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CONTINUITY OF THE DIRECT AND INVERSE PROBLEMS IN ONE-DIMENSIONAL SCATTERING THEORY AND NUMERICAL SOLUTION OF THE INVERSE PROBLEM

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Abstract

We propose an algorithm for computing the potential V(x) associated to the one-dimensional Schrödinger operator

$$E \equiv -\frac{d^2}{dx^2} + V(x) \qquad -\infty < x < \infty$$

from knowledge of the S-matrix, more exactly, of one of the reflection coefficients. The convergence of the algorithm is guaranteed by the stability results obtained for both the direct and inverse problems.

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1. Introduction

The one-dimensional scattering problem appears not only within the context of quantum mechanics, cf. Landau [12], but also in the mathematical description of a series of other physical phenomena. These include the reflection of electromagnetic waves by various media -- such as a plasma, cf. Szu et al. [16], the ionosphere, cf. Kay [9], a dielectric slab, cf. Portinari [15] -- or the propagation of waves in transmission lines, cf. Colin [3]. One-dimensional scattering plays a role also in the study of long water waves in a channel, due to its relationship to the Korteweg-deVries equation, cf. Gardner et al. [8].

The results contained in the first part of this paper give partial answers to the question raised by Sabatier in [3], namely "How stable are scattering problems, and in which sense?" This question is a basic concern to anyone who is either seeking numerical approaches to these problems, or dealing with experimental data. In our case, we were led to this investigation in the course of our search for an efficient numerical procedure for the one-dimensional inverse scattering problem. The numerical results we obtained are described in the second part of the paper.

The organization of this paper is as follows:

In Section 2 we recall some basic results on one-dimensional scattering and introduce the terminology we will be using in the other sections.

In Section 3 we state and prove some continuity results for both the direct and inverse problems. We also point out how these results can be applied for the radial problem, and restate a previous result by Marchenko.

Section 4 is devoted to the proof of some technical lemmas needed in Section 3.

In Section 5 we discuss some numerical results, while in Section 6 we include for the sake of completeness the proof of some of the basic results of one-dimensional scattering theory.

2. Background

In this section we explain briefly the basic facts of one-dimensional scattering, introducing at the same time the notation and terminology we adopted.

Consider the stationary Schrödinger operator

$$E \equiv -\partial^2 + q , \qquad \partial \equiv d/dx$$

on $L^2(\mathbb{R})$. We assume that the real potential $q(x)$ tends to zero sufficiently fast as $x \to \pm\infty$. Then it is known that the continuous spectrum of E comprises the positive axis $\mathbb{R}^+$, and has multiplicity 2. In addition, there may exist a finite number of negative eigenvalues λ_j.

With each point k^2, k real, in the continuous spectrum, one can associate a two-dimensional space of generalized eigenfunctions $y(x,k)$. These are solutions of the eigenvalue equation

$$(2.1) \qquad -y'' + qy = k^2 y .$$

They do not belong to $L^2(\mathbb{R})$ but can be shown to be bounded. Indeed, there exist constants

$$A_\pm \equiv A_\pm(y) , \qquad B_\pm \equiv B_\pm(y)$$

such that

$$(2.2) \qquad \lim_{x\to\pm\infty} y(x,k) - \{A_\pm e^{ikx} + B_\pm e^{-ikx}\} = 0 ,$$

i.e., the solutions of (2.1) behave asymptotically as the solutions of the unperturbed equation

$$- y'' = k^2 y \ .$$

Conversely, given any pair of constants (α,β), there exists a unique solution of (2.1) for which

$$A_+ = \alpha \ , \qquad B_+ = \beta \ .$$

Of course, A_-, B_- may also be prescribed arbitrarily.

The pairs of constants (A_-, B_+) and (A_+, B_-) are called the *incoming* and *outgoing components* of the solution $y(x,k)$. This terminology is motivated as follows:

If y is a solution of (2.1), then

$$u(x,t) \equiv e^{-ikt} \, y(x,k)$$

is a time-harmonic solution of the perturbed wave equation

$$u_{tt} - u_{xx} + qu = 0 \ .$$

The terms

$$A_{\pm} \, e^{ikx} \, e^{-ikt} = A_{\pm} \, e^{ik(x-t)}$$

and

$$B_{\pm} \, e^{-ikx} e^{-ikt} = B_{\pm} \, e^{-ik(x+t)}$$

represent waves moving to the right or to the left, respectively. We call a wave *incoming* if it moves from $\pm \infty$ towards the origin, and outgoing if it moves away from the origin towards $\pm \infty$.

It turns out that the incoming components (A_-, B_+) determine the outgoing components (A_+, B_-) uniquely. The operator relating them is given by a 2×2 matrix called the <u>scattering matrix</u> and denoted by S:

$$S\begin{pmatrix} A_- \\ B_+ \end{pmatrix} = \begin{pmatrix} A_+ \\ B_- \end{pmatrix} . \tag{2.3}$$

The S-matrix depends only on k, $S \equiv S(k)$, and not on the particular solution y being considered.

