SINGULAR POTENTIALS AND ANALYTIC REGULARIZATION IN CLASSICAL YANG - MILLS EQUATIONS

C. G. BOLLINI J. J. GIAMBIAGI

Centro Brasileiro de Pesquisas Físicas
Av. Wenceslau Braz, 71 - Rio de Janeiro -
Brasil

and

J. TIOMNO

Departamento de Física - Pontificia Universidade
Católica CxP 38071 - Rio de Janeiro -
Brasil

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ABSTRACT

The class of instanton solutions with "extension" parameter λ^2 positive is extended to λ^2 negative. The nature of the singular sphere of radius $|\lambda|$ is analized in the light of the analytical regularization method. This leads to well defined solutions of the Yang - Mills equations. Some of them are sourceless (" ± i o" and "Vp"), others correspond to currents concentrated on the sphere of singularity ("+" and "-"). Although the equations are non-linear, the "Vp" solutions turns out to be the real part of the "± i o" solutions. The anzats of t'Hooft for the superposition of instantons is used to sum the contributions corresponding to λ^2 with positive and negative signs. A subsequent limiting process allows then the construction of solutions of the "multipole" type.

The general situation of potentials having a denominator D, with a corresponding surface of singularity at D=0, is also considered in the same light.

1 - INTRODUCTION

A. A. Belavin et.al.[1] and G 't Hooft[2] have given an important solution of the homogeneous Yang-Mills equation (euclidean metric):

$$(1) \quad A_\mu = -2i \frac{\sigma_{\mu\nu} x_\nu}{x^2+\lambda^2} \quad ; \quad F_{\mu\nu} = 4i \frac{\lambda^2 \sigma_{\mu\nu}}{(x^2+\lambda^2)^2}$$

$$(2) \quad \sigma_{ij} = \frac{1}{2} \varepsilon_{ijk} \sigma_k \, , \, \sigma_{i4} = \frac{1}{2} \sigma_i \, , \, \sigma_{\mu\nu} = - \sigma_{\nu\mu}$$

(1) is a solution for any real value of the constant λ. It is natural to consider also complex values of λ, in particular imaginary ones. In this case ($\lambda^2 = -|\lambda^2|$) we have a singular sphere of radius $|\lambda|$. Outside this sphere (1) continues to be a solution, but it is not clear what happens at the surface itself.((1) is not well-defined there). We must give additional rules in order to complete the definition of the potential.

In section 2 we define the "$\pm$ i o" prescription. In section 3 we study the "outside" and "inside" solutions which are implied by the "$\pm$" potentials. In section 4, the principal value potential("Vp") is introduced via a combination of the "+" and "-"prescriptions. In section 5 we find the topological numbers of those solutions. In section 6 we briefly treat the linear denominator case. Once the use of negative values of λ^2 has been made clear and mathematically justified, it is then possible, following 't Hooft [3] method[4] [5] to superimpose instantons with positive and negative λ^2, forming the analogue of what could be called a "dipole instanton". This is done in section 7, where it is pointed out that the procedure can be extended to higher "multipoles".

Finally in section 8, we discuss the general case of potentials, having a denominator D, which are singular at a surface defined by the equation D=0. All this singularities can be treated by following the method of analytic regularization, which goes back to M.Riesz[6] methods[7] together with the distribution theory of Guelfand - Shilov[8].

Appendix A contains some formulae from reference[8] which are extensively used in the text. Appendix B is an alternative (and equivalent) way to treat the singularities.

2 - " ± i o" Potentials

When λ is pure imaginary in (1), we have a singularity at the sphere $x^2 = |\lambda^2|$. A natural way to attach a well defined meaning to this singularity is to approximate the imaginary axis from the right, for instance. For λ's with Re $\lambda=0$, (1) is a sourceless solution. Thus we take

$$\lambda = \lambda_1 + i\lambda_2 \qquad (\lambda_1 > 0)$$

$$\lambda^2 = \lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2 \rightarrow -\lambda_2^2 \pm i\,o \qquad (\text{for } \lambda_1 \rightarrow 0)$$

The ± sign corresponds to $\lambda_2 \gtrless 0$.

