QUANTIZATION IN CURVED SPACE

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1. INTRODUCTION

Feynman's path integral formalism is a very powerful and intuitive quantization technique. When this procedure is extended to Riemannian manifolds, the mixing in the Hamiltoman of functions of coordinates and momenta introduces difficulties and ambiguities, the ordering of the operators that provides the transition from classical to quantum mechanics being the basic problem one has to face.

If it is assumed that the system should be invariant under point transformations it is easy to verify that writing the Hamiltoman in a clearly Hermitian form is not enough. The second order differential operators must be an invariant operator in a Riemann space as

$$\Delta = g^{-1/2}\partial_\alpha \; g^{1/2} \; g^{\alpha\beta} \; \partial_\beta \, , \qquad (1.1)$$

where g is the determinant associated to the metric tensor. The operator is not obtained trivially from the classical Hamiltonian (1).

One could now identify

$$H_K = - \frac{\hbar^2}{2m} \; \Delta \qquad (1.2)$$

but this proposal is not satisfactory, as soon as one imposes more requirements to be satisfied by the Hamiltonian operator. It leads, nevertheless, to an invariant ground state(a constant) and seems to take into account all Kinematic effects.

For example, one expects that a Green's function can be defined for such systems. The Green's function or propagator approach, as a method of quantization, has been discussed by Pauli.[2]

The finite-time Green's function is defined according to Pauli, as a path integral over elementary semi-classical propagators, elementary meaning that they correspond to very small times $\delta t \to 0$. The elementary propagators, in this sence, have been proposed by Van Vleck[3], using as a guide the correspondence principle. Most quantum-mechanical systems obey Pauli's hypothesis, the functional integral over the elementary propagators satisfying the Schroedinger equation and the initial condition, namelly it is a δ- function in the coordinates for $\delta t \to 0$.

This procedure is a method of quantization because it defines a differential operator which is the quantum-mechanical analog of the classical Hamiltonian.

Assuming that the invariant second order differental operator Δ exausts the Hamiltonian, Pauli's quantization procedure is contradictory, unless the classical action is supplemented by an additional term, a quantum-mechanical potential proportional to the fully contracted Riemann curvature tensor, $R(q)$. If this term is not included in the Lagrangean it turns out that, consistently, an extra term appears in the Hamiltonian operator for the system, as shown by De Witt[4], in the form of a quantum mechanical potential

$$H = H_K + \frac{\hbar^2}{12m} R \tag{1.3}$$

Notice that H cannot be obtained from H_K by any reordering of the operators.

Cheng[5], using the definition of propagator strickly in Feyman's sense, i.e., without Van Vleck's preexponential factor, obtained a quantization where the potential function is twice that shown in eq.(1.3).

This proposal also violates, already in terms of order $\hbar$, the criterium based in Van Vleck's arguments.

One argument that favours De Witt's work is that the Hamiltonian proposed is invariant under conformal transformations[1]. It also obeys the correspondence principle without the necessity of any modification of the classical Lagrangean.

In the present work we define a quantization procedure which has the merit of being fully canonically invariant. It is based on the theory of Faddeev-Fradkin[6,7] for Hamiltonian constrained systems and leads exactly to De Witt's proposal.

The quantization of a particle moving in a Riemannian manifold of dimension n is obtained by first embeding this manifold in a larger Euclidean space of dimension $n + m$. The quantization is performed, subject to m constraints, reducing the system to the original space. This procedure is not equivalent to the introduction of very strong potentials associated to the forbiden directions that eventually go to infinite, since the divergent zero point energies associated with the collapse the m extra degrees of freedom, which introduce ambiguities, never appear.

In the first part of this work the treatment of the constraints is performed using Dirac's theory for degenerate

Lagrangeans[8].

In the second part Faddeev-Fradkin quantization is proposed for the constrained system. An elementary propagator, canonically invariant, is obtained for the system as a phase integration. In the spirit of Pauli's procedure the integration over the momenta is performed, in the semi-classical approximation.

This means that we follow for every interval the classical geodesical trajectory connecting the initial to the final point. The Hamiltonian we start wich is the classical one.

The result is shown to agree with De Witt's quantization for curved spaces.

If the manifold is compact, additional complications are introduced in the problem. The propagator in this case must be constructed taking into account the various classical trajectories leading to equivalent points, added with appropriate relative phases. This possibility is not discussed in this work, the the reader being refered to Marinov's work, ref.(1).