We can attribute a physical meaning to the elements $s_{ij}(k)$ of S:

Let ψ_- denote the solution of (2.1) which is a wave of unit amplitude coming in from $-\infty$, i.e., $A_- = 1$, $B_+ = 0$. (The existence of such a ψ_- can be proved rigorously.) Then by (2.3) the outgoing components of ψ_- are

$$A_+ = s_{11} , \qquad B_- = s_{12} .$$

Now $s_{11} = A_+$ is the amplitude of a wave travelling to the right, i.e., in the same direction as the above incoming wave, while $s_{12} = B_-$ is the amplitude of a wave travelling in the opposite direction. The latter is thus <u>reflected</u>, while the former is <u>transmitted</u>. Accordingly, we will call

$s_{11}(k)$: <u>transmission coefficient from the left</u>,
$s_{21}(k)$: <u>reflection coefficient from the left</u>,
$s_{12}(k)$: <u>reflection coefficient from the right</u>,
$s_{22}(k)$: <u>transmission coefficient from the right</u>.

S has the following important properties, valid for real k:

(i) $s_{11} = s_{22}$

(ii) S is unitary

(iii) $s_{ij}(-k) = \overline{s_{ij}(k)}$

(iv) $s_{11}(k)$ can be continued to the half-plane Im $k > 0$ as a nonvanishing meromorphic function with simple poles at the points $i\kappa_j$, where $-\kappa_j^2 = \lambda_j < 0$ are the eigenvalues of E

(v) $\lim\limits_{|k|\to\infty} s_{11}(k) = 1$

(vi) $\lim\limits_{|k|\to\infty} s_{ij}(k) = 0$, for $i \neq j$.

Property (i) allows us to refer to $s_{11} = s_{22}$ as <u>the</u> transmission coefficient. From now on, we will denote it by $t(k)$, the reflection coefficients from left and right by $r^-(k)$ and $r^+(k)$, respectively, while $r(k)$ will stand for either $r^-(k)$ or $r^+(k)$. Thus using this notation,

(2.4) $$S = \begin{pmatrix} t(k) & r^+(k) \\ r^-(k) & t(k) \end{pmatrix} .$$

Based on the unitarity of the S-matrix and on the uniqueness result mentioned after (2.2), we see that a solution of (2.1) is uniquely determined by any one of the pairs of <u>asymptotic components</u>

$$(A_-,B_+),\ (A_+,B_-),\ (A_-,B_-),\ (A_+,B_+) .$$

Now we will give the proof of properties (i)-(iv).

By definition of ψ_-,

$$(2.5)_- \qquad \psi_-(x,k) \sim \begin{cases} e^{ikx} + r^-(k)e^{-ikx}, & x \sim -\infty \\ s_{11}(k)e^{ikx}, & x \sim +\infty \end{cases}$$

and similarly

$$(2.5)_+ \qquad \psi_+(x,k) \sim \begin{cases} e^{-ikx} + r^+(k)e^{ikx}, & x \sim +\infty \\ s_{22}(k)e^{-ikx}, & x \sim -\infty \end{cases}$$

For any real $x > 0$, Green's formula holds:

$$(2.6) \quad 0 = \int_{-x}^{x} \left\{\psi_+(L - k^2)\psi_- - \psi_-(L - k^2)\psi_+\right\}$$

$$= \left\{\psi_-\psi_+' - \psi_+\psi_-'\right\}\Big|_{-x}^{x} .$$

It is possible to show that one can differentiate relations (2.5) to obtain the asymptotic behavior of ψ_-' and ψ_+'. By using these expansions we get

$$(2.7)_+ \qquad \lim_{x\to +\infty} \psi_-\psi_+' - \psi_+\psi_-' = -2ik\, s_{11}(k)$$

$$(2.7)_- \qquad \lim_{x\to -\infty} \psi_-\psi_+' - \psi_+\psi_-' = -2ik\, s_{22}(k) .$$

(i) follows from (2.6) and (2.7).

To prove (ii), take the complex conjugate of $(2.5)_{\pm}$:

$$(2.8) \quad \bar{\psi}_{\pm}(x,k) \sim \begin{cases} e^{\pm ikx} + \overline{r^{\pm}(k)}\, e^{\mp ikx}, & x \sim \pm\infty \\ \overline{t(k)}\, e^{\pm ikx}, & x \sim \mp\infty \end{cases}$$

and note that $\bar{\psi}_+$ has incoming components $(\bar{t},\overline{r^+})$ and outgoing components $(1,0)$, while $\bar{\psi}_-$ has incoming components

$(\overline{r^-}, \bar{t})$ and outgoing components (0,1). So by (2.3) and (2.4),

(2.9) $$t\,\overline{r^-} + r^+\bar{t} = 0$$

$(2.10)_-$ $$|r^-|^2 + |t|^2 = 1$$

$(2.10)_+$ $$|t|^2 + |r^+|^2 = 1\ ,$$

giving us (ii).

Next compare (2.8) with

$$\psi_{\pm}(x,-k) \sim \begin{cases} e^{\pm ikx} + r^{\pm}(-k)e^{\mp ikx}\ , & x \sim \pm\infty \\ t(-k)e^{\pm ikx} & x \sim \pm\infty. \end{cases}$$

Observe that $\bar{\psi}_-(x,k)$ and $\psi_-(x,-k)$ have the same outgoing components. The incoming components of $\bar{\psi}_+(x,k)$ and $\psi_+(x,-k)$ also coincide. Since a solution is characterized by either its incoming or its outgoing components, we must have

$$\psi_{\pm}(x,-k) = \bar{\psi}_{\pm}(x,k)\ .$$

Therefore

(2.10a) $$r^{\pm}(-k) = \overline{r^{\pm}(k)}$$

and

(2.10b) $$t(-k) = \overline{t(k)}\ ,$$

which proves (iii).