For simplicity we shall work with the positive sign. So, instead of (1), we shall write:

$$(3) \qquad A_\mu^{(+\,io)} = -2\,i\,\frac{\sigma_{\mu\nu}\,x_\nu}{(x^2 - \lambda^2 + io)}$$

We know from the results obtained by Guelfand and Shilov[8], that the distribution:

$$(4) \qquad (x^2 - \lambda^2 + i\,o)^{\alpha}$$

is a well defined functional analytic (entire) in the parameter α. All powers and derivatives are well defined. In particular we can compute the field $F_{\mu\nu}$ and the current corresponding to (3).

$$(5) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$(6) \qquad F_{\mu\nu}^{(+\,i\,o)} = 4i\,\frac{\lambda^2\,\sigma_{\nu\mu}}{(x^2 - \lambda^2 + i\,o)^2}$$

(7) $J_\mu = \partial_\nu F_{\nu\mu} + \left[A_\nu , F_{\nu\mu} \right]$

(8) $J_\mu^{(+ i o)} = 0$

Eq. (8) follows simply from the self-duality of (6) .

3 - " + " and " - " Potentials.

Due to (8) the potential (3) is a complex solution of the Yang-Mills equation. Eq. A.7 shows explicitely its real and imaginary parts.

In order to study the real part of (3), defined through A.8, A.7 and A.1, A.2, we shall consider the potential:

(9) $A_\mu^{(\alpha)+} = - 2 i \sigma_{\mu\nu} x_\nu (x^2 - \lambda^2)_+^\alpha$,

which is identically zero for $x^2 < \lambda^2$. (cf. A.1).

Field and current corresponding to (9) can easily be computed:

(10) $F_{\nu\mu}^{(\alpha)+} = 4 i \sigma_{\mu\nu} \left[x^2 - (x^2 - \lambda^2)_+^{-\alpha}\right] (x^2 - \lambda^2)_+^{2\alpha} +$

$+ 4 i (\sigma_{\nu\tau} x_\mu - \sigma_{\mu\tau} x_\nu) x_\tau \left[\alpha (x^2 - \lambda^2)_+^{\alpha-1} + (x^2 - \lambda^2)_+^{2\alpha} \right]$.

(11) $J_\mu^{(\alpha)+} = - 8 i \sigma_{\mu\nu} x_\nu \left[3 (x^2 - \lambda^2)_+^{2\alpha} + 3 \alpha (x^2 - \lambda^2)_+^{\alpha-1} + \right.$

$\left. + \alpha(\alpha-1) x^2 (x^2 - \lambda^2)_+^{\alpha-2} - 2x^2 (x^2 - \lambda^2)_+^{3\alpha}\right]$.

If we take naïvely the limit $\alpha \to -1$, the second member of (11) gives zero. However one must be cautious in taking that limit, as the distribution $(x^2 - \lambda^2)_+^\alpha$ has poles at negative integer values for α. When A.3 is taken into account in (9), (10) and (11), we can see that all three expressions give simple poles when $\alpha \to -1$. The residues at the poles are distributions concentrated on the surface of the sphere $x^2 = \lambda^2$. Furthermore, the finite part of (11) (at $\alpha=-1$)

is a derivative of a δ-function. In fact, by taking

$$(12)\qquad \text{P.f. } \Psi(\alpha)\Big|_{\alpha=-n} = \frac{d}{d\alpha}(\alpha+n)\ \Psi(\alpha)\Big|_{\alpha=-n} ,$$

and using A.9 (for n=1), we get:

$$(13)\qquad \text{P.f } J_\mu^{(\alpha)+}\Big|_{\alpha=-1} = 12\ i\ \sigma_{\mu\nu}\ x_\nu\ \lambda^2\ \delta''(x^2-\lambda^2) \qquad \text{(see A.5)}$$

We see then that, in the limit $\alpha \to -1$, the source of (9) is concentrated on the surface of the sphere $x^2 = \lambda^2$.

A similar procedure can be followed with the potential (compare with (9)):

$$(14)\qquad A_\mu^{(\alpha)-} = +\,2\ i\ \sigma_{\mu\nu}\ x_\nu\ (x^2-\lambda^2)_-^{\alpha}.$$

Where $(x^2-\lambda^2)_-^{\alpha}$ is defined by A.2, and is zero outside the sphere of radius λ.

