizable models, and that one meets analogues of this algebraic structure (which is somewhere intermediate between the algebras generated by the Heisenberg fields and that of the incoming fields and hence may be viewed as an algebraic QFT counterpart of the fictitious [3] interaction picture) in general local quantum physics.

6.4 Open Ends and Outlook

The really difficult problem in a constructive approach build on modular localization is the passing from the net of localized subspaces to a net of von Neumann subalgebras (educated guess: subfactors of von Neumann type III_1 as in the free field case). Here the most important issue is uniqueness. The net of localized subspaces is uniquely determined by the TCP-operator. Therefore the question of uniqueness of nets of operator algebras can be rephrased as: does the weak locality equivalence class contain maximally only one local Borchers class? We think that by a judicial use of spacelike (anti)commutativity, a given S-matrix ($\Leftrightarrow$a given Θ) will maximally allow (up to isomorphisms) one field system or algebraic net. The only argument we have in the moment is : $S = 1$

$\curvearrowright$free field Borchers class [19]. The essential step in the argument is the use of crossing symmetry and the Watson relations which link physical cuts in the analytically continued formfactors (matrix elements of a representative local field in the basis of the incoming multiparticle spaces) to the S-matrix. The fact that analytic on shell p-space properties, as one needs them for such arguments (e.g. crossing symmetry as an analyticity statement and not just a suggestive formal statement abstracted from LSZ extrapolation formulas), are difficult to obtain from the locality and spectral principles of QFT one can try to counteract by (as in the d=1+1 factorizable case) assuming "maximal analyticity" i. e. only taking physically motivated singularities into account. On the one hand this is certainly reasonable in a constructive approach and on the other hand we expect that the same modular theory which underlies the modular localization approach will also lead to a much better understanding of on shell analytic p-space properties. The present analytic techniques result from the so-called JLD-representation for matrix elements of causal commutators. We expect the exploration of modular localization concepts to give more powerful analytic results.

Some speculative remarks on the problem of associating a net of von Neumann algebras with a net of localization spaces may be helpful at this point. In the theory of operator algebras the Araki-Connes concept of the "natural cone" allows to construct a von Neumann algebra from the knowledge of the split of H_R into positive cones $\pm C_+$:

$$H_R = C_+ - C_+ \tag{6.56}$$

In the case of factorizable models the modular localization principle which leads to $H_R(W)$- and H_R(double cone)-subspaces uses the *real (on shell) particle conservation* and gave rise to (albeit new and subtle) quantum mechanical on shell Bethe Ansatz problems. However the local fields (or nets of von Neumann algebras) in such theories are known to have a very rich (non-quantum mechanical!) *virtual ("off-shell") particle creation and annihilation structure*. The same is expected in a theory of "free" anyons and plektons as opposed to free Bosons and Fermions. According to the arguments presented in this paper, this richness must occur in this last (unfortunately poorly understood) step from modular localized subspaces to (Einstein-) local algebras[7]. Connected with this is the already mentioned uniqueness problem i.e. the inverse scattering problem for local nets, which we will investigate in a future publication [19].

Finally we want to make some speculative remarks on how one imagines an iterative approach which unlike perturbation theory (which uses interaction densities expressed in the form of Wick-polynomials and their time-ordering) is based on modular concepts. One would start with a Heisenberg Ansatz for a relativistic S-matrix:

$$S^{(0)} = e^{i\eta}, \quad \eta = \sum \int \eta_n(x_1 \ldots x_2) : \phi_{in}(x_1) \ldots \phi_{in}(x_n) : \tag{6.57}$$

[7]This is also step from which I would expect a profound understanding of on shell crossing symmetry.

where the coefficient function should be Poincaré-invariant and cluster-connected, but yet without those complicated analytic p-space properties of multidimensional dispersion theory. In order to make at least some formal intuitive contact with the standard perturbative implementation of interaction via invariant free field polynomials one may choose for η the same expression. This Ansatz already assures (via Θ and J) the modular wedge localization (at almost no analytic costs) which leads to mass shell analyticity in certain rapidity variables (which form families corresponding to the family of wedges) as in the factorizable case ??. However the nontriviality of intersections of wedge spaces and the required existence of a map from the net of modular localized subspaces to a net of von Neumann algebras suggests to iteratively correct $S^{(0)}$ by $S^{(1)}, S^{(2)}$..etc. in order to achieve an *increasing amount of localization*[8]. Unlike the bootstrap-formfactor problem of the previous section for which a candidate for a "local", crossing symmetric S-matrix fulfilling the "maximal" p-space analyticity was known at the start of the formfactor program, such a contemplated iterative approach based on modular localization would constitute a true ("heterotic field theory-S-matrix") *mixed bootstrap approach*. It would be at least as removed from perturbation theory and functional integrals as the bootstrap-formfactor program is from Lagrangian QFT. The interaction could then receive its name not from Lagrangians but rather from the e.g. polynomial Ansatz for η.

At this point the question arises what, if any, is the relation between this modular approach and the standard one build on quantization. Here it is helpful to think of a kind of field theory-adapted *"Murphy's law"*: if there is neither a proof nor a an intrinsic structural reason for a conjecture in QFT, then it is wrong. Although this law does not follow from the physical principles, I do not know a single exception to it in QFT. So the idea, that behind the renormalized Gell-Mann Low formula or functional integrals (even with all their instanton corrections!) or the Bogoliubov-Shirkov axiomatics in terms of a space-time localized formal expression $S(g)$, there is an interacting theory based on those concepts would be an illusion, except for those low-dimensional models (φ^4_2 etc. where there is a connection through Borel resummability) for which one has a proof. Perturbation theory remains an infinitesimal deformation theory with no possibility for globalization. Despite its non-existence (invoking Murphy's law) of $S(g)$, from a formal physical point of view it is close to the modular invariant S (section 4 of [4]. Whereas standard perturbation theory maintains the linear structures as locality in every order but links unitarity with the non-perturbative existence (futile according to Murphy's law in QFT), any inductive approach based on modular properties is unitary in each step of the induction, but acquires sharp locality only in the limit (which is also the limit in which. the modular S acquires the scattering interpretation).

The present modular framework is not applicable to zero mass theories for which the LSZ incoming fields vanish i.e. a scattering matrix cannot be de-

[8] The problem seems to be vagely reminiscent of a selfconsistent Hartree-Fock iteration with the iteratively improved interaction being the analog of the $S^{(n)}$'s, and the zero order bilinear (mean field) interaction corresponding to an $S^{(0)}$ in the form of a Heisenberg ansatz6.57.

fined ("Infraparticles" as opposed to Wigner particles). Whereas for e.g. chiral conformal theories this is no problem since they are scale-invariant limits of massive theories, theories as QED (which presumably do not allow such a limiting description) require conceptual modifications. The modular approach is uncompromising but thankful for new conceptional challenges. It flourishes on the weaknesses, paradoxes and contradictions of the standard approach. Even at the risk of sounding immodest, one may hope to overcome the narrowing of QFT at the beginning of the 70^{ies}, when (as a result of enforcing classically based concepts as the "gauge principle" via quantization) very unfortunately noncommutative real time field theory was forced to become a footnote of euclidean field theory. Besides recent (in my opinion artificial) attempts to get out of the euclidean malaise by compensating the loss with a doses of noncommutative geometry[9], there has been a renewed interest in real time QFT. In addition to the formfactor-bootstrap program presented in this work, this is illustrated by recent progress on QFT in curved space-time for which the euclidean approach is physically senseless [23] (albeit mathematically interesting). We hope that our ideas on modular localization may also contribute to find a way out of the present stalemate and particularization of QFT. In any case our confidence at the moment is more based on its unifying point of view and its fundamental modular structure leading to new mathematical equations with the promise of a deep relation to Bethe Ansatz structures. Anybody with some knowledge of nonperturbative QFT will have no problem in recognizing that one is dealing here with the most fundamental structures which QFT has to offer.

[9]Instead of trying to escape the euclidean trap by physical brute force methods (employing ideas from nomcommutative geometry) one should remember that QFT has a natural noncommutative structure and *allows for a euclidean formulation and a Feynman-Kac representation only under very special circumstances.*

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Chapter 7

Introduction to Algebraic QFT

7.1 Some Useful Theorems

A representation (π, U) of a C^*-algebra and an automorphism group $(\mathcal{A}, \alpha_t)$ is a representation π of $\mathcal{A}$ in a Hilbert space H together with a strongly continuous unitary representation U of $\mathbf{R}$ in H:

$$U(t)\pi(A)U(t)^{-1} = \pi(\alpha_t(A)), \quad U(t) = e^{iHt}, \; H \geq 0 \tag{7.1}$$

For such representations the following theorems hold.

Theorem 17 *(Reeh-Schlieder) Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ be a local net with translation symmetry and $\mathcal{N}$ a space-time region such that there is $\mathcal{N}_0 \subset \mathcal{N}$ and a neighbourhood of zero $\mathcal{V}$ with: $\mathcal{N} \supset \mathcal{N}_0 + \mathcal{V}$ and additivity: $\bigvee_x \mathcal{A}(\mathcal{N}_0 + x) = \mathcal{A}''$. It follows that: $\overline{\mathcal{A}(\mathcal{N})\psi} = \overline{\mathcal{A}\psi}$ for all $\psi \in \mathcal{D}(e^{aP})$, $a \in V^\uparrow$*

Proof. One shows that for any vector $\phi \perp \mathcal{A}(\mathcal{N}) \curvearrowright \phi = 0$

For any such vector we have $(\phi, \alpha_{x_1}(A_1)....\alpha_{x_n}(A_n)\psi) = 0$, or in terms of boudary values of analytic functions in the tube:

$$\lim_{x_i \to z_i} (\phi, e^{iz_1 P} A_1 e^{i(z_2 - z_1)P} A_2 .... A_n e^{-(iz_n + a)P} e^{aP} \psi) = 0$$

where the limit is taken from inside the tube $\mathcal{T}^{(n,a)}$: $Im(z_{i+1} - z_i) \in V^\uparrow$, $z_0 = 0$, $z_{n+1} = ia$ in which the matrix element is an analytic function $F_{\phi,\psi}(z_1....z_n)$. Since it vanishes in an open set on the boudary it is (thanks to the generalized Schwartz reflection principle) identically zero. This proves the theorem since ϕ is then orthogonal to a dense set of vectors and hence on all vectors $\curvearrowright$cyclicity of ψ.