To prove (iv), consider the solutions ϕ_+ and ϕ_- of (2.1) characterized by

$(2.11)_{\pm}$ $$\phi_{\pm}(x,k) \sim e^{\pm ikx}\ , \qquad x \sim \pm\infty.$$

We can show that these solutions may be extended to the upper half-plane as analytic functions, with $(2.11)_{\pm}$ still holding in that region (see Section 6). The Wronskian $W[\phi_+,\phi_-]$ of these solutions is also an analytic function for $\operatorname{Im} k > 0$. Since (2.1) does not involve the first derivative of y, the Wronskian of any fixed pair of solutions is a constant and therefore

$$W[\phi_+,\phi_-] = \lim_{x\to\pm\infty} W[\phi_+,\phi_-] \ . \tag{2.12}$$

From (2.3) and (2.4) we can deduce

$$\phi_+(x,k) \sim \frac{1}{t(k)}\, e^{ikx} + \frac{r^-(k)}{t(k)}\, e^{-ikx} \ , \qquad x \sim -\infty, \tag{2.13}$$

for real k. Applying $(2.11)_-$ and (2.13) to (2.12) we get

$$W[\phi_+,\phi_-] = -\, 2ik/t(k) \ .$$

This implies that $t(k)$ may be extended to $\operatorname{Im} k > 0$ as a nonvanishing meromorphic function whose only poles are zeros of $W[\phi_+,\phi_-]$.

Observe that for $\operatorname{Im} k > 0$, the solutions ϕ_+ and ϕ_- decay exponentially, the first for $x \to +\infty$, the second for $x \to -\infty$. If for some $k = k_0$, with $\operatorname{Im} k_0 > 0$

$$W[\phi_+,\phi_-] = 0 \ ,$$

then

$$\phi_+(x,k_0) = \alpha\phi_-(x,k_0)$$

identically. This implies that these functions have an exponential decay for $x \to +\infty$ as well as for $x \to -\infty$. Thus k_0^2 is an eigenvalue of E and from the self-adjointness of E we conclude that $k_0 = i\kappa_j$, $\kappa_j > 0$.

To complete the proof of (iv) we still have to show that the poles of $t(k)$ are simple. The proof of this fact, as well as of (iv) and (vi) will not be presented here. They can be found in Friedrichs [7] or Faddeev [6].

Next we show that S is completely determined by either of its off-diagonal elements and the discrete spectrum $\{-\kappa_j^2\}$ of E. Indeed, equation $(2.10)_\pm$ enables us to get $|t|$ from the knowledge of $|r|$. To determine all of t, we form the function

$$(2.14) \qquad T(k) = t(k)\prod\left(\frac{k-i\kappa_j}{k+i\kappa_j}\right);$$

T(k) is analytic and does not vanish on the upper half-plane while for real k

$$|T(k)| = |t(k)|$$

and by (v):

$$T(k) \to 1 \quad \text{as} \quad k \to \pm\infty .$$

By applying the Cauchy integral formula to log T(k), we get

$$\log T(k) = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{\log |T(s)|}{s-k}\, ds , \qquad \text{Im } k > 0.$$

Using (2.10) and (2.14) we obtain

$$(2.15)\quad t(k) = \prod\left(\frac{k+i\kappa_j}{\overline{k+i\kappa_j}}\right) \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\,(1-|r(s)|^2)}{s-k}\,ds\right\},$$

for Im k > 0, while for k real

$$t(k) = \lim_{\varepsilon\to 0^+} t(k+i\varepsilon)\ .$$

Knowing t and one of the reflection coefficients, the other reflection coefficient may be computed from (2.9), and thus the S-matrix is determined.

If we know the potential $q(x)$, we can determine r as a function of k, k real. The study of the way the S-matrix depends on q(x) is called the direct problem.

In many physical situations, it is difficult, or impossible, to measure q directly, whereas one of the reflection coefficients is suitable to be measured. Bargmann discovered that neither of the reflection coefficients alone contains enough information for the unique determination of the potential q, even if we know the point spectrum of E, Cf. [2]. This lack of uniqueness can be overcome if one knows also the normalizing constants on the right m_j^+, or on the left m_j^-. They are introduced, for each eigenvalue $\lambda_j = -\kappa_j^2$, as the inverse of the L^2 norm of the eigenfunctions $\phi_+(x, i\kappa_j)$ or $\phi_-(x,i\kappa_j)$, respectively.

The inverse problem is the search of information about q from the knowledge of one of the reflection coefficients, the corresponding normalizing constants and the point spectrum of E.

3. Stability Results

In what follows, unless otherwise stated, our functions will be real valued and measurable, defined on either the real line $\mathbb{R}$ or on the positive axis $\mathbb{R}^+$. We will use the notation:

$$\|f\|_{(1)} \equiv \int |x\, f(x)|\, dx$$

$$\|f\|_{(2)} \equiv \sup \left\{x^2 |f(x)|/2\right\},$$

where both the integral and the supremum are taken on $\mathbb{R}$ or $\mathbb{R}^+$, whichever the domain of f.