2. THE CLASSICAL CONSTRAINED HAMILTONIAN

Assume a particle of unitary mass moving in a Euclidean space of dimension $n + m$. The Lagrangean is writen as (summation over repeated indeces is assumed),

$$L = \frac{1}{2} \dot{q}^i \dot{q}^i \ , \quad i = 1, 2, \ldots n + m \tag{2.1}$$

Let us impose m orthogonal constraints

$$\phi^{j}(\vec{q}) = 0 \ , \quad j = 1, 2, \ldots m \ , \tag{2.2}$$

$$\text{with } |\nabla\phi^{j}| \neq 0 \text{ and } \frac{\partial\phi^{j}}{\partial q^{i}} \frac{\partial\phi^{k}}{\partial q^{i}} = |\nabla\phi^{j}|^{2} \ \delta_{jk} (\text{no sum over } j) \tag{2.3}$$

The Lagrangean now becomes

$$L = \frac{1}{2} \dot{q}^{i} \dot{q}^{i} + \alpha_{j} \phi^{j} \ , \quad i = 1, 2, \ldots n + m, \ j = 1,2,\ldots m \tag{2.4}$$

Here α_j are the usual Lagrange multipliers, to be taken as a new dynamical variables. We have constructed a $(n + 2m)$ dimension coordinate space. Since $\dot{\alpha}_j$ are absent from the Lagrangean, the momenta β^j, canonically conjugated to α_j must vanish identically.

$$\beta^{j} = \frac{\partial L}{\partial \dot{\alpha}_{j}} \equiv 0 \ , \quad j = 1, 2, \ldots m \tag{2.5}$$

Let us construct the Dirac Hamiltonian of such system; Writing

$$H_D = \frac{1}{2} p_i \, p_i - \alpha_j \, \phi^j - \mu_j \, \beta^j \tag{2.6}$$

The particle moves, now in a $2n + 4m$ phases space.

We must impose that $\beta^j \equiv 0$. This means that all time derivatives of β^j must vanish, generating new sets of constraints and finally, reducing the system to a $2n$ dimensional phase space. All constraints we shall be dealing with are second class in the sense of Dirac, and every second class constraint reduces the phase space in one dimension. A second class constraint χ means that $\{\chi, H_D\} \not\equiv 0$, the curly brackets meaning Poisson brackets.

The constraint equation are, j = 1, 2, ... m,

$$\chi^{j} = \beta^{j} = 0$$

$$\chi^{j+m} = \{\beta^{j}, H_D\} = \dot{\beta}^{j} = \phi^{j} = 0$$

$$\chi^{j+2m} = \ddot{\beta}^{j} = \frac{\partial\phi^{j}}{\partial q^{k}} p_k = 0$$

$$\chi^{j+3m} = \dddot{\beta}^{j} = \left\{ \frac{\partial\phi^{j}}{\partial q^{i}} \frac{\partial\phi^{k}}{\partial q^{i}} \alpha_k + \frac{\partial^2\phi^{j}}{\partial q^{i}\partial q^{k}} p_i \, p_k \right\} = 0$$

$$\chi^{j+4m} = \ddddot{\beta}^{j} = p_i \frac{\partial}{\partial q^{i}} \left(\frac{\partial\phi^{j}}{\partial q^{\ell}} \frac{\partial\phi^{k}}{\partial q^{\ell}} \alpha_k + \frac{\partial^2\phi^{j}}{\partial q^{\ell}\, q^{k}} p_\ell \, p_k \right) + $$

$$+ 2\alpha_\ell \frac{\partial\phi^{\ell}}{\partial q^{i}} \frac{\partial^2\phi^{j}}{\partial q^{i}\partial q^{k}} p_k - \mu_k \frac{\partial\phi^{k}}{\partial q^{i}} \frac{\partial\phi^{j}}{\partial q^{i}} = 0 \qquad (2.7)$$

Since, by hypothesis, $\mathrm{Det}\left(\frac{\partial\phi^{k}}{\partial q^{i}} \frac{\partial\phi^{j}}{\partial q^{i}} \right) = \mathrm{Det}|\nabla\phi^{j}|^2 \neq 0$. the last set of equation can be solved for μ_j.

The equations $\chi^{j+3m} = 0$ can, similarly, be solved for α_j. This eliminates 2m dimensions associated to the conjugated variables α_j and β^{j}. We are left with a 2m + 2n dimensional phase space.