The field and current for (14) are:

$$(15)\qquad F_{\nu\mu}^{(\alpha)-} = 4\ i\ \sigma_{\mu\nu}\left[x^2 + (x^2-\lambda^2)_-^{-\alpha}\right](x^2-\lambda^2)_-^{2\alpha} +$$

$$+\,4\ i\ (\sigma_{\nu\tau}\ x_\mu - \sigma_{\mu\tau}x_\nu)\ x_\tau\ \left[\alpha\ (x^2-\lambda^2)_-^{\alpha-1} + (x^2-\lambda^2)_-^{2\alpha}\right].$$

$$(16)\qquad J_\mu^{(\alpha)-} = -\,8\ i\ \sigma_{\mu\nu}\ x_\nu\Big[\ 3\alpha\ (x^2-\lambda^2)_-^{\alpha-1} + 3\ (x^2-\lambda^2)_-^{2\alpha}$$

$$-\alpha\ (\alpha-1)\ x^2(x^2-\lambda^2)_-^{\alpha-2} + 2x^2(x^2-\lambda^2)_-^{3\alpha}\Big].$$

Taking into account A.4, we again find that, in the limit $\alpha \to -1$, the source is concentrated on the sphere.

$$(17)\qquad \text{P.f. } J_\mu^{(\alpha)-}\Big|_{\alpha=-1} = -\,12\ i\ \lambda^2\ \sigma_{\mu\nu}\ x_\nu\ \delta''(x^2-\lambda^2).$$

It is worth mentioning that, in the limit $\alpha \to -1$, the source (17) is equal and opposite to (13).

4 - "Vp" Potential

We introduce the definition,

(18) $$A_\mu^{(\alpha)} = A_\mu^{(\alpha)+} + A_\mu^{(\alpha)-}$$

I.e.:

(19) $$A_\mu^{(\alpha)} = -2\, i\, \sigma_{\mu\nu}\, x_\nu \left[(x^2-\lambda^2)_+^\alpha - (x^2-\lambda^2)_-^\alpha\right]$$

For $\alpha \to -1$ the pole parts of the " + " and " - " distributions cancel each other, leaving only a finite result.

(20) $$A_\mu^{Vp} = -2\, i\, \sigma_{\mu\nu}\, x_\nu \left[x^2 - \lambda^2\right]^{-1} ,$$

which is a real solution.

Where $\left[x^2 - \lambda^2\right]^{-1}$ is given by A.8 and coincides with Cauchy's principal value.

It is easy to compute the field and current for the potential (19). The " + " and " - " distributions do not interfere with one another as one of them is zero when the other is not. For this reason the field corresponding to (19) is a superposition of (10) and (15).

(21) $$F_{\nu\mu}^{(\alpha)} = F_{\nu\mu}^{(\alpha)+} + F_{\nu\mu}^{(\alpha)-} .$$

Analogously:

(22) $$J_\mu^{(\alpha)} = J_\mu^{(\alpha)+} + J_\mu^{(\alpha)-}$$

where $J_\mu^{(\alpha)+}$ and $J_\mu^{(\alpha)-}$ are given by (11) and (16).

Taking now the limit $\alpha \to -1$, we find that the current is zero.

The corresponding field is given by (21) (with (10) and (15)). For $\alpha \to -1$ we get:

$$(23) \qquad F^{Vp}_{\nu\mu} = 4\, i\, \sigma_{\mu\nu}\, \lambda^2 \left[x^2 - \lambda^2\right]^{-2} ,$$

where the "finite part" $[\ ]^{-n}$ is given by A.8.

Note that (23) is self-dual, showing again that the field is sourceless.

According to A.7, the potential (20) and the field (23) are respectively the real parts of (3) and (6). We see then that the "$\pm$ i o" potential (3) is a complex solution of the homogeneous Yang-Mills equation, whose real part (20) is also a solution. (see also reference [9]).

Note that in (20) and (23), the square brackets are in fact labels for the limiting processes implied by the analytic regularization method. If one takes (20) directly in the Yang-Mills equation, one finds ambiguities due to the existence of products which are not well defined. (See also ref. [11]).

5 - Topological number

The gauge fields so far considered ((6), (10), (15) and (23)) are all singular on the sphere $x^2 = \lambda^2$.