P ≡ the set of piecewise continuous functions $q(x)$ defined on $\mathbb{R}$ and satisfying

$$\int_{-\infty}^{\infty} (1 + |x|)\ |q(x)|\ dx < \infty \ ;$$

a function q in P will be called a <u>potential</u>.

$R^{\pm}$ ≡ the sets of continuous complex functions $r^{\pm}(k)$ defined on $\mathbb{R}$ and such that:

(i) $r^{\pm}$ have <u>real</u> Fourier transforms

$$(3.1)_{\pm} \qquad F_{\pm}(t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} r^{\pm}(k)\ e^{\pm 2ikt}\ dk$$

that are absolutely continuous and whose derivatives $F'_{\pm}(t)$ satisfy

$$\pm \int_{a}^{\pm\infty} (1 + |t|)\ |F'_{\pm}(t)|\ dt \le C_a < \infty,$$

for all real a.

(ii) $|r(k)| < 1$, for $k \neq 0$; if $|r(0)| = 1$, then $r(0) = -1$. (Note that $|r(0)| \leq 1$, by continuity; and by (i), $r(-k) = \overline{r(k)}$, so that $r(0)$ is real.)

(iii) $r(k) = O(1/|k|)$, $|k| \sim \infty$.

We will refer to the functions in R as <u>reflection coefficients</u>.

As remarked in the previous section, a set of <u>scattering data</u> is a triple

$$s^{\pm} \equiv (r^{\pm}, \kappa, m^{\pm})$$

where r is a reflection coefficient, $\kappa \equiv (\kappa_j)$ and $m \equiv (m_j)$ are N-tuples of positive numbers, N being a nonnegative integer and the κ_j's being all distinct. The collection of all such scattering data will be denoted by $S^{\pm}$.

The assumptions made above on the potentials are nearly the weakest that we can make and still have a scattering theory. Indeed, the integrability of q implies the asymptotic behavior of the solutions of (2.1), as described in (2.2), while the existence of the first moment of q implies the finiteness of the point spectrum of E (see Section 6).

On the other hand, the conditions imposed on the scattering data are also nearly as mild as they could be, as is assured by the following result, cf. [4]:

<u>Theorem 1</u> (Faddeev). There exists a one-to-one correspondence between the set of potentials P and either set of scattering data S^{+} or S^{-}.

We now state the two main continuity results we obtained. Observe that we restrict ourselves to sets smaller than P and S.

Let P^+ be the set of potentials q such that:

(a)$_+$ each $q \in P^+$ vanishes on some half-line $\{x \leq \alpha\}$;

(b) $q(x) = o(x^{-2})$, $|x| \sim \infty$;

(c) the operator E associated with q has no point spectrum.

Theorem 2 (Stability of the Direct Scattering Problem). Take in P^+ the distance corresponding to the norm

$$(3.2) \qquad \|q\| \equiv \|q\|_\infty + \|q\|_{(1)} + \|q\|_{(2)} .$$

Then the direct scattering mapping

$$q \to r^+ ,$$

$q \in P^+$, $r^+ \in L^2(\mathbb{R})$ is continuous.

Remark. To get an analogous result for r^-, introduce the set P^- , requiring instead of condition (a)$_+$:

(a)$_-$ each $q \in P^-$ vanishes on some half-line $\{x \geq \alpha\}$.

In order to state a continuity result for the inverse mapping, let us introduce two new sets:

Let $R_1^{\pm}$ denote the sets of reflection coefficients $r^{\pm}$ for which $F'_{\pm}$ are bounded and satisfy

$$F'_{\pm}(t) = o(|t|^{-3}) , \qquad |t| \sim \infty,$$

where $F'_{\pm}$ are defined in (3.1).

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and the reflection coefficients $r^{\pm}$, and are defined in

Theorem 3 (Stability of the Inverse Scattering Mapping). Take in $\mathcal{R}_1^{\pm}$ the distance associated with the norm

$$\|r\| \equiv \|F'\|_{\infty} + \|F'\|_{(1)} ,$$

and in $\mathcal{P}$ the distance associated with uniform convergence on compact sets. Then the inverse scattering mappings

$$\mathcal{R}_1^{\pm} \to \mathcal{P}$$

are continuous in the above topologies.

Remarks. (a) Actually, the mapping $\mathcal{R}_1^{+} \to \mathcal{P}$ is continuous in the topology of uniform convergence on half-lines $\{x \geq \alpha\}$, while $\mathcal{R}_1^{-} \to \mathcal{P}$ is continuous in the topology of uniform convergence on $\{x \leq \alpha\}$.

(b) The same continuity results hold if we fix an integer $N > 0$ and consider scattering data

$$s \equiv (r,\kappa,m)$$

with N-tuples κ and m, $r \in \mathcal{R}_1$, and take

$$\|s\| \equiv \|F'\|_{\infty} + \|F'\|_{(1)} + \|\kappa\| + \|m\| ,$$

where the norms for κ and m are $\mathbb{R}^N$ norms.