Taking now locality into account we conclude that if $A\psi = 0$ for a vector ψ as above and $A \in \mathcal{A}(\mathcal{N}') \curvearrowright BA\psi = AB\psi = 0$ for all $B \in \mathcal{A}(\mathcal{N})$ and hence $A = 0$ on a dense set $\curvearrowright A \equiv 0$ or in words: ψ is cyclic and separating.

This is a characteristic property in QFT of finite energy states (in particular the vacuum Ω) with respect to local algebras $\mathcal{A}(\mathcal{O})$ such that $\mathcal{O}'$ is not empty i.e. it does not hold in standard Q.M. If not handled with great care, one can easily get into pitfalls with causality[1].Most of the more sophysticated "violations of causality" are due to some conceptual misunderstanding of QFT. Literally speaking the R-S theorem says that by applying suitable operators which have some time duration in a spatially limited laboratory, one can approximate a state which describes the instanteneous creation of matter "behind the moon". A closer look reveals that the higher members of such an approximation sequence require more and more exotic (increasing energy-momentum costs) pieces of hardware. This suggests that the limited localization in phase space (i.e. a field theoretic analogon of the finite number of degrees of freedom of standard QM) becomes relevant for the cost balance. Indeed the precise formulation of this idea in the form of the "nuclearity property" of degrees of freedom has turned out to be extraordinarily useful. Theories which do not obey this requirement as e.g. those with a e.g. "Hagedorn temperature" are physically ill.

Another technically important property is the following "property B" which is due to Borchers.

Theorem 18 *Let E be a local projector $E \in \mathcal{A}(\mathcal{O})$. Then there exists an isometry V localizable in a possibly slightly bigger region $\tilde{\mathcal{O}} \supset \mathcal{O}$ with $E = VV^*$.*

Somebody with some knowledge of von Neumann's classification theory of factors and its refinement of type III factors by Connes and Haagerup[2], will have no difficulty in realyzing, that if it would not be for the possible enlargement of $\mathcal{O}$, the statement $E = VV^*, V^*V = 1$ (isometry) yields $E \sim 1$, i.e. the projectors are "infinite" $\curvearrowright \mathcal{A}(\mathcal{O})$ is type III. Indeed all explicitely known local QFT algebras are factors of type III_1 in the refined classification theory. The subscript 1 stands for "hyperfinite" which is somewhat loosely speaking a property of approximability by finite degrees of freedoms (prerequisite for lattice approximations) and which can be shown to arise from the QFT phase space structure encapsulated in the nuclearity requirement. It holds for the local algebras but does not necessarily apply to globalizations as $\mathcal{A}_{univ}$. Type III are the "biggest" von Neumann factors in the sense that they absorbe any tensor factor. For the wedge region one can actually prove that $\mathcal{A}(W)$ is a type III_1 factor (factor algebras are very natural also in physics since they substitute the notion of irreducibility in those cases where their intrinsic impurity prevents irreducibility). On the opposite end of the side one finds type II factors which are absorbed into any other tensor factor. The latter are however big enough in order to incorporate all the generalizations of group symmetry which recently surfaced in V. Jones inclusion theory of subfactors. In algebraic QFT the intertwiner algebras associated with the charge sectors of observable algebras are of

[1] Some of the more "spectacular" published claims about appearant causality violation have been reviewed in B.Schroer, "Reminiscences about many pitfalls and some successes of QFT within the last three decades", Reviews in Mathematical Physics, Vol.7, No.4 (1995), page 651.

[2] An account from the point of view of a theoretical physicist can be found in Haag's book.

this kind (combinatorical or topological QFT).

In connection with the limitation of energy-momentum and the nuclearity formalism it is convenient to have a mathematical framework which makes precise the concept of energy-momentum transfer. This was done by the mathematical physicist Borchers and the mathematician Arveson. The idea is to first introduce a notion of spectrum of the automorphism. The automorphism α_t of the C^*-algebra may be extended via 7.1to the envelopping von Neumann algebra $\pi(\mathcal{A})''$. It should not lead to any confusion if we stay sometimes with the same symbols for the extended objects. With the help of L.Schwartz test functions $f \in \mathcal{S}(\mathbf{R})$ we form $\alpha_f(A) = \int dt f(t)\alpha_t(A)$. It is easy to see that the extended automorphism, and therefore α_f, maps also the von Neumann extension into itself (since it commutes with elements from the commutant $\pi(\mathcal{A})'$ inside matrix elements). One now defines the (Arveson-)spectrum of $A \in \pi(\mathcal{A})''$ as:

$$spec_\alpha(A) = \left\{\omega \in \mathbf{R} \mid \forall nbhds\, N\, of\, \omega, \exists\, f \in \mathcal{S}(\mathbf{R})\, s.t. supp\tilde{f} \subset N,\ \alpha_f(A) \neq 0\right\} \tag{7.2}$$

The size of the individual $\alpha_f(A)$-contributions is evidently limited by $suppf$. We can manufacture operators A with $spec_\alpha(A) \in I, I$ given, by smoothening a given B with $f, supp\tilde{f} \subset I$:

$$A := \alpha_f(B)$$

The (algebraic) subspaces with energy transfer $\geq E$ are defined as:

$$\mathcal{A}_E = \{A \in \mathcal{A} \mid sp_\alpha(A) \subset [E, \infty]\} \tag{7.3}$$

The usefulness of these concepts begins to show up if one relates this with projection operators in the Hilbert space H of the representation (π, U) :

$$P_E := proj\ on \bigcap_{E'<E} \mathcal{A}_{E'}H \tag{7.4}$$

One may now associate a "hamiltonian" with α_t :

$$\begin{aligned} \mathbf{H} &= \int E dP_E,\ V(t) := e^{i\mathbf{H}t} = \int e^{iEt} dP_E \\ &\curvearrowright\ \alpha_t(A) = V(t)AV(t)^{-1} \end{aligned} \tag{7.5}$$

Since: $\pi(\mathcal{A})'\mathcal{A}_E H \subset \mathcal{A}_E H$, we find:

$$\begin{aligned} A' P_E &= P_E A' P_E = P_E A', \\ &\curvearrowright\ [P_E, A'] = 0\ \forall A' \in \pi(\mathcal{A})' \end{aligned}$$

or using more physical words: the infinitesimal generator $\mathbf{H}$ of the symmetry α_t may always be chosen in such a way that $\mathbf{H}$ is associated to the algebra (the Arveson-Borchers theorem). The algebraically determined $\mathbf{H}$ is called the "minimal" generator. Although the general situation, unlike the well-known

explicite expression (Sugawara) for the chiral translation generator in chiral conformal QFT, does not lead to a concrete functional expression, nevertheless this should be interpreted as the generalized analogon of the situation known in conformal QFT.. The generalization to abelian groups with several parameters should be obvious.

In the same vein, but taking in addition locality into account and using more powerful analyticity tools ("edge of the wedge techniques"), one proves the following four interesting theorems (H:J: Borchers "Translation Group and Particle Representations in Quantum Field Theory" Springer Lecture Notes in Physics 1996).

Theorem 19 *(Locality and the shape of the spectrum) Let* $\{\mathcal{A}(\mathcal{O}), \mathcal{A}, \mathbf{R}^d, \alpha\}$ *be a local net with translation symmetry and positive energy. Let* $V(a)$ *denote the above minimal positive energy representation. Then the lower bound of specV is Lorentz-invariant.*

This is the counterpart of the classical fact that causal propagation can only be satisfied with L-covariant equations. As a result of this inexorable link in the classical theory, Einstein never separated the issue of L-invariance from causality.

Theorem 20 *(General cluster property) Let* $\{\mathcal{A}(\mathcal{O}), \mathcal{A}, \mathbf{R}^d, \alpha_a\}$ *be a local net as before and* ω *a tranlation invariant state and* $\{\pi, H, \Omega, U(a)\}$ *the GNS-representation with* $U(a)\Omega = \Omega$. *Denote by* P_0 *the projector onto the subspace of pointwise invariant vectors i.e.* $\Omega \in P_0 H$. *Assume furthermore that the center of* $\pi(A)''$ *is pointwise invariant under* α_a. *Then we have the following relation:*

$$\begin{aligned} & \lim_{\lambda\to\infty} (\Omega, \pi(A_1)\pi(\alpha_{\lambda b}B_1)\pi(A_2)...\pi(A_n)\pi(\alpha_{\lambda b}B_n)\Omega) \qquad (7.6) \\ = \; & (\Omega, \pi(A_1A_2...A_n)P_0\pi(B_1B_2...B_n)\Omega) \\ = \; & (\Omega, \pi(B_1B_2...B_n)P_0\pi(A_1A_2...A_n)\Omega), \quad b \text{ spacelike} \end{aligned}$$

Naturally some of the A_i, B_j may be identity operators $\mathbf{1}$ which allows to have $\#A \neq \#B$. In the case of the unique vacuum and a spectral mass gap one may prove the strong approach of the right hand side (faster than any inverse power in λ) which is the standard form of the cluster property. This is then the starting point of the derivation of scattering theory.

Theorem 21 *(Additivity of spectrum) Let* $\{\pi, H, U(a)\}$ *be a factor representation (a von Neumann algebra with trivial center:* $\mathcal{Z} = \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$*) of a theory of local observables fulfilling the spectrum condition and assume that* $U(a)$ *is the minimal representation. Then if* p_1 *and* p_2 *are in specP, so is* $p_1 + p_2$. *moreover if the mass spectrum consists of a discrete part* $m_0 < m_1 < ...$ *and a continuum starting at* $m_c > m_i$ *then:*

$$3m_0 \geq m_c$$

The expected relation $m_c = 2m_0$ remains still unproven.