For the calculation of the topological number of (6) one must take the integral over all space of:

$$(24) \qquad \mathrm{Tr}\,\{ F^{(io)}_{\mu\nu}\, \tilde{F}^{(io)}_{\mu\nu} \} = \frac{-96\,\lambda^4}{(x^2 - \lambda^2 + io)^4}$$

One could think that the already mentioned singularity renders the integral divergent. However, it is well known, and easy to check, that when the denominator is $(x^2 + \lambda^2)^{+4}$, the topological number comes out to be one independently of the value of λ^2. It is then natural to expect that the integration of (24) will give also the same result as it is obtained by a continuation in λ to the imaginary axis. Nevertheless to be consistent, one should like to obtain it by direct calculation using A.6,

$$(x^2 - \lambda^2 + io)^{\alpha} = (x^2 - \lambda^2)^{\alpha}_{+} + e^{i\pi\alpha}(x^2 - \lambda^2)^{\alpha}_{-} ,$$

as a functional analytic in α, and then take the limit $\alpha \to -4$.

To do that, we shall consider a "trial" function $\Psi(x^2)$, which is 1 for $x^2 < \Lambda^2$ and zero for $x^2 > \Lambda^2$. Of course this is not a proper trial function, but it can be approximated as much as we want by infinite differentiable functions. We have then:

$$((x^2 - \lambda^2 + io)^{\alpha}, \Psi) = \pi^2 \int_{\lambda^2}^{\Lambda^2} dx^2\, x^2 (x^2 - \lambda^2)^{\alpha} +$$

$$+ \pi^2 e^{i\pi\alpha} \int_0^{\lambda^2} dx^2\, x^2 (\lambda^2 - x^2)^{\alpha} =$$

$$= \pi^2 \left[\frac{(\Lambda^2 - \lambda^2)^{\alpha+2}}{\alpha + 2} + \lambda^2 \frac{(\Lambda^2 - \lambda^2)^{\alpha+1}}{\alpha + 1} \right] +$$

$$+ \pi^2 e^{i\pi\alpha} \left[- \frac{(\lambda^2)^{\alpha+2}}{\alpha + 2} + \lambda^2 \frac{(\lambda^2)^{\alpha+1}}{\alpha + 1} \right].$$

Where a factor $2\pi^2$ comes from angular integration.

For $\alpha = -4$ we have:

$$(25) \qquad ((x^2 - \lambda^2 + io)^{-4}, \Psi) = -\pi^2 \left[\frac{1}{2} (\Lambda^2 - \lambda^2)^{-2} + \frac{\lambda^2}{3} (\Lambda^2 - \lambda^2)^{-3} \right] +$$

$$+ \frac{\pi^2}{6} \lambda^{-4} .$$

(25) gives the result of the integration over a sphere of radius Λ. If we now take limit $\Lambda \to \infty$ we get

$$(26)\quad \lim_{\Lambda\to\infty} \left((x^2 - \lambda^2 + io)^{-4}, \Psi \right) = \int d^4 x \, (x^2 - \lambda^2 + io)^{-4} = \frac{\pi^2}{6\lambda^4}$$

So, for the topological number we obtain:

$$(27)\quad -\frac{1}{16\,\pi^2} \int F^{(io)}_{\mu\nu} \tilde{F}^{(io)}_{\mu\nu} d^4 x = \frac{96\lambda^4}{16\pi^2} \cdot \frac{\pi^2}{6\lambda^4} = 1 .$$

One can also repeat the calculation using a Gaussian function $e^{-\frac{x^2}{a^2}}$, which is a true trial function. The result, for $a \to \infty$, reproduces (27).