This is the essence of Dirac's method: The reduction of dimensionality of the system through the constraint equations.

We are still left with 2m second class constraints χ^{m+j} and χ^{2m+j}. This must reduce futher the dimensionality of the space. To accomplish this reduction in a more transparent way we perform a canonical transformation generated by

$$F_2 = f^{i} \, P_i \qquad (2.8)$$

Where f^{i} are functions of the old coordinates and P_i are the new momenta.

We define f^i as follows:

$f^i(\vec{q}) = Q^i(\vec{q})$ $i = 1, 2, \ldots n$, are the coordinates defined on the initial n-dimensional manifold and

$f^i(\vec{q}) = \phi^{i-n}(\vec{q})$, $i = n + 1, \ldots n + m$, are the m constraints.

The old momenta are given by

$$p_i = \frac{\partial F_2}{\partial q^i} = \frac{\partial f^j}{\partial q^i} P_j \qquad (2.9)$$

Multiplying Eq.(2.9) by $\frac{\partial \phi^k}{\partial q^i}$, for $m + n \geq j \geq n + 1$ we may solve for P_j,

$$P_j = \frac{1}{(\nabla\phi^j)^2} \frac{\partial f^j}{\partial q^i} p_i \text{ , no sum over j.} \qquad (2.10)$$

Notice that these are equivalent to the constraint equations χ^{2m+j}, $j = 1, 2, \ldots m$.

Therefore,

$$P_j = 0, \; j = n + 1, \ldots n + m. \qquad (2.11)$$

The new conditions are, for $j = n + 1, \ldots n + m$,

$$Q^j = \frac{\partial F_2}{\partial P_j} = \phi^{j-n} = 0 \text{ ,} \qquad (2.12)$$

Consequently, $P_{n+1}, \ldots P_{n+m}, Q^{n+1}, \ldots Q^{n+m}$ can also be eliminated from H_D reduces the system to a 2n-dimensional phase space as required.

No zero point energy is to be associated with the m constraints since these degrees of freedom have been excluded from Hamiltonian, definitely.

The Hamiltonian is obtained directly as

$$H_D = \frac{1}{2} g^{ij}(Q) P_i P_j + \frac{1}{2} G^{k\ell} P_k P_\ell - \alpha_s \phi^s - \mu_s \beta^s , \quad (2.13)$$

where $g^{ij} = \frac{\partial f^i}{\partial q^k} \frac{\partial f^j}{\partial q^k}$ is the contravariant metric tensor for the manifold, $1 \le i,j \le n$, $G^{k\ell}$ is diagonal, $n + 1 \le \ell , k \le n + m$ and $1 \le s \le m$.

3. QUANTIFICATION

In order to obtain the Green's function for the system we shall adopt Faddeev-Fradkin[7,8] quantization procedure for constrained systems.

The elementary propagator is writen in phase space as the canonically invariant quantity.

$$\delta G_F = \exp \left\{ \frac{i}{\hbar} \int_t^{t+\varepsilon} \left[(P_i \dot{Q}^i + \beta^j \dot{\alpha}_j - H_D) dt \right\} d\mu_t \quad (3.1)$$

Here $d\mu_t$ is the non-trivial metric

$$d\mu_t = \prod_{s=1}^{4m} \delta (\chi^s) A^{1/2} \prod_{j=1}^{m} d\alpha_{jt} \, d\beta_t^j \prod_{i=1}^{n+m} \frac{dQ_t^i \, dP_{ti}}{(2\pi h)^n} , \quad (3.2)$$

which takes care of all the constraints in the δ-functions, and

$$A^{1/2} = \left[\mathrm{Det} \left\{ \chi^r , \chi^s \right\} \right]^{1/2} = \Pi_i \left| \nabla \phi_i \right|^4 \quad (3.3)$$

Integrating over the appropriate 4m variables in order to eliminate the δ-function we are left with

$$\delta G_F = \prod_{i=1}^{n} \left\{ \exp \int_t^{t+\varepsilon} \frac{i}{\hbar} \left[P_i Q^i - H_{cL} \right] dt \right\} \frac{dP_i \quad dQ_t^i}{(2\Pi h)^n} \quad , \tag{3.4}$$

where

$$H_{cL} = \frac{1}{2} g^{ij} \quad P_i P_j , \; 1 \leq i,j \leq n$$

is the classical Hamiltonian for the particle.