Theorem 22 *(Absence of classical fields) There exist no classical field theories (i.e. abelian algebras) which fulfill the spectrum condition.*

Theorem 23 *(Algebraic generalization of Bohr-Rosenfeld observation on e.m. fields) Let* $\{\mathcal{A}(\mathcal{O}), \mathcal{A}, \mathbf{R}^4, \alpha\}$ *be a theory of local observables and define:*

$$\mathcal{A}(a) = \bigcap_{a \in \mathcal{O}} \mathcal{A}(\mathcal{O})$$

Then $\mathcal{A}(a) \subset \mathcal{Z}(\mathcal{A})$.

Clearly this may be interpreted as a generalization of the nonexistence of finite e.m. quantum field strength at a point i.e. the necessity for smearing quantum fields. This theorem which is due to Wightman has generalizations to subsets of Minkowski space. For spacelike 3-d hypersurfaces and for time-like segments the analogously defined algebras are nontrivial and equal to the algebras of their causal completions i.e. $\mathcal{A}(\mathcal{O}'')$

7.2 Abstracting Principles from Standard Setting

If we use fields in standard QFT in order to define local nets of field algebras $\{\mathcal{F}(\mathcal{O})\}$, we find the following properties[3]:

- $\tilde{P}$−covariance, positive energy and uniqueness of the vacuum.

  $\exists$ a strongly continuous representation U of the covering of the Poincaré group $P_+^\uparrow$:

  $$U(L)\mathcal{F}(\mathcal{O})U(L)^{-1} = \mathcal{F}(L\mathcal{O}) \tag{7.7}$$

  and the generators P_μ of the translations satisfy the spectrum condition $spec \in V^\uparrow$ with $P\Omega = 0$, Ω being the unique vacuum.

- $\exists$ a compact (global gauge) group[4] G and stronly continuous faithful representation U of G which commutes with the Poincaré group (factorization of internal and external symmetries) s.t.: $U(g)\mathcal{F}(\mathcal{O})U(g)^{-1} = \mathcal{F}(\mathcal{O}), \quad U(g)\Omega = \Omega$

- $\exists \kappa \in G$ of order two i.e. $\kappa^2 = 1$ s.t. with $\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_-$, $\alpha_\kappa(\mathcal{F})_\pm = \pm\mathcal{F}$ and spacelike separated $\mathcal{O}_1$and $\mathcal{O}_2$; the following graded (or "twisted") locality relation holds:

$$\begin{aligned} \{\mathcal{F}_-(\mathcal{O}_1), \mathcal{F}_-(\mathcal{O}_2)\} &= 0 \\ [\mathcal{F}_+(\mathcal{O}_1), \mathcal{F}_-(\mathcal{O}_2)] &= 0 = [\mathcal{F}_+(\mathcal{O}_1), \mathcal{F}_+(\mathcal{O}_2)] \end{aligned} \tag{7.8}$$

[3] The reader interested in technical and conceptual details should follow the historical path and look at Haag's book and the original articles.

[4] Always global gauge group, because local gauge groups are not related to symmetries in the same theory.

We write this in the condensed form:

$$\begin{aligned} \mathcal{F}(\mathcal{O}') &= \mathcal{F}(\mathcal{O})^{tw}, \quad \mathcal{F}(\mathcal{O})^{tw} := V\mathcal{F}(\mathcal{O})' \\ K \text{ s.t. } K\mathcal{F}(\mathcal{O})K^{-1} &= \kappa(\mathcal{F}(\mathcal{O})) \end{aligned} \tag{7.9}$$

- Additivity: $\mathcal{F}(\mathcal{O}) = \bigvee_i \mathcal{F}(\mathcal{O}_i), \quad \mathcal{O} = \bigcup \mathcal{O}_i$
- Haag (twisted) Duality:

$$\begin{aligned} \mathcal{F}(\mathcal{O}) &= \mathcal{F}(\mathcal{O}')^{tw} \curvearrowright \\ \mathcal{A}(\mathcal{O}')' \quad | \quad & H_0 = \mathcal{A}(\mathcal{O})''|_{H_0}, \ \mathcal{A}(\mathcal{O}) := \mathcal{F}(\mathcal{O}) \cap U(G)' \end{aligned} \tag{7.10}$$

where the observable algebra is defined by this invariance principle and the von Neumann algebra of a noncompact region, as the causal complement of a double cone $\mathcal{O}'$, are defined in terms of an additive covering by double cones together with von Neumann closure:

Comments: although these properties have an enormous plausibility, I recommend to look up the proofs. The conclusion $\curvearrowright$does not hold in the case of d=1+1 where the order-disorder duality makes its appearance (see chapter 3, section 7). The "quantum intuition" aquired from standard QT (as well as from Lagrangian quantization) is treacherous in QFT, an area for which a good intuition still needs to be developped.

The notation $|_{H_0}$ denotes the restriction to the vacuum sector H_0: $U(G)H_0 = H_0$ pointwise in H_0. What is referred to as the observable algebra in these notes is not $\mathcal{A}$ in H but rather the smaller (irreducible) algebra $\mathcal{A}|_{H_0}$. The gauge invariant part can also be obtained via the conditional expectation (averaging with compact group):

$$\begin{aligned} m(F) \quad &: = \int_G dg\alpha_g(F), \quad \int_G dg = 1 \\ \text{properties} \quad &: (1)\ m(\mathcal{F}(\mathcal{O})) = \mathcal{A}(\mathcal{O}) \\ &\quad (2)\ m \text{ is normal,i.e. } \sigma - cont. \\ &\quad (3)\ m \text{ commutes with } \alpha_g \end{aligned} \tag{7.11}$$

The continuity property (2) allows the continuation of m to all operators $B(H)$. We obtain $\mathcal{F}'' = B(H), \mathcal{A}'' = m(\mathcal{F}'')$ and hence $\mathcal{A}'' = m(B(H))$ as well as $\mathcal{A}' = U(G)''$

This gives us the desired tensor decomposition of the Hilbert space:

$$H = \bigoplus_\sigma H_\sigma \otimes H'_\sigma \tag{7.12}$$

where the first factor H_σ is the irreducible representation space for the irreducible representation $U_\sigma(G)$ of the internal symmetry group and H'_σ denotes

its (infinite dimensional) multiplicity space which is an irreducible representation space of $\mathcal{A}$ correspoding to π_σ . With other words we have:

$$\begin{array}{rcl} A & | & H_\sigma \otimes H'_\sigma = 1_{H_\sigma} \otimes \pi_\sigma(A) \quad A \in \mathcal{A} \\ U(g) & | & H_\sigma \otimes H'_\sigma = U_\sigma(g) \otimes 1_{H'_\sigma} \quad g \in G \end{array} \tag{7.13}$$

$\mathcal{A}$ in H contains generally many other irreducible representations π_σ besides the vacuum representation, and the primer into the theory of superselection sectors consists in classifying these, in particular to understand what properties they share. For this purpose we introduce minimal projectors in the algebra $U(G)''$:

$$E = \int dg U(g)\,(\phi, U_\sigma(g^{-1})\phi) \quad \phi \in H_\sigma \text{ arbitrary, } \|\phi\| = 1 \tag{7.14}$$

Since according to the Reeh-Schlieder theorem $\mathcal{F}(\mathcal{O})$ acts cyclically on Ω, we always find elements $F \in \mathcal{F}(\mathcal{O})$ with $EF\Omega \neq 0$. The definition:

$$T\psi = EF\psi \quad \psi \in H_0$$

determines a partial intertwiner $T : H_0 \to EH$ with the intertwining property:

$$T\pi_0(A) = \pi_E(A)T, \quad A \in \mathcal{A}(\mathcal{O}') \tag{7.15}$$

The reader easily checks that the vectors $T\Omega \in H_E$ and $|T|\Omega \in H_0$ (since $T^*T : H_0 \to H_0$) have the same expectation values on $\mathcal{A}(\mathcal{O}')$ i.e. induce the same partial states. Using the Reeh-Schlieder cyclicity one shows that there are sufficiently many partial intertwiners such that the set of states over $\mathcal{A}(\mathcal{O}')$ in all representation obtained from the decomposition of $\pi(\mathcal{A})$ on H agree i.e. the restriction of the net $\mathcal{A}$ to $\mathcal{A}(\mathcal{O}')$ gives the same folium (see mathematical appendix) of states independent of the charge sector σ.

Theorem 24 *All irreducible subrepresentations π_σ satisfy the (DHR) condition:*

$$\pi_\sigma \,|_{\mathcal{A}(\mathcal{O}')} = \pi_0 \,|_{\mathcal{A}(\mathcal{O}')} \tag{7.16}$$

i.e. the representations of the observable algebra (obtained from an invariance principle on the field algebra) are unitarily equivalent in the causal complement of any space time region which admits a nontrivial causal complement.

This is taken as a definition of (DHR) compactly localizable representations for an arbitrary observable net.

7.3 Starting the Reverse: the DHR Endomorphisms

The previous DHR localization condition may be now be taken as the starting point of the most important part of algebraic QFT: the DHR superselection

theory. Let us start with the classification of abelian (simple) sectors because they are also simpler in the everyday use of the word. In order to appreciate the following definitions, one should think of one-dimensional representations of a group G which form a subcategory of representations closed under compositions (i.e. the projectors E_σ on $H_\sigma \otimes H'_\sigma$ of the previous section are elements of the center $\mathcal{Z}(U(G'')) = \mathcal{Z}(\mathcal{A}'')$ or $U(g)E_\sigma = E_\sigma U(g) = \chi(g)E_\sigma$). In order to understand this property in terms of observables $\mathcal{A}$ (without $\mathcal{F}$) only, we convince ourselves that the representation π_σ satisfies the Haag Duality property, which we up to now only met in connection with the vacuum representation:

$$\pi(\mathcal{A}(\mathcal{O}'))'' = \pi(\mathcal{A}(\mathcal{O}))' \cap \pi(\mathcal{A})'' \tag{7.17}$$

Comment: the left hand side is the von Neumann algebra generated by $\mathcal{A}(\mathcal{O}_1)$ for all $\mathcal{O}_1 \subset \mathcal{O}'$. For π irreducible the relation is often written as $\pi(\mathcal{A}(\mathcal{O}')) = \pi(\mathcal{A}(\mathcal{O}))'$. Replacing $=$ by $\subset$ we have the Einstein causality relation, therefore 7.17represents a (maximal as i8t turns out) strengthening of causality.