To prove these theorems we introduce the functions $B_{\pm}(x,y)$. They establish the link between the potential q and the reflection coefficients $r^{\pm}$, and are defined in terms of the functions $\phi_{\pm}$ introduced in $(2.11)_{\pm}$, specifically as Fourier transforms with respect to k:

$$(3.3)_{\pm} \qquad \phi_{\pm}(x,k)\, e^{\mp ikx} = 1 + \int_{-\infty}^{\infty} B_{\pm}(x,y)\, e^{\pm 2iky}\, dy \ .$$

We have observed in Section 2 that $\phi_{\pm}$ are analytic for Im k > 0, and it can be shown that

$$\phi_{\pm}(x,k)\, e^{\mp ikx} = 1 + O(1/|k|) \ , \qquad |k| \sim \infty.$$

So we deduce from the Paley-Wiener theorem that

$$B_{\pm}(x,y) = 0 \ , \qquad \pm\, y \geq 0.$$

It can be proved that

$$(3.3a)_{\pm} \qquad B_{\pm}(x,y) = 2K_{\pm}(x,2y+x),$$

where $K_{\pm}$ satisfy

$$\{\partial_{yy} - \partial_{xx} + q(x)\}\, K_{\pm}(x,y) = 0$$

and

$$(3.3b)_{\pm} \qquad \frac{d}{dx}\, K_{\pm}(x,\ \pm x) = \mp q(x)/2.$$

The functions $K_{\pm}$ have the following important property: if $v_0(x,k)$ satisfies

$$-\, v'' = k^2 v$$

then

$$y_{\pm}(x,k) \equiv v_0(x,k) \pm \int_{x}^{\pm\infty} K_{\pm}(x,y)\, v_0(y,k)\, dy$$

are solutions of (2.1).

The relationship between the functions $B_{\pm}$ and the potential $q(x)$ is expressed by the equations

$$(3.4)_{\pm} \quad B_{\pm}(x,y) = \pm \int_{x+y}^{\pm\infty} q(t)\, dt + \int_0^y dz \int_{x+y-z}^{\pm\infty} q(t)\, B(t,z)\, dt\ , \quad \pm y \geq 0$$

while the so-called Marchenko equations

$$(3.5)_{\pm} \quad \Omega_{\pm}(x+y) + B_{\pm}(x,y) \pm \int_0^{\pm\infty} \Omega_{\pm}(x+t+y)\, B_{\pm}(x,t)\, dt = 0, \quad \pm y \geq 0$$

relate $B_{\pm}$ to the scattering data, since Ω is defined as

$$(3.6)_{\pm} \qquad \Omega_{\pm}(t) \equiv F_{\pm}(t) + 2 \sum m_j^{\pm} e^{\mp 2\kappa_j t}.$$

The derivation of both equations will be given in Section 6.

To study the direct mapping, first we solve (3.4), which is a Volterra equation for $B(x,y)$. Once B is determined, we set $y = 0$ in (3.5), regarding now B as the kernel. The equation we obtain for Ω is again of Volterra type:

$$(3.7)_{\pm} \quad \Omega_{\pm}(x) + B_{\pm}(x,0) \pm \int_x^{\pm\infty} B_{\pm}(x,t-x)\, \Omega_{\pm}(t)\, dt = 0\ .$$

When considering the inverse mapping, we regard Ω as given and solve equations (3.5) for B, observing that these are a family of Fredholm equations, where x enters as a parameter. Once B is determined, we can get q from (3.4) by setting $y = 0$ and differentiating with respect to x:

$$(3.8)_{\pm} \qquad q(x) = \mp\, \partial_1 B(x,0).$$

Consequently, we have to analyze the continuity properties of the chain of mappings

$$q \leftrightarrow B \leftrightarrow \Omega \leftrightarrow r \; .$$

The proof of Theorem 2 is based on Claims 1-3 of Section 4, which give the continuity of $q \to \Omega_+$. Since in the absence of the point spectrum, $\Omega = F$, the Fourier transform of r, we can use the unitarity of the Fourier transform to obtain the continuity of $q \to r^+$.

Theorem 3 is a direct consequence of Claims 4 and 5 in Section 4. We remark that, for the sake of brevity in the statement of this theorem, we required

$$F'(t) = o(|t|^{-3}) \; , \qquad |t| \sim \infty,$$

instead of the weaker conditions

$$\|F'\|_{(1)} < \infty$$

$$F_{\pm}(t) = o(t^{-2}) \; , \qquad t \sim \pm\infty \; ,$$

which are the ones we actually use.

We end this section with two observations:

First, results similar to Theorems 2 and 3 can be obtained for the radial problem, since equations analogous to (3.4) and (3.5) hold in this case, cf. [1] and [5].

Second, by using Theorem 4 of Section 6 we can restate a previous stability result by Lundina and Marchenko, Cf. [14], with hypothesis only on the scattering data, avoiding a priori assumptions on the potentials.

4. Technical Lemmas

In this section we will denote by $L^{(1)}(D)$ or $L^{(2)}(D)$ the sets of functions f on D for which

$$\|f\|_{(1)} < \infty \quad \text{or} \quad \|f\|_{(2)} < \infty ,$$

respectively; by $X(D)$ the set of bounded functions f that belong to $L^{(1)}(D)$ and decay like

$$f(x) = o(x^{-2}) , \qquad |x| \sim \infty.$$

We also use:

$$M \equiv R^+ \times R^+.$$

Claim 1. For any q in $L^1(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+)$, the equation

$$\text{(4.1)} \quad B(x,y) = \int_{x+y}^{\infty} q(t)\, dt + \int_0^y dz \int_{x+y-z}^{\infty} q(t)\, B(t,z)\, dt, \quad x,y \geq 0$$

has a unique solution B. This solution belongs to $C_0(M)$, its first partial derivative $\partial_1 B$ belongs to $L^1(M)$, and the mapping

$$S: q \to B$$

is continuous with respect to the norms

$$|||q||| \equiv \|q\|_1 + \|q\|_{(1)} ,$$

$$|||B|||_p \equiv \|B\|_\infty + \|B(\cdot,0)\|_p + \|\partial_1 B(\cdot,0)\|_1 , \qquad 1 \leq p \leq \infty.$$

Also, if q is continuous, so are $\partial_1 B$ and $\partial_2 B$.