To obtain the propagator in coordinate space we make use of the semi-classical approximation. This means that the exponential term, which is a functional of $Q(\xi)$ and $P(\xi)$, is evaluated over the classical trajectories geodesics leaving $Q(t)$ and reaching $Q'(t + \varepsilon)$,

In this sense, given $Q(t)$ and $Q'(t + \varepsilon)$, the classical value for $P(\xi)$ is fixed and the integral over dP_{it},the initial momenta, is evaluated in the stationary phase aproximation. This corresponds to a particular canonical transformation, the dynamical transformation in which the old variable are $Q(t)$, $P(t)$ and the new variables are $Q(t)$ and $Q'(t + \varepsilon)$.The generator of this transformation is the classical action $S(Q',Q)$. The integral over the momenta turns out to be[9]

$$\delta G_F(Q',Q) = \frac{\Pi dQ_t^j}{(2\Pi i\hbar)^{n/2}} \left| \det \frac{\partial^2 S}{\partial Q'^k \partial Q^j} \right|^{1/2} \exp \left(\frac{i}{\hbar} S(Q',Q) \right) \tag{3.5}$$

Notice that the pre-exponential factor

$$- \frac{\partial^2 S}{\partial Q'^k \partial Q^j} = \left[\frac{\partial P_t^j}{\partial Q'^k_{t+\varepsilon}} \right] \tag{3.6}$$

illustrate the change of variable mentioned above.

To compare eq.(3.5) with De Witt's propagator one must consider the fact that De Witt uses the natural metric[4], i.e., the wave funtion normalization

$$\int \psi^{+}_{DW} \quad \psi_{DW} \sqrt{g(Q)}\; d^{n}Q = 1 \tag{3.7}$$

while here, as in Faddeev's proposal,

$$\int \psi^{+}_{F} \quad \psi_{F}\; d^{n}Q = 1 \tag{3.8}$$

Therefore,

$$\psi_{F}(Q') = \int \partial\; G_{F}\; (Q',Q)\; \psi_{F}(Q) \tag{3.9}$$

and in De Witt

$$\psi_{DW}\;(Q') = \int \delta G_{DW}\;(Q',Q)\;\sqrt{g(Q)}\;\psi_{DW}(Q) = \int \frac{dQ\;\sqrt{g(Q)}}{(2\Pi i h)^{n/2}}$$

$$\left|\det \frac{\partial^{2} S}{\partial Q'^{i}\,\partial Q^{j}}\right|^{1/2} \exp\left[\frac{i}{h}\; S(Q',Q)\right] g^{-1/4}(Q')\; g^{-1/4}(Q). \tag{3.10}$$

Comparing eq.(3.9) and (3.10) we verify that

$$\delta G_{F}\;(Q',Q) = g^{+1/4}(Q')\; g^{+1/4}(Q)\;\delta G_{DW}\;(Q'Q)\;. \tag{3.11}$$

But for the wave function normalization, the theories are identical. They have therefore the same physical content.

Making use of the elementary propagator we may calculate the Schroedinger equation for the particle. The procedure is very similar to that of ref.(5), the only difference remaining in the pre-exponential factor (see appendix I),

$$\left|\det \frac{\partial^2 S}{\partial Q'^i \partial Q^j}\right| = \frac{g^{1/2}(Q)g^{1/2}(Q')}{\varepsilon^n}\left|1 + \frac{R_{ij}}{6}(Q'^i - Q^i)(Q'^j - Q^j)\right|$$

The calculation, long but straigh forward, leads to

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\; g^{ij}\,\partial_j\psi) + \frac{\hbar^2}{12} R\,\psi, \qquad (3.12)$$

which is the Schroedinger equation for the particle.

3. CONCLUSION

The quantization of systems moving in Riemannian manifold is performed by first embeding the space into a larger Euclidean space. The reduction to the original manifold is performed using Dirac's treatment of degenerate Lagrangean and Faddeev-Fradkin quantization technique for constranied Hamiltonian systems.

The integrations over momenta, being performed over semi-classical trajectories, lead to De Witt's quantization of particles moving in curved space.

The conclusion drawn here is that Faddeev's formal propagator, must be understood as a functional integral over semi-classical phase space trajectories of the elementary propagator. To eliminate the momenta over to obtain an elementary propagator in coordinate space, corresponding to a time interval ε, the momenta are integrated off by means of the trajectories corresponding to the classical Geodesical motion between the initial and final points.