A π_σ as obtained in the previous section by restriction from a field algebra $\mathcal{F}$ fulfills Haag Duality since $\mathcal{A}(\mathcal{O})' = (\mathcal{F}(\mathcal{O})' \cap U(G)')' = \mathcal{F}(\mathcal{O})' \vee U(G)''$ and acting with the projection by E_σ as well as with m from both sides (those actions commute) the $U(G)''$ is killed and we obtain: $\pi_\sigma(\mathcal{A}(\mathcal{O}))' = E_\sigma(\mathcal{F}(\mathcal{O})'E_\sigma = E_\sigma(\mathcal{F}(\mathcal{O})' \cap U(G)')E_\sigma = \pi_\sigma(\mathcal{A}(\mathcal{O}'))''$ where in the last step we used the twisted duality of $\mathcal{F}$. We will later see that representations of the observable net $\mathcal{A}$ fulfill Haag duality iff they correspond to simple sectors.

Let us now start to do the reverse, namely construct a charge carrying field algebra $\mathcal{F}$ from the observable algebra A and its DHR7.16 representations. We first must find some good mathematical concepts to classify the DHR localized representations. The unitary equivalence of $\pi(\mathcal{A}(\mathcal{O}'))$ with $\pi_0(\mathcal{A}(\mathcal{O}'))$ in7.16 garanties the existence of partial intertwiners i.e. isometries V: $H_0 \to H_\pi$ with:

$$V\pi_0(A) = \pi(A)V, \quad A \in \mathcal{A}(\mathcal{O}') \tag{7.18}$$

We define a representation $\hat{\pi}(\mathcal{A})$ in H_0 equivalent to $\pi(\mathcal{A})$ in H_π by:

$$\hat{\pi}(A) := V^{-1}\pi(A)V, \quad A \in \mathcal{A} \tag{7.19}$$

By construction this representation agrees with π_0 in $\mathcal{O}'$. For sufficiently large regions namely $\mathcal{O}_1 \supset \mathcal{O}$, the range of $\hat{\pi}$ is contained in that of π_0 i.e. $\hat{\pi}(\mathcal{A}(\mathcal{O}_1)) \subset \pi_0(\mathcal{A}(\mathcal{O}_1))$ and hence a fortiori $\hat{\pi}(\mathcal{A}) \subset \pi_0(\mathcal{A})$. This follows by using (vacuum) Haag Duality, namely: $[\pi_0(A'), \hat{\pi}(A)] = \hat{\pi}\,[A', A] = 0$ for $A' \in \mathcal{A}(\mathcal{O}'_1)$, $A \in \mathcal{A}(\mathcal{O})$ and $\curvearrowright \hat{\pi}(A) \in \pi_0(A(\mathcal{O}'_1)' \subset \pi_0(\mathcal{A})$ by Haag duality. Therefore ρ defined by:

$$\rho := \pi_0^{-1} \circ \hat{\pi}, \quad \rho : \mathcal{A} \to \mathcal{A} \tag{7.20}$$

is an endomorphism of the C^*algebra $\mathcal{A}$ with the following remarkable properties:

- ρ is localized in $\mathcal{O}$ ($\mathrm{loc}\rho \subset \mathcal{O}$), i.e. $\rho(A) = A$, $A \in \mathcal{A}(\mathcal{O}')$

- transportable, i.e. $\forall \mathcal{O}_1, \mathcal{O}_2$ with $\mathcal{O}_2 \supset \mathcal{O}_1 \cup \mathcal{O}$ $\exists U \in \mathcal{A}(\mathcal{O}_2)$ *s.t.*$AdU \circ \rho(A) = A$ for $A \in \mathcal{A}(\mathcal{O}_1')$
- $\rho(\mathcal{A}(\mathcal{O}_1)) \subset \mathcal{A}(\mathcal{O}_1) \quad \forall \mathcal{O}_1 \supset loc\rho$

The very simple proof of these properties is left to the reader. We follow Haag and call the set of such $\rho's$ Δ, and denote by $\Delta(\mathcal{O})$ the subset of $\rho's$ with $loc\rho \in \mathcal{O}$.

In the constructive approach based on the observable net and its endomorphisms with the above properties, one defines the *sectors* as the equivalence classes of $\rho's$ modulo inner automorphisms. The following structural investigation of localized transportable endomorphisms is independent of the dimensionality of the QFT i.e. holds as well for low dimensional theories. Let us first look at abelian sectors which by definition are equivalence classes of automorphism i.e. $\rho's$ with $\rho(\mathcal{A}) = \mathcal{A}$.

Theorem 25 *ρ is automorphism$\Leftrightarrow$ $\pi_\rho = \pi_0 \circ \rho$ is Haag dual$\Leftrightarrow$ ρ^2 is irreducible (no branched fusion)$\Leftrightarrow$ Ind$[A : \rho(A)] = 1$ (trivial Jones index)*

The reader should try to prove it for himself and consult Haags book if he needs more than 5 lines.

In algebraic QFT the Jones index enters through the statistics operators ε which we explain briefly in the sequel. They are special intertwiners ("Verketter" in the sense of Schur). An intertwining operator is a $V \in B(H)$ which links a representation $\pi_0\rho$ with a subrepresentation of $\pi_0\sigma$ i.e. $V \cdot \pi_0\rho(a) = \pi_0\sigma(a) \cdot V$. In case that ρ is irreducible, the two representations are equivalent and the intertwining operator becomes a "charge transporting" operator. By Haag duality[5] one obtains $V = \pi_0(T)$ with $T \in \mathcal{A}$ and the intertwining relations:

$$T\rho(A) = \sigma(A)T \quad \forall\ A \in \mathcal{A} \tag{7.21}$$

The space of self-intertwiners: $\rho \to \rho$ is the commutant $\rho(\mathcal{A})'$ of $\rho(\mathcal{A})$ in $\mathcal{A}$ and by Schur's lemma, equal to the scalars C iff ρ is irreducible. Therefore, when ρ is irreducible, the linear space of intertwiners: $\rho \to \sigma$ is a *Hilbert space within the algebra of local observables* with the inner product $(T_1, T_2) := T_1^* T_2$. The notation for the space of intertwiners T from σ to ρ is $T \in (\rho, \sigma)$.

For every pair of DHR endomorphisms there is a unitary local intertwiner $\varepsilon(\rho, \sigma) : \rho\sigma \to \sigma\rho$ i.e.$\varepsilon \in (\sigma\rho, \rho\sigma)$. This flip operator is called the *statistic operator*. The collection of statistics operators is uniquely determined by the coherence with local intertwiners and among themselves:

$$\begin{aligned} \varepsilon(\sigma_1, \sigma_2)\sigma_1(T_2)T_1 &= T_2\rho_2(T_1)\varepsilon(\rho_1, \rho_2) \quad \forall\ T_i : \rho_i \to \sigma_i \\ \varepsilon(\rho_1\rho_2, \sigma) &= \varepsilon(\rho_1, \sigma)\rho_1(\varepsilon(\rho_2, \sigma)) \\ \varepsilon(\rho, \sigma_1\sigma_2) &= \sigma_1(\varepsilon(\rho, \sigma_2))\varepsilon(\rho, \sigma_1) \end{aligned} \tag{7.22}$$

[5] In the reverse approach which starts from the observable algebra $\mathcal{A}$, the Haag duality is postulated for the vacuum representation. if it does not hold for the original net, one passes to the dual net $\mathcal{A}^d$ which fulfills Haag duality by construction.

together with the "initial conditions"

$$\begin{aligned} \varepsilon(\rho, id) &= \varepsilon(id, \rho) = 1 \\ \varepsilon(\rho, \sigma) &= 1 \quad \text{whenever } \sigma < \rho \end{aligned} \tag{7.23}$$

where $\sigma < \rho$ means locσ is in the spacelike complement on the left of locρ. The Artin braid relation is a special consequence of the above coherence relations

$$\begin{aligned} & \rho_3(\varepsilon(\rho_1, \rho_2))\varepsilon(\rho_1, \rho_3)\rho_1(\varepsilon(\rho_2, \rho_3)) \\ = \; & \varepsilon(\rho_2, \rho_3)\rho_2(\varepsilon(\rho_1, \rho_3))\varepsilon(\rho_1, \rho_2) \end{aligned} \tag{7.24}$$

In particular, by assigning the local operators $\rho^{i-1}(\varepsilon(\rho, \rho))$ to the standard Artin generators σ_i of the braid group B_n (see remarks and figure in chapter1) we obtain a unitary representation of the braid group in $\mathcal{A}$ wich we call the statistics of the endomorphism ρ for reasons which will become evident soon.