We would like to point out that it is also possible to use directly Guelfand-Shilov's definition of integrals for certain functionals (Reference [8] page 65), namely:

$$(28)\quad \int_0^\infty z^\alpha \, dz = 0 . \qquad \text{(any complex } \alpha)$$

With this definition:

$$\int (x^2 - \lambda^2 + io)^\alpha d^4 x = \pi^2 \int_{\lambda^2}^\infty dx^2 \, x^2 (x^2 - \lambda^2)^\alpha +$$

$$+ \pi^2 e^{i\pi\alpha} \int_0^{\lambda^2} dx^2 \, x^2 \, (\lambda^2 - x^2)^\alpha =$$

$$= \pi^2 \int_0^\infty dz \, (z^{\alpha+1} + \lambda^2 z^\alpha) + \pi^2 e^{i\pi\alpha} \int_0^{\lambda^2} dz \, (-z^{\alpha+1} + \lambda^2 z^\alpha)$$

$$= \pi^2 e^{i\pi\alpha} (\lambda^2)^{\alpha+2} \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+2} \right) = \frac{\pi^2 (\lambda^2)^{\alpha+2} e^{i\pi\alpha}}{(\alpha+1)(\alpha+2)}$$

which, for $\alpha = -4$, reproduces again (26), (27).

With the same procedure we can compute the topological numbers of the previously discussed solutions. The "+" field(10) turns out to have topological number zero. All the others("-","Vp","± io")have topological number equal to one.

Of course, if instead of using the self-dual matrix $\sigma_{\mu\nu}$, we use the antiself-dual one $\tilde{\sigma}_{\mu\nu}$, the signs of the Chern number would be reversed.

6 - Solutions with linear denominators

It is well known [10] that by choosing

$$(29) \qquad \rho = \frac{1}{(x.n+\lambda)}$$

in the ansatz proposed by t'Hooft, Wilzced, and Corrigan-Fairlie [3] [4] [5], we get the self dual solution:

$$(30) \qquad A = -i \frac{\sigma_{\mu\nu}\, n_\nu}{(x.n+\lambda)}$$

$$(31) \qquad F_{\mu\nu} = -i \frac{\sigma_{\mu\nu}}{(x.n+\lambda)^2}$$

It can also be directly verified that with

$$(32) \qquad A_\mu = +i \frac{\sigma_{\mu\nu}\, n_\nu}{(x.n+\lambda)}$$

we get the anti-self dual solution:

$$(33) \qquad F_{\mu\nu} = \frac{i}{(x.n+\lambda)^2}(2\,\sigma_{\mu\rho}\, n_\rho\, n_\nu - 2\,\sigma_{\nu\rho}\, n_\rho\, n_\mu - \sigma_{\mu\nu} n^2)$$

These solutions are all singular at the plane

$$(34) \qquad x.n+\lambda = 0$$

In order to give a meaning to those singular expression, in a way similar to the cases already examined, we shall compute the fields and currents for the " + " and " - " potentials.

(35) $$A_{\mu}^{(\alpha)+} = - i \; s \; \sigma_{\mu\nu} n_{\nu} (x.n+\lambda)_{+}^{\alpha} \qquad (s^2 = 1)$$

(36) $$A_{\mu}^{(\alpha)-} = + i \; s \; \sigma_{\mu\nu} \; n_{\nu} (x.n+\lambda)_{-}^{\alpha} \qquad (s^2 = 1)$$

Which lead to:

(37) $$F_{\nu\mu}^{(\alpha)\pm} = i \; \sigma_{\mu\nu} n^2 (x.n+\lambda)_{\pm}^{2} \; +$$

$$+ \; i \; (\sigma_{\nu\tau} \; n_{\mu} \; - \sigma_{\mu\tau} n_{\nu} \;) \; n_{\tau} \left[(x.n+\lambda \;)_{\pm}^{2\alpha} + s \; \alpha (x.n+\lambda)_{\pm}^{\alpha-1} \right]$$

(38) $$J_{\mu}^{(\alpha)\pm} = \pm \; i \; s \; n^2 \; \sigma_{\mu\nu} \; n_{\nu} \; \left[2 \; (x.n+\lambda)_{\pm}^{3\alpha} - \alpha(\alpha-1) (x.n+\lambda)_{\pm}^{\alpha-2} \right].$$

It is easy to see that outside the singular plane the current is zero for $\alpha = -1$. So, the source (if it exists) is concentrated on the plane $x.n+\lambda=0$.

As was previously done, we can now define the potentials with the labels "± io" "Vp", and in the present case also " | |", which is defined by (compare with (18)):

(39) $$A_{\mu}^{(\alpha)||} = A_{\mu}^{(\alpha)+} - A_{\mu}^{(\alpha)-} \quad .$$

The " ± io" and Vp " potentials are sourceless, while " ± " and " || " are solutions of the inhomogeneous equation with current concentrated on the plane $x.n+\lambda=0$.