APPENDIX

In this appendix we derive the Schrodinger equation from the elementary propagator proposed in this work.The expansion is essentially the same as that found in ref.(5).

A wave function at time $t + \varepsilon$ is related to the wave function at time t by the propagator

$$\psi(q'',t+\varepsilon) = \int \kappa(q'', q', \varepsilon)\, \psi(q',t)\, \sqrt{g(q')}\, d^n q' \qquad (A.1)$$

where

$$\kappa(q'',q',\varepsilon) = \frac{1}{(2\Pi i\hbar)^n} \frac{1}{[g(q'')]^{1/4}} \frac{1}{[g(q')]^{1/4}} D^{1/2} \exp\{\frac{i}{\hbar} S(q'',q',\varepsilon)\} \qquad (A.2)$$

The action S is the stationary value of

$$\int_t^{t+\varepsilon} L(q'(t'), q(t'))\, dt' \qquad (A.3)$$

for

$q'' = q(t+\varepsilon)$, $q' = q(t)$. For a free particle

$$L = \frac{1}{2} g_{ij} (q(t))\, \dot{q}^i \dot{q}^j . \qquad (A.4)$$

The determinant of Van Vleck, D, is given by

$$D = \frac{[g(q")]^{1/2}\ [g(q')]^{1/2}}{\varepsilon^n}\left[1+\frac{R_{ij}}{6}(q"^i - q'^i)(q"^j - q'^j)\right] \tag{A.4}$$

Here R_{ij} is the Ricci tensor, connected to R by,

$$R = g^{ij}\ R_{ij}$$

In order to calculate the propagator K we must use the equations of motion derived from the Lagrangean,

$$\ddot{q}^k = \left\{ \begin{matrix} k \\ \alpha \quad \beta \end{matrix} \right\} \dot{q}^\alpha\ \dot{q}^\beta \tag{A.6}$$

and,

$$g_{mj}\ \ddot{q}^j = -\frac{1}{2}\left(\frac{\partial g_{m\beta}}{\partial q^\alpha} + \frac{\partial g_{m\alpha}}{\partial q^\beta} - \frac{\partial g_{\alpha\beta}}{\partial q^m}\right)\ \dot{q}^\alpha\ \dot{q}^\beta \tag{A.7}$$

The last equation is obtained from the definition of the Christöffel symbols (and use of (A.6)).

$$\left\{ \begin{matrix} k \\ \alpha \quad \beta \end{matrix} \right\} = \left(\frac{\partial g_{m\alpha}}{\partial q^\beta} + \frac{\partial g_{m\beta}}{\partial q^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial q^m}\right)\ \frac{g^{km}}{2} \tag{A.8}$$

It is easy to prove that, using the result above, that

$$\frac{d}{dt}\left(\frac{1}{2}\ g_{ij}\ \dot{q}^i\ \dot{q}^j\right) = 0\ , \tag{A.9}$$

Which exhibits the energy conservation

From (A.9), the action may be calculated directly

$$S = \frac{1}{2}\left[g_{ij}\ (q(t+\varepsilon))\ \dot{q}^i\ (t+\varepsilon)\right]\ \varepsilon \tag{A.10}$$

Moreover, using the expansion of $\dot{q}$ as

$$\dot{q}^i(t) = \dot{q}^i\,(t+\varepsilon) - \varepsilon\ddot{q}^j\,(t+\varepsilon) + \frac{\varepsilon^2}{2}\dddot{q}^i(t+\varepsilon) + \ldots \qquad (A.11)$$

and (A.6), we obtain

$$S(q'',t+\varepsilon\; q',t) = \frac{1}{2\varepsilon}(g_{ij})\Big[\Delta q^i\,\Delta q^j - \begin{Bmatrix} i \\ mn \end{Bmatrix}\,\Delta q\,\Delta q^m\,\Delta q^n + \frac{1}{4}\begin{Bmatrix} i \\ mn \end{Bmatrix}\begin{Bmatrix} j \\ \alpha\beta \end{Bmatrix}\Delta q^m\,\Delta q^n\,\Delta q^\alpha\,\Delta q^\beta + \frac{1}{3}\left(\frac{\partial}{\partial q^\ell}\begin{Bmatrix} i \\ mn \end{Bmatrix} + \begin{Bmatrix} i \\ \alpha\ell \end{Bmatrix}\begin{Bmatrix} \alpha \\ mn \end{Bmatrix}\right)\Delta q^j\,\Delta q^\ell\,\Delta q^m\,\Delta q^n\Big] \qquad (A.12)$$