We introduce a conjugate endomorphism $\bar{\rho}$ to ρ by demanding that $\bar{\rho}\rho$ contains the vacuum sector, i.e. that there exists an intertwiner $R \in (id, \bar{\rho}\rho)$ which induces a standard left inverse ϕ of ρ

$$\phi(A) = R^* \bar{\rho}(A) R \quad \forall\, A \in \mathcal{A} \tag{7.25}$$

with finite statistics. Here we recall that the left inverse of an endomorphism ρ of $\mathcal{A}$ is a normalized positive linear map satisfying the relation $\phi(\rho(A)B\rho(C)) = A\phi(B)C$. It is called regular if it is of the above form, and standard if in addition the statistics parameter $\lambda_\rho := \phi(\varepsilon(\rho, \rho)) \in \rho(\mathcal{A})'$ is a nonvanishing multiple of a unitary which depends only on the sector $[\rho]$. A sufficient condition for the existence of a standard left-inverse and therefore of a conjugate is that there is *some* left-inverse with statistics parameter $\lambda_\rho \neq 0$ ("finite statistics") and that ρ is translation covariant with positive energy condition. the uniqueness of the standard left inverse is a consequence of its definition. Any theory with a mass gap possesses a standard left inverse. The standard left inverse of ρ turns out to be a trace on $\rho(\mathcal{A})'$. The inverse modulus of λ_ρ is called the *statistical dimension* $d(\rho) \equiv d_\rho \geq 1$. One easily proves that $\lambda_\rho = \lambda_{\bar{\rho}}$. For irreducible $\rho's$ we have $\lambda_\rho = \frac{\kappa_\rho}{d_\rho}$ with κ_ρ being the *statistical phase*. If one computes this numbers using the field formalism presented below, one finds $d_\rho = dim H_\rho$ and $\kappa_\rho = \pm 1$ for Bosons/Fermions. In fact for d=3+1 the statistics operator is easily shown on general grounds to fulfill $\varepsilon^2 = 1$ (absence of monodromies) which leads to permutation group statistics.The concepts are much richer in the case of braid group statistics. Even in that case one succeeds to prove the identity of the spin phase with the above statistics phase.

We will not enter a presentation of V. Jones inclusion theory, but just mention that $\mathrm{Ind}[A : \rho(A)] = d_\rho^2$, i.e. the index is the square of the statistical dimension.

In order to understand the reconstruction in the case of genuine endomorphisms i.e. for $\rho's$ with $\mathrm{Ind}[A : \rho(A)] > 1$, we need some more conceptual preparation. This is obtained by briefly returning to the field algebra $\mathcal{F}$ in the

case where $\mathcal{A}$ is the fixed point algebra under a nonabelian G. In that an irreducible endomorphism ρ with $\pi_0 \circ \rho \simeq \pi_\rho$ and $\mathrm{loc}\rho \subset \mathcal{O}$ gives via $\omega = \omega_0 \circ \rho$ a pure state localized in $\mathcal{O}$. The big Hilbert space H in which $\mathcal{F}$ acts has many vectors which induce ω :

$$H_\omega = \left\{ \phi \in H \mid (\phi, A\phi) = \omega(A) \|\phi\|^2 \right\} \tag{7.26}$$

As the notation already anticipates, H_ω is a Hilbert space (i.e.linear combinations again belong to H_ω) a fact which is easily verified using the purity of ω. Its dimension is equal to the dimension of the $H_\rho-$ tensor factor namely d_ρ. Physical intuition tells us that such vectors in H_ω can be created from the vacuum by applying charge-carrying fields in $\mathcal{F}$. In fact we have:

Theorem 26 *Every $\phi \in H_\omega$ determines uniquely a field operator $\psi \in \mathcal{F}(\mathcal{O})$ with $\psi^*\Omega = \phi$ and the intertwining property $\psi A = \rho(A)\psi$, $A \in \mathcal{A}$..*

Encouraged by the intertwining relation in the previous theorem we define the following linear subspace of $\mathcal{F}$:

$$H_\rho = \{\psi \in \mathcal{F} \mid \psi A = \rho(A)\psi, A \in \mathcal{A}\} \tag{7.27}$$

The notation suggests the structure of a Hilbert space. Indeed for two vectors $\psi_i, i = 1, 2$ we have the following scalar product:

$$\psi_1^*\psi_2 \in \mathbf{C} \cdot \mathbf{1} \tag{7.28}$$

The reason is that the inclusion of $\mathcal{A} \in \mathcal{F}$ is irreducible i.e. $\mathcal{A}' \cap \mathcal{F} = \mathbf{C} \cdot \mathbf{1}$. This follows from $\mathcal{A}' \cap \mathcal{F} = U(G)'' \cap \mathcal{F}$ and the statement that for any element F_0 with $F_0 \mid_{H_0} = c\mathbf{1}_{H_0}$ from the latter algebra the conditional expectation of F^*F with $F := F_0 - c\mathbf{1}$ vanishes: $0 = \pi_0(m(F^*F)) \curvearrowright m(F^*F) = 0$ since π_0 is faithful. But the expectation values of m in any vector state $\phi \in H$ may be written as an average with a positive integrand:

$$\begin{aligned} 0 &= (\phi, m(F^*F)\phi) = \int dg (\phi, \alpha_g(F^*F)\phi) \curvearrowright \\ \alpha_g(F^*F) &= 0 \curvearrowright F^*F = 0 \curvearrowright F_0 = c\mathbf{1} \quad qed \end{aligned} \tag{7.29}$$

This phenomenon of finding Hilbert spaces of isometries inside von Neumann algebras always occurs for $B(H)$ with H infinite dimensional.

Theorem 27 *For any set of field operators $(F_i)_{i=1\ldots d_\rho} \in \mathcal{F}(\mathcal{O})$ transforming like an irreducible tensor representation U_ρ there exists a $\rho \in \Delta(\mathcal{O})$ and a $B \in A(\mathcal{O})$ s.t. $F_i = B\psi_i$ with $(\psi_i) \in H_\rho$ a orthonormal system of isometries spanning H_ρ. The endomorphism ρ is implemented by the $\psi_i's$:*

$$\rho(F) = \sum_{i=1}^{d_\rho} \psi_i F \psi_i^* \tag{7.30}$$

$$\textit{with the left inverse } \phi(F) = \frac{1}{d_\rho} \sum_{i=1}^{d_\rho} \psi_i^* F \psi_i, \textit{ i.e. } \phi(\rho(F)) = F$$

The most interesting and useful emerging structure is the so called Cuntz algebra O_d i.e. the unique C^*−algebra generated by a family of isometries (ψ_i) with a full range i.e. $\sum \psi_i \psi_i^* = \mathbf{1}$. A detailed investigation (not done here) reveals that this is a Z-graded simple (no two-sided ideals) C^*algebra. Doplicher and Roberts found that this algebra O_d is the perfect model for a characterization of the group dual which is appropiate for internal symmetries in QFT. The reason is that since each compact group G is a subgroup of some $U(d)$ for sufficiently large N, there is a natural action α on O_d (summation convention):

$$\alpha_g(\psi_i) = \psi_j g_{ji}, \quad \text{unitary in } H_d \tag{7.31}$$

The tensor product structure is naturally contained in O_d since $H^k \simeq H \otimes ... \otimes H$. The fixed point algebra :

$$O_G = \{A \in O_d \mid \alpha_g(A) = A \; \forall g \in G\}$$

gives rise to an inclusion $O_G \subset O_d$ which, in analogy with Galois theory is expected to encode the group structure. It naturally contains all the intertwiners of tensor representations $T : H^{\otimes r} \to H^{\otimes s}$. In terms of endomorphisms these may be characterized without tensor products purely algebraically:

$$T\rho^r(A) = \rho^s(A)T, \quad A, T \in O_G \tag{7.32}$$

This should be compared with the "classical" Tanaka-Krein theory of group duals in terms of representation spaces and intertwiners. In QFT based on observables $\mathcal{A}$ only, one only knows the $\rho's$ of $\mathcal{A}$ and neither the $H'_\rho s$ nor $\mathcal{F}$. So the question how to construct from an algebra of intertwiners a bigger algebra with a group action is a "baby version" of the QFT symmetry problem: how to reconstruct the symmetry from its shadow it leaves on the observables (in analogy to the famous problem of Marc Kac: how to hear the shape of a drum?). For a successful treatment we must make sure that our representation category i.e. the endomorphisms and their intertwiners are big enough in order to contain the conjugates (antiparticle representations in QFT). This is easily achieved by securing the existence of a faithful selfconjugate representation because the tensor products of such a representation contain every irreducible representation. Let us briefly look at the special case $SU(d)$. The first tensor power which contains the identity representation is H^d, explicitly the first invariant is:

$$S = \frac{1}{\sqrt{d!}} \sum_{P \in S_d} sign(P) \psi_{P(1)} .... \psi_{P(d)} \tag{7.33}$$

Hence:

$$\hat{\psi}_i = \frac{1}{\sqrt{(d-1)!}} \sum_{\substack{P \in S_d \\ P(1)=i}} sign(P) \psi_{P(2)} .... \psi_{P(d)} \tag{7.34}$$

fulfill $\hat{\psi}_i = \sqrt{d} \psi_i^* S$ and therefore is a basis in H_d^{conj} and $\bar{\rho}(A) = \sum_{i=1}^d \hat{\psi}_i A \hat{\psi}_i$. Thus $O_{SU(d)}$ contains all irreducible representations of $SU(d)$ and every intertwiner. But how do we recognize that a C^*-algebra is isomorphic to $O_{SU(d)}$?

The answer is surprisingly simple: in addition to the operator S we must find a copy of the (infinite) permutation group S_∞. The model theory $O_{SU(d)} \subset O_d$ as such a presentation:

$$\begin{aligned} \varepsilon(P) &= \sum \psi_{(\alpha)}\psi^*_{(\alpha_P)}, \quad P \in S_n \subset S_\infty, \quad S_n \hookrightarrow S_{n+1} \qquad (7.35) \\ (\alpha) &= (\alpha_1, ..., \alpha_n), \ (\alpha_P) = (\alpha_{P(1)} .... \alpha_{P(n)}) \end{aligned}$$

where we used a multiindex notation. In particular the formula for the basic transposition is:

$$\varepsilon((12)) = \sum_{i,j} \psi_i \psi_j \psi_i^* \psi_j^* = \pm\rho(U^*)U, \quad \pm : F/B \qquad (7.36)$$

where U is an auxilary charge transporter: $\psi_i' = U\psi_i$ which shifts the charge into $\mathrm{loc}\psi_i' \subset (\mathrm{loc}\psi_i)'$.