7 - "Multipole" solutions

To clarify the ideas we shall recall 't Hooft [3] method [4] [5] for two instantons.

We chose

(40) $$\rho = 1 + \frac{\lambda_1^2}{x^2} + \frac{\lambda_2^2}{(x-x_2)^2} \quad ,$$

from which we get the potential :

(41) $$A_\mu = i\, \sigma_{\mu\nu}\, \partial_\nu\, \ell n\, \rho \ .$$

Now, once the idea of a negative λ^2 has been acepted, we can choose in (40), $\lambda_2^2 = -\lambda_1^2$, and take the limit $x_2 \to 0$, $\lambda_2^2 \to \infty$ in such a way that $\lambda_2^2\, x_{2\mu} = P_\mu$ is a constant vector. It is then easily seen that:

(42) $$\rho \to 1 + \frac{P.x}{x^4} \ .$$

where P defines a "dipole moment" for the above mentioned two instantons case.

From (42) we obtain:

(43) $$A_\mu = i\, \frac{\sigma_{\mu\nu} Q_\nu}{D}$$

with:

(44) $$D = x^4 + x.P$$

and

(45) $$Q_\nu = P_\nu - 4\, \frac{P.x}{x^2}\, x_\nu$$

After a straightforward calculation we get:

(46) $$F_{\nu\mu} = i\, \sigma_{\mu\nu}\, \frac{Q^2}{D^2} + i\, \frac{\sigma_{\mu\tau}}{D^2} \left[D\, Q_{\tau,\nu} - Q_\tau\, (Q_\nu + D_\nu) \right]$$
$$- i\, \frac{\sigma_{\nu\tau}}{D^2} \left[D\, Q_{\tau,\mu} - Q_\tau\, (Q_\mu + D_\mu) \right] .$$

It is not difficult to check, using (44) and (45) in (46), that $F_{\nu\mu}$ is anti-self dual, implying that we have a solution of the homogeneous equation, outside the surface D=0.

It is easy to see that we can still have (43) as a solution if we change (44) to:

(47) $$D = \lambda x^4 + x.P \quad ,$$

for arbitrary λ, including $\lambda = 0$.

The method can be generalized to higher "multipoles". For instance, with

(48) $$\rho = 1 + \frac{x.Q.x}{x^6} \quad ,$$

we have again (43), but now:

(49) $$D = x^6 + x.Q.x \quad ,$$

(50) $$Q_\nu = 2\, Q_{\nu\mu}\, x_\mu - 6x_\nu \frac{x.Q.x}{x^2} \quad .$$

(43) and (46), (with (49) and (50)), provides us with a new solution of the sourceless equation, except perhaps at the surface D=0.

8 - Regularized singular solutions.

The solutions found in section 7, are singular at the surface defined by:

(51) $$D = 0$$

Where D is de denominator of the potential (43).

In order to give a well defined meaning to expressions such as (43) and (46), (containing a denominator D for which (51) has a real solution) we shall use the methods described in previous paragraphs.

We can give several prescriptions for the definition of the singularity. All of them can be built up from the " + " and " - " distributions which have the following definition [8]:

(52) $$D_+^\alpha = D^\alpha \quad \text{for} \quad D > 0, \qquad = 0 \quad \text{for} \quad D < 0.$$

(53) $$D_- = 0 \quad \text{for} \quad D > 0 \qquad = |D|^\alpha \quad \text{for} \quad D < 0.$$

Both (52) and (53) are meromorphic functionals in the parameter α (see ref. [8], chapter 3 section 4.)

We now follow the usual pattern:

(54) $$A_\mu^{(\alpha)+} = i\, \sigma_{\mu\nu}\, Q_\nu\, D_+^\alpha\ .$$

(55) $$A_\mu^{(\alpha)-} = -i\, \sigma_{\mu\nu}\, Q_\nu\, D_-^\alpha\ .$$

(56) $$A_\mu^{(\alpha)\pm\, io} = A_\mu^{(\alpha)+} \pm e^{i\pi\alpha}\, A_\mu^{(\alpha)-}.$$

(57) $$A_\mu^{(\alpha)\,Vp} = A_\mu^{(\alpha)+} + A_\mu^{(\alpha)-}\ .$$

For $\alpha \to -1$, (56) and (57) are solutions of the homogeneous equation, while (54) and (55) have sources concentrated on the surface (51).