Proof: Consider the operators

$$V_q\colon \beta \to C(x,y) \equiv \int_0^y dz \int_{x+y-z}^{\infty} q(t)\ \beta(t,z)\ dt$$

for q in $L^1(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+)$. Then,

$$C(x,y)\ /\|\beta\|_\infty \le \int_0^y \left(\int_{x+y-z}^{\infty} |q(t)|\ dt\right) dz = \int_0^y dz \int_{x+z}^{\infty} |q(t)|\ dt$$

$$\le \int_0^\infty dz \int_z^\infty |q(t)|\ dt = \int_0^\infty |q(t)|\ dt \int_0^t dz = \|q\|_{(1)}\ ,$$

so that $\|V_q\beta\|_\infty \le \|q\|_{(1)}\ \|\beta\|_\infty$; i.e., V_q is a bounded operator on $L^\infty(M)$ which depends continuously on $q \in L^1(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+)$.

Now,

$$\|V_q^2\beta\| = \sup_{u,v} \left| \int_0^u dy \int_{u+v-y}^{\infty} dx\ q(x) \int_0^y dz \int_{x+y-z}^{\infty} dt\ q(t)\ \beta(t,z)\right|$$

$$\le \int_0^u dy \int_{v+y}^{\infty} dx\ |q(x)| \left(\int_x^\infty |tq(t)|\ dt\right) \|\beta\|_\infty$$

$$\le \|\beta\|_\infty \int_0^\infty x|q(x)| \left(\int_x^\infty t|q(t)|\ dt\right) dx = \left(\|q\|^2_{(1)}/2\right)\|\beta\|_\infty,$$

and, in the same way,

$$\|V_q^m\beta\|_\infty \le \left(\|q\|^m_{(1)}/m!\right)\|\beta\|_\infty\ ,$$

which implies that on $L^\infty(M)$, $(I+V_q)^{-1}$ exists and $\|(I+V_q)^{-1}\| \le \exp \|q\|_{(1)}$. By the continuity of the inversion mapping on the algebra of bounded invertible operators (with the uniform operator topology), $(I+V_q)^{-1}$ depends continuously on q.

Let

$$Q(x,y) \equiv \int_{x+y}^{\infty} q(t)\, dt .$$

Then $\|Q\|_\infty \le \|q\|_1$, and thus, since (4.1) may be rewritten as

$$B = (I-V_q)^{-1} Q ,$$

the continuity of the mapping S from $L^1(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+)$ to $L^\infty(M)$ follows.

Now Q is continuous, and so is $V_q Q$; consequently

$$B = \sum_{m=0}^{\infty} (V_q)^m Q$$

is also continuous, as this series converges in the $L^\infty(M)$ sense. Since

$$B(x,0) = \int_x^\infty q(t)\, dt ,$$

and

$$\partial_1 B(x,0) = -q(x) ,$$

we have

$$\|B(\cdot,0)\|_1 \leq \|q\|_{(1)} ,$$

$$\|B(\cdot,0)\|_\infty \leq \|q\|_1 ,$$

$$\|\partial_1 B(\cdot,0)\|_1 = \|q\|_1 .$$

and therefore the continuity of S in the sense stated above is proven.

To show the other properties of B, let us introduce the functions

$$\xi(x) \equiv \int_x^\infty |q(t)| \, dt ,$$

and

$$\eta(x) \equiv \int_x^\infty |q(t)| t \, dt .$$

Then

$$|Q(x,y)| \leq \xi(x+y) ,$$

and

$$|(V_q Q)(x,y)| \leq \int_0^y dz \int_{x+y-z}^\infty |q(t)| \, \xi(t+z) \, dt$$

$$\leq \xi(x+y) \int_0^y dz \int_{x+z}^\infty |q(t)| \, dt$$

$$\leq \xi(x+y) \int_x^\infty |q(t)| \, dt \int_0^{t-x} dz \leq \xi(x+y) \, \eta(x) .$$

By modifying the previous estimates for V_q^m in this fashion, we get

$$|(V_q^m Q)(x,y)| \le \xi(x+y)[\eta(x)]^m/m!$$

so that the solution of (4.1) satisfies

$$|B(x,y)| \le \xi(x+y) \exp \eta(x) \le \xi(x+y) \exp \|q\|_{(1)} .$$

This inequality implies that B vanishes at ∞.