One easily checks that the ε, S and ρ are related by:

$$\begin{aligned} SS^* &= E_d := \frac{1}{d!} \sum_{P \in S_d} sign(P)\varepsilon(P) \\ S^*\rho(S) &= (-1)^{d-1}d^{-1}\mathbf{1} \\ \rho(\varepsilon(P)) &= \varepsilon(P'), \quad P' \in S_{n+1}, \quad P'(1) = 1, \ P'(i+1) = P(i) \ i = 1....n \\ \phi(\varepsilon(P)) &= \begin{cases} \varepsilon(P) & P(1) = 1 \\ \frac{1}{d}\varepsilon(P') & P(1) \neq 1, \quad P'(i) = ((1P(1))P)(i+1) \end{cases} \end{aligned}$$

Here E_d is the antisymmetric representation projector in the S_d group algebra and ρ and its left inverse ϕ implement right and (partial) left shifts on S_∞. The algebra $O_{SU(d)}$ is generated by these permutation group $\varepsilon(P)$ and S-intertwiners. If $G \subset SU(d)$ then $O_G \supset O_{SU(d)}$ and therefore there are more generators.

We will give the DR characterization of G without proof:

Theorem 28 *Let $\hat{O}$ be a simple C^*-algebra with an endomorphism $\hat{\rho}$ and a unitary representation $\hat{\varepsilon}$ of S_∞ with the following properties:*

(i) $\hat{\varepsilon}(P) \in (\hat{\rho}^n, \hat{\rho}^n)$, $P \in S_n$

(ii) $\hat{\varepsilon}((12...n+1))\hat{T} = \hat{\rho}(\hat{T})\hat{\varepsilon}((12...m+1))$, $\hat{T} \in (\hat{\rho}^n, \hat{\rho}^n)$

(iii) $\exists \hat{S} \in (id, \hat{\rho}^d)$ *with* $\hat{S}^*\hat{S} = \mathbf{1}$, $\hat{S}^*\hat{\rho}(\hat{S}) = (-1)^{d-1}\frac{1}{d}\mathbf{1}$ *and* $\hat{S}\hat{S}^* = \hat{E}_d$

(iv) $\hat{O}$ *is generated by the intertwiners* $\hat{T} \in (\hat{\rho}^n, \hat{\rho}^m)$, $n, m \in \mathbf{N}$

Then there is a closed subgroup $G \in SU(d)$ (unique up to conjugation) and an embedding of $\hat{O}$ in O_d with $\hat{O} \in O_G$, s.t. $\rho\,|_{O_G} = \hat{\rho}, \hat{\varepsilon} = \varepsilon$, and $\hat{S} = S$ hold.

The basic idea on which the proof relies is actually reasonably simple: one takes a kind of amalgamated product B of $\hat{O}$ with O_d amalgamated over $O_{SU(d)} \subset \hat{O}$, i.e.we look for an algebra with the following relations:

- $\psi_i A = \hat{\rho}(A)\psi_i, \quad A \in \hat{O}, \quad \psi_i \in O_d, \ i = 1....d$
- $\hat{\varepsilon}(P) = \varepsilon(P)$

- $\hat{S} = S$

The $SU(d)$ action on $\mathcal{B}$ is:

$$\alpha_g(\psi_i) = \sum_i \psi_i g_{ji}, \quad \alpha_g(A) = A$$

For the special case that $\hat{O}$ is generated by $\hat{\epsilon}(P)$ and $\hat{S} \curvearrowright \hat{O} = O_{SU(d)} \subset O_d$ and $\mathcal{B}$ is obviously O_d, as expected. If $\hat{O}$ is genuinly bigger, there exist intertwiner $\hat{O} \ni T \notin O_{SU(d)}$. The operators $\hat{T}_{(\alpha)(\beta)} := \psi^*_{(\beta)} T \psi_{(\alpha)}$ commute with $A \in \hat{O}$ and hence $T_{(\alpha)(\beta)} \in \hat{O} \cap \mathcal{B}$. Actually such operators are automatically in the center of $\mathcal{B}$ i.e. $\hat{O}' \cap \mathcal{B} = \mathcal{Z}(\mathcal{B})$. This follows from the invariance of $\hat{O}' \cap \mathcal{B}$ under the action of $SU(d)$ which means that this subalgebra consists of invariant $SU(d)$ tensors F_i. With $F_i, i = 1....n$ being a tensor multiplet we determine an orthonormal basis $\tilde{\psi}_i, i = 1....n$ and obtain the representation:

$$F_i = B\tilde{\psi}_i, \quad B = \sum_i F_i \tilde{\psi}^*_i$$

If we could show that the F_i commute with the generators of the Cuntz algebra, we would be done. But this is an easy computational result of :

$$B \in (\hat{\rho}^n, id), \quad \tilde{\psi}_i \in (id, \rho^n)$$

which according to the assumption (ii) of the previous theorem yields:

$$\begin{aligned} \hat{\rho}(B) &= B\hat{\epsilon}(n+1, ..., 1) \\ \rho(\tilde{\psi}_i) &= \epsilon(1, ..., n+1)\tilde{\psi}_i \end{aligned}$$

and hence $\psi_j F_i = \psi_j B \tilde{\psi}_i = \hat{\rho}(B)\rho(\tilde{\psi}_i)\psi_j = ... = \phi_i \psi_j$. qed.

Let us now return to the problem of construction of the field algebra. A helpful and informative intermediate construction is the introduction of the so-called "reduced field bundle". This is a bimodule over $\mathcal{A}$ which allows to use the $\rho's$ in a direct manner.

Definition 5 *As our Hilbert space in which we define the reduced field bundle we take the direct sum of vacuum Hilbert spaces $H_\alpha := (\alpha, H_0)$ and define operators (e, A) with $A \in \mathcal{A}$, $e = \{(\rho_\beta, \rho\rho_\alpha)\}$ the set of "charge edges" i.e triples of source charge ρ_α, transferred charge ρ and range charge ρ_β. All the irreducible endomorphisms are taken from a preselected set with one endomorphism ρ_α per sector $[\alpha]$. We the define:*

$$F(e, A) \cdot (\alpha, \psi) = (\rho_\beta, \pi_0(T_e^* \rho_\alpha(A)) \cdot \psi)$$

where $T_e \in \mathcal{A}$ are intertwiners from the space $T_e \in (\rho_\beta, \rho\rho_\alpha)$.

The $F's$ generate a C^*algebra $\mathcal{F}_{red}$ in $H = \oplus_\alpha H_\alpha$

This reduced field bundle only agrees with the field algebra of the standard approach if G is abelian. Its Hilbert space lacks the group theoretic multiplicities incorporated in the formula [?] and the net inclusion $\mathcal{A} \subset \mathcal{F}_{red}$ is reducible and its index is the square of the index of the irreducible inclusion $\mathcal{A} \subset \mathcal{F}$ (thinking of a finite G). For d=3+1 this is the parastatistics description which deals with higher Young tableaux but without an internal symmetry group. The parastatistics fields are more noncommutative and do not allow an interpretation in terms of "quantization" (e.g. they have no Lagrangians). In case of the nonabelian braid group statistics of chiral conformal field theory and d=2+1 plektons this is the only available description. There the F_{red}-algebra is also called the "exchange algebra"(Rehren-Schroer).

In case of d=3+1 theories and for the subcategory of permutation group statistics sectors in d<3+1 there exists the famous canonical Doplicher-Roberts construction of a genuine field algebra in the sense of the beginning of this section.

Let us first mention the special case of only automorphism and assume d=3+1 i.e. permutation group statistics. Since for automorphisms $d_\rho = 1$, the permutation group representation is abelian, we are dealing with Bosons/Fermions. In this case some very simple modifications of the reduced field bundle will give the field algebra. The interested reader is refereed to Haag's book.

For the above mentioned nonabelian representations one applies the DR theorem [?] about the construction of the group from a subalgebra of the Cuntz algebra with a distinguished endomorphism. Without loss of generality we may assume that there exists a ρ with statistical dimension d s.t. $id \subset \rho^d$ which can always be achieved by adding conjugates. A C^*-algebra $\hat{O}$ as needed in the theorem may be obtained via the inductive limit of intertwiner spaces:

$$^0\hat{O} := \bigcup_{n,m\geq 0} (\rho^n, \rho^m)$$

where the induction uses the embedding $(\rho^n, \rho^m) \to (\rho^{n+1}, \rho^{m+1}) : T \to T \times 1_\rho$. This leads to a natural composition of two operators S and T by embedding both in a space sufficiently shifted to the right. The algebra contains the statistics operators $\varepsilon(P)$ (still in bosonized form) as well as an isometry $S \in (id, \rho^d)$, $SS^* = E_d$. The endomorphism ρ acts in a natural way on $^0\hat{O}$ and $\phi(.) = S^*\rho(.)S$ defines a left inverse of ρ. The properies of $\phi, \rho, \varepsilon(P)$ and S are easily checked by computation. $^0\hat{O}$ has a unique C^*-norm and no ideal (i.e. $\hat{O}$ is simple). Now the DR theorem leads to the identification with a subalgebra $O_G \subset O_d$ with a subgroup G of $SU(d)$ which is determined up to conjugation. The field algebra is now simply the free product of the observable algebra A with the Cuntz algebra O_d modulo the following relations:

$$\psi_i A = \rho(A)\psi_i, \quad A \in \mathcal{A}$$

7.4 Remarks on Broken Symmetries

The idea of spontaneusly broken symmetries originated during th 60^{ies} in Lagrangian QFT (Goldstone, Nambu). There were parallel developments in condensed matter physics in which case the understanding of the phase transitions in the Heisenberg model was the main goal. Already at the end of this decade there was a general model independent understanding within the framework of QFT^{ies} pocessing conserved quantum Noether currents independent of their (Lagrangian or non-Lagrangian) origin. The main theorem of this more general approach was the relation of the nonexistence of the global charges (as a result of large distance infrared divergencies in the spatial integrals over currents) as a result of the occurrence of "Goldstone Bosons" in the energy-momentum spectrum. The nice feature of these rigorous methods is that they apply to composite "Goldstones"(no Lagrangian field i.e. beyond the family of Goldstone Lagrangian for which a perturbative approach to the broken phase is possible) as well.