APPENDIX A

A.1 $$(x^2-\lambda^2)_+^{\alpha} = \begin{cases} (x^2-\lambda^2)^{\alpha} & \text{if } x^2 > \lambda^2 \\ 0 & \text{if } x^2 < \lambda^2 \end{cases}$$

A.2 $$(x^2-\lambda^2)_-^{\alpha} = \begin{cases} (\lambda^2-x^2)^{\alpha} & \text{if } x^2 < \lambda^2 \\ 0 & \text{if } x^2 > \lambda^2 \end{cases}$$

A.1 and A.2 are well defined distribution, analytic in α with poles for $\alpha = -n$ (n, positive integer). Near these poles we have:

A.3 $$(x^2-\lambda^2)_+^{\alpha} = \frac{(-1)^{n-1}\,\delta^{(n-1)}(x^2-\lambda^2)}{(n-1)!\,(\alpha+n)} + \left[x^2-\lambda^2\right]_+^{-n} + 0(\alpha+n)$$

A.4 $$(x^2-\lambda^2)_-^{\alpha} = \frac{\delta^{n-1}(x^2-\lambda^2)}{(n-1)!\,(\alpha+n)} + \left[x^2-\lambda^2\right]_-^{-n} + 0(\alpha+n)$$

Where $\left[x^2-\lambda^2\right]_\pm^{-n} = \mathrm{Pf}\,(x^2-\lambda^2)_\pm^{\alpha}\Big|_{\alpha=-n}$

$$\delta^k(u) = \frac{d^k\,\delta(u)}{du^k}$$ and

A.5 $$\delta^{(k-1)}(\lambda^2-x^2) = \frac{(-1)^{k-1}}{2^k\,\lambda\,x^{k-1}}\left[\delta^{k-1}(x-\lambda) - \delta^{k-1}(x+\lambda)\right]$$

We can now define

A.6 $$(x^2-\lambda^2\pm io)^{\alpha} = (x^2-\lambda^2)_+ + e^{\pm i\pi\alpha}\,(x^2-\lambda^2)_-^{\alpha}$$

It is easily seen from A.3 and A.4 that this new distribution is analytic everywhere. In particular

A.7 $$(x^2-\lambda^2\pm io)^{-n} = \left[x^2-\lambda^2\right]^{-n} + i\,\pi\,\frac{(-1)^n}{(n-1)!}\,\delta^{(n-1)}(x^2-\lambda^2)$$

where

A.8 $$\left[x^2 - \lambda^2\right]^{-n} = \left[x^2 - \lambda^2\right]_+^{-n} + (-1)^n \left[x^2 - \lambda^2\right]_-^{-n}$$

which, for n=1 gives

$$\mathrm{Vp}\ (x^2 - \lambda^2)^{-1} = (x^2 - \lambda^2)_+^{-1} - (x^2 - \lambda^2)_-^{-1}$$

It is possible to show that (Reference [8] p.347)

A.9 $$(x^2 - \lambda^2)\quad \delta^{(n)}(x^2 - \lambda^2) + n\ \delta^{(n-1)}(x^2 - \lambda^2) = 0$$

Note also that if we multiply A.8 times a similar expression with m in place of n, the result is not well defined. However, multiplying $(x^2 - \lambda^2 + io)^{-n}$ times $(x^2 - \lambda^2 + io)^{-m}$ using A.7, and then taking the real part of the result, we get:

A.10 $$\left[x^2 - \lambda^2\right]^{-n-m} = \left\{ \left[x^2 - \lambda^2\right]^{-m} \cdot \left[x^2 - \lambda^2\right]^{-n} - \right.$$

$$\left. - \frac{\pi^2\ (-1)^{n+m}}{(n-1)!\,(m-1)!}\ \delta^{(m-1)}(x^2 - \lambda^2)\quad \delta^{(n-1)}(x^2 - \lambda^2) \right\}$$

As a matter of fact, each of the terms on the right hand side is meaningless; however the complete combination is well defined (see reference [11])