By differentiating (4.1) with respect to x we get

$$(4.2) \quad \partial_1 B(x,y) = -q(x+y) - \int_0^y q(x+y-z)\, B(x+y-z,z)\, dz$$

outside a null set. Therefore

$$\|\partial_1 B\|_1 \le \int_0^\infty dx \int_0^\infty |q(x+y)|\, dy + \int_0^\infty dx \int_0^\infty dy \int_0^y |q(x+y-z) B(x+y-z,z)|dz$$

$$\le \|q\|_{(1)} + \exp\left(\|q\|_{(1)}\right) \int_0^\infty dx \int_0^\infty \xi(x+y)dy \int_x^{x+y} |q(z)|\, dz$$

$$\le \|q\|_{(1)} + \exp\left(\|q\|_{(1)}\right) \int_0^\infty dx \int_0^\infty dy \int_y^\infty |q(t)|dt \int_x^\infty |q(z)|\, dz$$

$$= \|q\|_{(1)} + \exp\left(\|q\|_{(1)}\right) \|q\|_{(1)}^2 ;$$

i.e.,

$$\partial_1 B \in L^1(M) .$$

<u>Claim 2</u>. If $q \in X(\mathbb{R}^+)$ in addition to the conclusions of Claim 1, we have that

$$B \in L^p(M) , \qquad 2 \le p \le \infty ,$$
$$\partial_1 B \in L^p(M) , \qquad 1 \le p \le \infty ,$$

and that $S: q \to B$ is continuous with respect to the norms

$$|q| \equiv \|q\|_\infty + \|q\|_{(1)} + \|q\|_{(2)}$$

$$|B|_p \equiv \|B\|_p + \|B(\cdot,0)\|_p + \|\partial_1 B(\cdot,0)\|_p + \|\partial_1 B\|_\infty, \quad 2\le p\le\infty.$$

Proof: If we consider q_1 and q_2 satisfying the hypothesis, and if B_1 and B_2 are their corresponding images by S, we obtain from (4.2) that

$$\|\partial_1(B_1-B_2)\|_\infty \le \|q_1-q_2\|_\infty + \|q_2\|_1 \|B_1-B_2\|_\infty + \|B_1\|_\infty \|q_1-q_2\|_1 .$$

This implies that

$$\partial_1 B \in L^1(M) \cap L^\infty(M) ,$$

and that $\partial_1 B$ depends continuously on q, in the $L^\infty(M)$ norm. We can also conclude that

$$\partial_1 B(x,0) = -q(x) \in L^p(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+), \qquad \text{for } 1\le p\le\infty.$$

All that is left to show is the continuous dependence of B on q, in any $L^p(M)$ norm, for $p \ge 2$. To get this result, let us show first that V_q is a compact operator on $L^1(M)$ which depends continuously on q:

$$\|V_q\beta\|_1 \le \int_0^\infty dx \int_0^\infty dy \int_0^y dz \int_{x+y-z}^\infty |q(t)\ \beta(t,z)|\ dt$$

$$= \int_0^\infty dt \int_0^\infty |q(t)\ \beta(t,z)|\ dz \int_z^{t+z} dy \int_0^{t+z-y} dx$$

$$= \int_0^\infty dt \int_0^\infty |q(t)\ \beta(t,z)|\, t^2/2dz \ \le\ \|q\|_{(2)} \|\beta\|_1 .$$

Now, for $R \to +\infty$

$$\int_R^\infty dx \int_R^\infty dy \left| \int_0^y dz \int_{x+y-z}^\infty q(t)\ \beta(t,z)\ dt \right|$$

$$\leq \int_R^\infty dt \int_0^\infty dz\ |q(t)\ \beta(t,z)| \int_z^{t+z} dy \int_0^{t+z-y} dx$$

$$= \int_R^\infty t^2 |q(t)|/2dt \int_0^\infty |\beta(t,z)|\ dz$$

$$\leq \|\beta\|_1 \sup_{t \geq R} |q(t)t^2/2| \to 0$$

uniformly for $\|\beta\|_1 \leq 1$.

In the same way we prove that

$$\lim_{h^2+k^2 \to 0} \|(V_q\beta)(x+h,y+k) - (V_q\beta)(x,y)\|_1 = 0$$

uniformly for $\|\beta\|_1 \leq 1$, so that the set

$$\{V_q\beta;\ \|\beta\|_1 \leq 1\}$$

is precompact in $L^1(M)$ by the Kolmogorov-Frechet theorem, and therefore V_q is compact on $L^1(M)$.

Assume now that for some $\lambda \neq 0$ and some $\beta \in L^1(M)$,

$$V_q\beta = \lambda\beta\ .$$

Then since

$$\|V_q\beta\|_\infty \leq \|q\|_\infty \|\beta\|_1\ ,$$

we conclude that $\beta \equiv 0$. This implies that $(I-V_q)$ is invertible on $L^1(M)$.

Thus, as an operator on either $L^1(M)$ or $L^\infty(M)$, $(I-V_q)^{-1}$ is bounded and depends continuously on $q \in X(\mathbb{R}^+)$, so that the same must also hold for $L^p(M)$, $1 < p < \infty$.