Algebraic QFT offers an even more profound physical picture which we are going to explain in the sequel. The starting point is the DR reconstruction theory of the previous section. That theory always deals with unbroken symmetry G because only those transformations are in a one to one correspondence with the superselection sectors. Where to look for the bigger spontaneously broken group Γ? The answer is contained in the breakdown of the vacuum Haag duality of $\mathcal{A}$ (J.E.Roberts 1974). The physical reason for this is that certain operators which, if one only looks at their local properties, carry charges and transform according to Γ-multipletts globally condense into the vacuum sector. We have met a special case of this phenomenon in connection with the d=1+1 order/disorder discussion in the last section of chapter 3. The main point there was that the original net violated the vacuum Haag Duality and the order/disorder fields were required precisely in order to restore it. By definition we called the field which did not belong to the original vacuum representation but has a nonvanishing vacuum expectation "disorder". It is the adjunction of this field, which enlarges the observable net $\mathcal{A}$ to the Haag dual net $\mathcal{A}^d$. Adjusting this to the situation at hand, we assume that our original observable net $\mathcal{A}$ is smaller than its unique dual extension $\mathcal{A}^d$ i.e.:

$$\mathcal{A} \subset \mathcal{A}^d \subseteq \mathcal{F} \tag{7.37}$$

where $\mathcal{F}$ is the unique DR field algebra determined by the superselection theory of $\mathcal{A}^d$. The DR group G is the unbroken gauge group and $\Gamma \supset G$ with Γ defined to be the group of automorphisms of $\mathcal{F}$ which leave $\mathcal{A}$ pointwise fixed. G is the unbroken part of Γ. The following theorem demonstrates the correctness of this interpretation.

Theorem 29 *(Buchholz et al.)*

(i) Each $\gamma \in \Gamma$ leaves $\mathcal{F}(\mathcal{O})$ for each $\mathcal{O} \in \mathcal{K}$ globally stable and is locally normal.

(ii) G is the $\mathcal{F}-$vacuum stabilizer in Γ

(iii) The normalizer of G in Γ is the invariance subgroup which act automorphically on $\mathcal{A}^d$

The Goldstone theorem, i.e. the prediction of a special kind of zero mass particle as a result of spontaneous symmetry breaking, only *follows* under more stringent conditions and the *standard situation of a conserved Noether current* is certainly one possibility. In order to understand better the physics involved, let us look at the vacuum expectation of the derivation defined by the generator δ of the automorphism, using one-parametric subgroup:

$$\delta(F) = \lim_{\lambda \to 0} \lambda^{-1}(\gamma_\lambda(F) - F)$$

The criteria for a spontaneous symmetry breaking in the general setting of algebraic QFT without or with Goldstone particles are then formulated in terms of behaviour of the vacuum expectation of $\delta(F)$ for increasing localization regions.

From Lagrangian field theories one knows another mechanism of symmetry breaking which was first conjectured and exemplified by Schwinger and then brought into a perturbative setting by Higgs. Since it needs the formalism of gauge theory its, intrinsic content has never been spelled out; up to this date there is no known property of the observable part of the theory which tells us that a massive particle received its mass in such a way. Most of the folclore around this mechanism is not quite correct. For example the idea that the mechanism could be thought of as a "fattened" Goldstone boson is contradicted by the Schwinger model because in d=1+1 there arent any Goldstone bosons but there is the Schwinger-Higgs mechanism. However this does not mean that there are no consistent nonperturbative conjectures which have some chance to be proven in algebraic QFT. To me the most appealing idea is to think of a would be charged field with a Maxwellian i.e. very nonlocal charge[6]

7.5 Chiral Conformal Algebraic QFT

Chiral conformal QFT has turned out to be an ideal theoretical laboratory for algebraic QFT. Not only conformal QFT has profited from this happy marriage, but the confidence in the algebraic method has also significantly increased. To cite a recent example, within chiral conformal QFT one was able to rigorously prove the equivalence of the standard approach using pointlike covariant fields with the net approach (work by Fredenhagen and Joers). This is important because in formulating the net approch one did not want to open Pandora's Box concerning the physical content, but rather only put the advanced theory of von Neumann algebras to the use for exploring the physical principles of local QFT.

Since the literature on the subject, even if restricted by the above guideline, is quite formidable, I will limit my attention to two points:

[6]Formally a semiinfinite extended object formed from a Dirac field of QED modified by a Mandelstam like A_μ-flux to spatial ∞ serves as a candidate for a local gauge invariant, but global U(1) charge carrying field.

- What is charge structure and quantum symmetry of the $\mathcal{A}_{uni}\,(S^1)$-compactification?
- How does one classify chiral conformal QFT?

Concerning the algebra $\mathcal{A}_{uni}(S^1)$ as opposed to the non-compact DHR quasilocal algebra $\mathcal{A}(\mathbf{R})$ we note that the net $\{\mathcal{A}(I)\}_{I\subset S^1}$ is not directed as the nets of double cones in Minkowski space towards infinity). Therefore would should think of a globalization which is different from the inductive limit. For this we use the following definition universal algebra $\mathcal{A}_{univ}$:

Definition 6 *$\mathcal{A}_{univ}$ is the C^* algebra which is uniquely determined by the system of local algebras $(A(I))_{I\in\mathcal{T}}$,$\mathcal{T}$ = family of proper intervalls $I\subset S^1$ and the following universality condition:*

(i) there are unital embeddings $i^I : \mathcal{A}(I)\to\mathcal{A}_{univ}$ s.t.

$$i^J\mid_{\mathcal{A}(I)} = i^I \quad if \quad I\subset J,\ I,J\in\mathcal{T}$$

and $\mathcal{A}_{univ}$ is generated by the algebras $i^I(\mathcal{A}(I))$, $I\in\mathcal{T}$;

(ii) for every coherent family of representations $\pi^I : \mathcal{A}(I)\to\mathcal{B}(H_\pi)$ there is a unique representation π of $\mathcal{A}_{univ}$ in H_π s.t.

$$\pi\circ i^I = \pi^I$$

The universal algebra inherits the action of the Moebius group as well as the notion of positive energy representation through the embedding.

The universal algebra has more global elements than the quasilocal algebra of the DHR theory: $\mathcal{A}_{quasi}\equiv\mathcal{A}\subset\mathcal{A}_{univ}$ with the consequence that the vacuum representation π_0 ceases to be faithful and the global superselection charge operators which are outer for $\mathcal{A}$ become inner for $\mathcal{A}_{univ}$. From this observation emerges the algebra of Verlinde which originally was obtained by geometric rather than local quantum physics arguments. The removal of a point ξ from S^1 (this removal recreates the infinity of $\mathcal{A}_{quasi}$) forces $\mathcal{A}_{univ}$ to shrink to $\mathcal{A}$.

Most of this new features can be seen by studying global intertwiners in $\mathcal{A}_{univ}$. Let $I,J\in\mathcal{T}$ and $\xi,\zeta\in I'\cap J'$ (i.e. two points removed from the complements) and choose ρ and σ s.t. $\mathrm{loc}\rho,\sigma\subset I$ and $\hat\rho\in[\rho]$ with $\mathrm{loc}\hat\rho\subset J$. Then the statistics operators $\varepsilon(\rho,\sigma)$ and $\varepsilon(\sigma,\rho)\in\mathcal{A}(I)\subset\mathcal{A}_\xi\cap\mathcal{A}_\zeta$ are the same (i.e. they don't need a label ξ or ζ) independently of whether we use the quasilocal algebra $\mathcal{A}_\xi$ or $\mathcal{A}_\zeta$ for their definition. By Haag duality a charge transporter $V : \pi_0\rho\to\pi_0\hat\rho$ lies both in $\pi_0(\mathcal{A}_\xi)$ and $\pi_0(\mathcal{A}_\zeta)$. However its preimages with respect to the embedding are different. In fact:

$$\begin{aligned} V_\rho &: = V_+^* V_- \text{ with } V_+\in\mathcal{A}_\xi,\quad V_-\in\mathcal{A}_\zeta \\ V_\rho &: \rho\to\rho \end{aligned}$$

is a global selfintertwiner which is easily shown to be independent of the choice of V and $\hat\rho$. The representation of the statistics operators in terms of the charge transporters $\varepsilon(\rho,\sigma)=\sigma(V_+)^*V_+$, $\varepsilon(\sigma,\rho)^*=\sigma(V_-)^*V_-$ leads to:

$$\sigma(V_\rho)=\varepsilon(\rho,\sigma)V_\rho\varepsilon(\sigma,\rho)\ \curvearrowright\ \pi_0\sigma(V_\rho)=\pi_0\left[\varepsilon(\rho,\sigma)\varepsilon(\sigma,\rho)\right] \qquad (7.38)$$

The first identity is very different from the relation between $\varepsilon's$ due to local intertwiners. The global intertwiner V_ρ is trivial in the vacuum representation thus showing its unfaithfulness with respect to $\mathcal{A}_{univ}$, it only "comes to live" in charged representations where it coalesces with monodromy operators. From its definition it is clear that it represents a charge transport once around S^1[17]. As a result of its existence, the monodromy which is defined as the above two-fold iteration of the braid generator, takes on some of its geometric meaning which it has e.g. in the theory of complex functions. The left hand side of the first equation in 7.38 expresses a transport "around" in the presence of another charge σ, i.e. a kind of charge polarization. Let us look at the invariant version of V_ρ namely the global "Casimir" operators $W_\rho = R_\rho^* V_\rho R_\rho : id \to id$. This operator lies in the center $\mathcal{A}_{univ} \cap \mathcal{A}'_{univ}$ and depend only on the class (=sector) $[\rho]$ of ρ. By explicite computation one shows that after the numerical renormalization $C_\rho := d_\rho W_\rho$ one encounters the fusion algebra:

$$\begin{aligned} (i)\ C_{\sigma\rho} &= C_\sigma \cdot C_\rho \\ (ii)\ C_\rho^* &= C_{\bar{\rho}} \\ (iii)\ C_\rho &= \sum_\alpha N^\alpha C_\alpha \ \ if\ \rho \simeq \oplus_\alpha N^\alpha \rho_\alpha \end{aligned}$$