APPENDIX B

We present here a perhaps more intuitive way to deal with the singularities produced by the zeros of the denominators. Let us introduce a regularizing function $\eta_\varepsilon(x)$ with the following properties:

$\eta_\varepsilon(x)$ and all its derivatives are null at the origin $\eta_\varepsilon(0)$, $\left.\frac{d^p \eta_\varepsilon(x)}{dx^p}\right|_{x=0} = 0$

(all p) . Furthermore:

$\eta_\varepsilon(x)_{x \geqslant \varepsilon} = 1$ and also $\left.\frac{d^p \eta_\varepsilon(x)}{dx^p}\right|_{x=\varepsilon} = 0$ (all p)

The actual form of $\eta_\varepsilon(x)$ is irrelevant, as long as it is differentiable any number of times. At the end of the calculations the limit $\varepsilon \to 0$ is to be taken, eliminating all pole terms in ε (regularization) , thus keeping only the finite part.

We start with the regularized potential

B.1 $$A^+_\mu = - 2 i \sigma_{\mu\nu} x^\nu (x^2 - \lambda^2)^{-1}_\varepsilon = - 2 i \sigma_{\mu\nu} x^\nu \frac{\eta_\varepsilon(Z)}{Z}$$

B.2 $$Z = (x^2 - \lambda^2)$$

we define

B.3 $$V = \frac{\eta_\varepsilon(Z)}{Z}$$

From B.1 we get

B.4 $$F^+_{\mu\nu} = -4 i \sigma_{\mu\nu} \left[(Z+ \Lambda^2) V^2 - V \right] + 4 i (\sigma_{\mu\tau} x_\tau x_\nu - \sigma_{\nu\tau} x^\tau x_\mu) \left[V^2 + \frac{d}{dZ} V \right] .$$

and from the Yang & Mills eqs.

B.5 $$J^+_\mu = -8 i \sigma_{\mu\nu} x^\nu \left\{ \left[(V^{-1} + \Lambda^2) \left(\frac{d^2}{dZ^2} V - 2 V^3 \right) \right] + 3 \left[V^2 + \frac{d}{dZ} V \right] \right\}$$

from which

$$\text{B.6} \qquad J^{+}_{\mu} = -8\, i\, \sigma_{\mu\nu}\, x^{\nu} \left\{ \Lambda^2 \left[\frac{\eta''_{\varepsilon}}{Z} - 2\,\frac{\eta'_{\varepsilon}}{Z^2} \right] + \left[\frac{\eta'_{\varepsilon}}{Z} + \eta''_{\varepsilon} \right] \right\}$$

In order to understand the distributions of the second term in the limit $\varepsilon \to 0$, we apply, for instance, the square bracket to a function $\Phi(x)$.

$$\left(\left[\frac{\eta''_{\varepsilon}}{Z} - 2\,\frac{\eta'_{\varepsilon}}{Z^2} \right], \Phi \right) = \int_{\varepsilon}^{\infty} \left\{ \frac{d^2}{dZ^2}\, \frac{\Phi(Z)}{Z} + 2\,\frac{d}{dZ}\,\frac{\Phi(Z)}{Z^2} \right\} dZ =$$

(where use has been made of the definition and properties of η_{ε})

$$= - \frac{d}{dZ}\,\frac{\Phi(Z)}{Z} \Bigg|_{Z=\varepsilon}^{\infty} - 2\,\frac{\Phi(Z)}{Z^2} = - \frac{\Phi'(\varepsilon)}{\varepsilon} + \frac{\Phi(\varepsilon)}{\varepsilon^2} - 2\,\frac{\Phi(\varepsilon)}{\varepsilon^2}$$

using $\Phi(\varepsilon) = \Phi(0) + \varepsilon\Phi'(0) + \frac{\varepsilon^2}{2}\,\Phi''(0)$ and dropping the pole terms to get the finite part, one finally get

$$\text{B.7} \qquad \mathrm{Pf} \left(\left[\frac{\eta''_{\varepsilon}}{Z} - 2\,\frac{\eta'_{\varepsilon}}{Z^2} \right], \Phi \right) \Bigg|_{\varepsilon \to 0} = - \frac{3}{2}\, \Phi''(0)$$

with a similar procedure, it is shown that the finite part of the second bracket is zero.

So, we get for the current the same result as given in form 13.

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