Where

$$\Delta q^i = q''^i - q'^i$$

Expanding $\sqrt{g(q(t))}$ and $\psi\,(q,t)$ about $t+\varepsilon$ similarly we obtain for eq.(A.1) after expanding also the exponential term,

$$\psi\,(q'',t+\varepsilon) = \int \exp\frac{i}{2\hbar\varepsilon}\,g_{j\kappa}\,\Delta q^j\,\Delta q^\kappa\,\Big[1 - \frac{i}{2\hbar\varepsilon}\,g_{j\kappa}\begin{Bmatrix} j \\ mn \end{Bmatrix}\Delta q^\kappa\,\Delta q^m\,\Delta q^n + \frac{i}{8\hbar\varepsilon}\,g_{j\kappa}\begin{Bmatrix} j \\ mn \end{Bmatrix}\begin{Bmatrix} \kappa \\ \alpha\beta \end{Bmatrix}\Delta q^m\,\Delta q^n\,\Delta q^\alpha\,\Delta q^\beta + \frac{i}{6\hbar\varepsilon}\,g_{j\kappa}\,\frac{\partial}{\partial q^\ell}\begin{Bmatrix} j \\ mn \end{Bmatrix} + \begin{Bmatrix} j \\ \alpha\ell \end{Bmatrix}\begin{Bmatrix} \alpha \\ mn \end{Bmatrix}\Delta q^\kappa\,\Delta q^m\,\Delta q^n\,\Delta q^\ell - \frac{g_{j\kappa}\;g_{st}}{8\hbar^2\;\varepsilon 2}\begin{Bmatrix} j \\ mn \end{Bmatrix}\begin{Bmatrix} s \\ \alpha\beta \end{Bmatrix}\Delta q^\kappa\,\Delta q^t\,\Delta q^m\,\Delta q^n\,\Delta q^\alpha\,\Delta q^\beta + \ldots\Big]$$

$$\Big[\sqrt{g(q(t+\varepsilon))} - \Delta q^j\,\frac{\partial\sqrt{g}}{\partial q^j} + \frac{1}{2}\,\Delta q^j\,\Delta q^\kappa\,\frac{\partial^2\sqrt{g}}{\partial q^j\,\partial q^\kappa} + \ldots\Big]$$

$$\left[\psi\,(q'',t+\varepsilon) - q^j \frac{\partial\psi}{\partial q^j} + \frac{1}{2}\,\Delta q^j\,\Delta q^\kappa\,\frac{\partial^2\psi}{\partial q^j\partial q^\kappa} + \dots\right]$$

$$\left[1 + R_{\mu\nu}\,\Delta q^\mu\,\Delta q^\nu + \dots\right] \frac{\Pi_j d\,(\Delta q^j)}{\sqrt{(2\Pi i\hbar\,\varepsilon)^n}} \qquad \text{(A.13)}$$

Making use repeatelly of the integral,

$$\int_{-\alpha}^{\alpha}\dots\int_{-\alpha}^{\alpha} \exp\left(\frac{i}{2\hbar\varepsilon}\,g_{j\kappa}\,\Delta q^j\,\Delta q^\kappa\right)\Delta q^{\alpha_1}\,\Delta q^{\alpha_2}\dots\Delta q^{\alpha_{2m}}\,d(\Delta q^1)\dots d(\Delta q^n) =$$

$$= (2\Pi i\varepsilon)^{n/2}\,g^{-1/2}(ih\varepsilon)^m\,g^{\alpha_1\alpha_2}\,g^{\alpha_3\alpha_4}\dots g^{\alpha_{2m-1},\alpha_{2m}} \quad +$$

all possible permutations,

we calculate eq.(A.13) in just order in ε, obtained

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{-\hbar^2}{12}\,R\psi - \frac{\hbar^2}{2}\,g^{mn}\,\frac{\partial^2\psi}{\partial q^m\,\partial q^n} - \frac{\hbar^2}{2}\,\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial q^m}(\sqrt{g}\,g^{mn})\,\frac{\partial\psi}{\partial q^n}\right.$$

(A.14)

or finally

$$ih\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\,\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial q^m}\left(\sqrt{g}\,g^{mn}\,\frac{\partial\psi}{\partial q^n}\right) + \frac{\hbar^2 R}{12}\,\psi \qquad \text{(A.15)}$$

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