Verlinde's modular algebra emerges upon forming matrices with row index equal to the label of the central charge and the column index to that of the sector in which it is measured:

$$S_{\rho\sigma} := \left|\sum_\gamma d_\gamma^2\right|^{-\frac{1}{2}} d_\rho d_\sigma \cdot \pi_0\sigma(W_\rho)$$

In case of nondegeneracy of sectors, which expressed in terms of statistical dimensions and phases means $\left|\sum_\rho \kappa_\rho d_\rho^2\right|^2 = \sum_\rho d_\rho^2$, the above matrix S is equal to Verlinde's matrix S which together with the diagonal matrix $T = \kappa^{-1} Diag(\kappa_\rho)$, with $\kappa^3 = (\sum_\rho \kappa_\rho d_\rho^2)/\left|\sum_\rho \kappa_\rho d_\rho^2\right|$ satisfies the modular equations of the genus 1 mapping class group

$$\begin{aligned} SS^\dagger &= 1 = TT^\dagger, \quad TSTST = S \\ S^2 &= C, \quad C_{\rho\sigma} \equiv \delta_{\bar{\rho}\sigma} \\ TC &= CT = T \end{aligned}$$

The matrix S is similar to the character matrix in section 2 of the first chapter. However in distinction to nonabelian finite groups (which also yield a *finite* set of charge sectors of the fixed point observable algebra) the present nonabelian sectors produce a symmetric "character" matrix S which signals a perfect auto-duality between charge measurers $\{Q\}$ and charge creators $\{\rho\}$. Furthermore

[7]Note that in $\mathcal{A}_{univ}$ which corresponds to a compact quantum world it is not possible to "dump" unwanted charges to "infinity"(as in the case for $\mathcal{A}_{quasi}$), but instead one encounters "polarization" effects upon charge transportation once around.

the algebra Q generated by the central charges and the action of the endomorphisms on those charges[8] do not contain the old group theoretical stuff since the phenomenon of charge "polarization" is completely blind against it; only monodromic properties are retained in Q. This strongly suggests to look for the new "quantum symmetry" property by investigating the structural properties of Q. As a generalization of S one finds for the $Q's$ in the presence of more than one polarization charges the entries of the higher genus mapping class group matrices. Closely related to these structures are the knot theoretical invariants of 3-manifolds. Some of these objects also have been seen by analyzing certain formal functional integrals with the hindsight of geometry and topology. But in the context of algebraic QFT the physical interpretation is totally different: the new properties have nothing to do with the "living space" (in the sense of quantum theoretical localization) of fields or algebras but are rather manifestations of the inexorable link between external (space-time) and internal symmetries which one encounters in low dimensional QFT. They generalize in some sense the angular momentum decompositions and are expected to be useful e.g. for the analysis of scattering of d=2+1 plektons. Also these ideas of linking "quantum symmetry" with a kind of universal mapping class group (containing all geni) are highly seducive, I did not yet find an convincing argument why one should read the numerical aspects of those polarization charges as entries of mapping class matrices.

It should be mentioned here that most attempts in the direction of quantum symmetry have been directed towards modified ("weak"..) Hopf algebras which is more in the spirit of the DR by looking for same "square root" of the inclusion of the observable algebra in the reduced field bundle. None of the present attempts was successful. Perhaps it is helpful for the reader to define success in this context. One expects from a useful quantum symmetry concept a clarification of the following two points:

- A better understanding why in low dimensions the link between external/internal symmetries is so strong whereas in d=3+1 there was no possibility to bring them together in any nontrivial way. This aims in particular at a better physical understanding of the 3-manifold invariants.
- A simplification of the problem of computing correlations of "free" plektons, i.e. the freest objects (in d=2+1 preferably with vanishing cross sections) which fulfill the new braid group statistics.

Concerning the classification of chiral conformal QFT's, it is reasonable to approach this problem in two steps:

- Classification of the physically admissable braid group representations which go with the category of finitely many localizable representations ("rational" representations).

[8] This action leads out of the center and generates a global subalgebra of $\mathcal{A}_{univ}$.

- Construction of representative 4-point functions for the different plektonic families.

The basic techniques for this two step approach is quite old (Rehren-Schroer, "Artin Braids and Einstein Causality" Nucl.Phys. **B 312**, 715 (1989)) and have been elaborated for the unitary braid group representations affiliated with the special family of the Jones, Temperly-Lieb algebras. The more general representations are those affiliated with the Hecke algebra and with the Bierman-Wenzl algebra. Even if one does not know anything about these mathematical construction, the principles of algebraic field theory will lead us there and even supply us with an argument in favour of completeness of these families.

Here a detailed account of the construction of these families by using the fusions of algebraic QFT will be given.

7.6 Constructive Aspects of Plektons

One particle modular localization structure of d=2+1 Wigner representations with non-semiinteger U(1)-spin. The structure of the multi-particle scattering space of plektons. "Free" plektons: real particle conservation in the presence of virtual (off shell) nonconservation. Spacelike cone (semiinfinite string) localization properties in d=2+1. First attempts at plektonic construction.

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Chapter 8

Tentative Resume and Outlook

The most fruitful times in theoretical physics are those of clash of principles. A good clash can lead to an enormous amount of progress as exemplified by the rapid emergence of QM as a result of Bohr's atomic model and the ensuing paradoxical situation.

The post electro weak stagnation in particle physics and QFT did not lead yet to such a clearcut clash. The present situation restricted to QFT seems to be somewhat similar to the years preceeding the progress of "renormalization" of QFT. Then there was no clearcut clash with fundamental principles but many physicist thought that the correct handling of the ultraviolet problem requires radical new inventions. In my opinion the present situation is the result of a total disequilibrium between two modes of thinking whose delicate balance is the hall-mark of good times in theoretical physics. I am thinking of Dirac's method build on mathematics and estethical appeal versus the Bohr-Heisenberg-Einstein approach based on conceptual analysis. The modern Diracian approach is that of inventions based particularly on the formal geometric extension of formalism leaving behind the physical principles which originally led to this formalism. Recent illustrations are the invention of supersymmetry with most contributions emphasizing some formal aspects of perturbation theory but little attention to problems of stability with temperature as mentioned in the introduction.

In such a situation it seems to me to be helpful to revisit those old ideas in a critical spirit which were essential to the conceptual development of QFT. Naturally those structures which are most "Diracian" i.e. geometrical esthetical inventions in the setting of functional integrals have received our sharpest critical scrutiny (in particular the gauge concept and its alleged spontaneous breaking) because they are farthest from the principles of local quantum physics. The reason for their tremendous popularity can not be understood in solely logical terms. This trend which narrowed the rich conceptual structure of nonperturbative QFT to some rituals practiced on some classical field space is immediatly

understandable if one excepts a secret (not admitted or subconscious) longing for a classical (euclidean) structure. The guilt feeling resulting from this indulgence is then often compensated by a dosis of noncommutative geometry or q-deformation. We hope that we convinced at least some readers that there is a third way which consists simply in following the logic of the physical principles instead of formalism a path taken first which considerable success by E.P.Wigner. There are three big unsovled (resp.partially solved) problems which we presented in these notes: the problem of understanding the non-Lagrangian constructive (bootstrap formfactor) approach and the standard approach in a common setting, the elaboration of a theory of free anyons end plektons and the clarification of "quantum symmetry" in low dimensional QFT with braid group statistics in particular its appearant inexorable link with space time symmetry as exemplified by the relation to invariants of 3-mannifolds.

We also emphasized in these notes that despite its conservative way of dealing with physical principles, algebraic QFT leads to a radical change of paradigm: instead of the Newtonian view of a space time filled with a material content one enters the Leibnizian reality created by relation (in particular inclusions) between "monades"(the type III_1 local von Neumann factors $\mathcal{A}(\mathcal{O})$ which as single algebras are nearly void of physical meaning). Related to this is a very new esthetics namely the art of comprimating relations between very big objects as type III_1 von Neumann factors[1] into extremely simple structures which is very reminiscent to the esthetics of the V. Jones subfactor theory. Another important distinction between the standard approach and algebraic QFT is that the former deals already in its formulation with global concepts as e.g. functional integrals whereas the latter is making contact with global aspects (as global topology) only in a later stage. Those aspects of the vacuum structure, which through the spontaneous breaking of localizable symmetries and superselection rules are related to local properties of the theory, are accounted for. However vacuum degeneracies without any visible local origin as e.g. the vacuum structure in the Seiberg-Witten Duality construction are presently out of reach by methods of algebraic QFT.

It was our intention to apply the concepts of algebraic QFT to those problems which in our view are not appropriately taken care of by the standard quantization formalism or which may even contain paradoxa and physically fruitful contradictions. Examples are the local gauge concept, QFT's in curved space-time, the structure of nonperturbative low dimensional QFT's, and the role of various forms of "Duality" as well as"Quantum Symmetry". Even in cases where definite answers are still missing, algebraic QFT certainly casts a different and physically interesting light on those problems. We reviewed the two notion of temperature, the standard one being generated by a heath bath and the second one by a loss of information through the creation of a horizon (the Hawking-Unruh temperature or mathematically: the Tomita KMS temper-

[1] Those von Neumann algebras are the biggest in the sense that they absorbe any tensor factor with another von Neumann algebra. Although V.Jones formulated his subfactor theory in terms of the smallest (type II_1 which gets absorbed by any other tensor factor), his theory applies with only a few modifications.

ature originating from the vacuum representation of local observable algebras). Certain noncausal symmetries, as the supersymmetry, are known to be unstable against the former and it is an interesting question if they suffer the same fate under the latter.