Now

$$\left|\int_0^\infty\int_0^\infty f(x,y)\ Q(x,y)\ dx\ dy\right| \le \int_0^\infty dx \int_0^\infty dy\ |f(x,y)| \int_{x+y}^\infty |q(t)|\ dt$$

$$= \int_0^\infty dx \int_x^\infty dt\ |q(t)| \int_0^{t-x} |f(x,y)|\ dy = \int_0^\infty dt|q(t)| \int_0^t dt \int_0^{t-x} |f(x,y)|dy$$

$$\le \int_0^\infty dt|q(t)| \sqrt{\int_0^t dx \int_0^{t-x} |f(x,y)|^2 dy}\ \ \sqrt{t^2/2} \le \frac{\|f\|_2}{\sqrt{2}} \|q\|_{(1)} ,$$

so that $Q \in L^2(M)$. This gives the continuity of the solution B of (4.1) in the sense of the $L^2(M)$ norm. But continuity in the L^2 and L^∞ sense implies continuity for any L^p norm, $2 \le p \le \infty$.

Claim 3. If $B = S(q)$ for some $q \in X(\mathbb{R}^+)$, S as defined in Claim 1, the equation

$$(4.3) \quad \Omega(x) + \int_x^\infty B(x,t-x)\ \Omega(t)\ dt + B(x,0) = 0 , \quad x \ge 0 ,$$

has a unique solution Ω. This solution belongs to $L^p(\mathbb{R}^+)$, for $1 \le p \le \infty$, and satisfies $\Omega(x) = o(1/x)$, for $x \to +\infty$; it is also differentiable, $\Omega' \in L^2(\mathbb{R}^+)$, and the mapping

$$T:\ S(X) \subset L^2(M) \to L^2(\mathbb{R}^+)$$
$$B \to \Omega$$

is continuous.

Remark. Stronger conclusions can be obtained for Ω', namely that

$$\Omega' \in L^{(1)}(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$$

which in particular implies that

$$\Omega' \in L^p(\mathbb{R}^+) \ , \qquad 1 \leq p \leq \infty.$$

Proof: Consider the operator

$$V_B : \omega \to C(x) \equiv \int_x^\infty B(x,t-x)\ \omega(t)\ dt \ , \qquad x \geq 0$$

with B in $S(X)$. We claim that V_B is of Volterra type on $L^\infty(\mathbb{R}^+)$. Indeed:

$$|C(x)|/\|\omega\|_\infty \leq \int_x^\infty \xi(t)\ \exp \eta(x)\ dt \leq \exp\left(\|q\|_{(1)}\right) \int_x^\infty \xi(t)\ dt$$

$$\leq \|q\|_{(1)} \exp \|q\|_{(1)} \ ,$$

$$|V_B C(y)|/\|\omega\|_\infty \leq \int_y^\infty \xi(x)\ \exp \eta(y)\ dx \int_x^\infty \xi(t)\ \exp \eta(x)\ dt$$

$$\leq \exp\left(2\|q\|_{(1)}\right) \int_y^\infty \xi(x)\ dx \int_x^\infty \xi(t)\ dt$$

$$\leq [\|q\|_{(1)} \exp \|q\|_{(1)}]^2/2 \ ,$$

and in general

$$|V_B^n \omega(x)|/\|\omega\|_\infty \leq [\|q\|_{(1)} \exp\|q\|_{(1)}]^n/n! \ .$$

Consequently, a solution for (4.3) exists, is unique, and is given by

$$\Omega = -\Sigma(-V_B)^n B(\cdot,0) ,$$

being thus a continuous function.

We show now that V_B is bounded if considered as an operator from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$, where $1 \leq p \leq q \leq \infty$.

We already know that V_B is bounded on $L^\infty(\mathbb{R}^+)$. It is also bounded on $L^1(\mathbb{R}^+)$:

$$|(V_B\omega)(x)| \leq \int_x^\infty |\omega(t)| \; \xi(t) \exp \eta(x) \; dt \leq \|\omega\|_1 \exp\left(\|q\|_{(1)}\right)\xi(x) ,$$

so that

$$\int_0^\infty |(V_B \;)(x)| \; dx \leq \|\omega\|_1 \exp\left(\|q\|_{(1)}\right) \int_0^\infty \xi(x) \; dx = \|\omega\|_1 \|q\|_{(1)} \exp\|q\|_{(1)} .^*$$

Thus V_B is bounded from L^p to L^q, for $p,q \in \{1,\infty\}$, $q \geq p$. This implies that the same holds for $p,q \in [1,\infty]$, $q \geq p$.

Assume now that

$$V_B\omega = \lambda\omega$$

for $\lambda \neq 0$, $\omega \in L^2$. Then

$$\|\lambda\omega\|_\infty = \|V_B\omega\|_\infty \leq \|V_B\|_{2,\infty}\|\omega\|_2 ,$$

so that $\omega \in L^\infty(\mathbb{R}^+)$, and consequently

$$\omega \equiv 0 .$$

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* Also $\|V_B\omega\|_\infty \leq \|B\|_\infty\|\omega\|_1$.

Thus V_B is one-to-one on $L^2(\mathbb{R}^+)$. V_B being the uniform limit of compact operators, we conclude that

$$(I+V_B)^{-1}: L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$$

is defined and bounded. Observe that for $2 \leq p < \infty$, $1/p + 1/p' = 1$,

$$\|V_B\omega\|_{p'} = \sup_{\|g\|_p=1} \int_0^\infty g(x)(V_B\omega)(x)\,dx ,$$

and thus,

$$(4.4) \quad \|V_B\omega\|_{p'} \leq \left(\int_0^\infty dx \int_x^\infty |B(x,t-x)|^{p'}\,dt\right)^{1/p'} \cdot \left(\int_0^\infty dx \int_x^\infty |\omega(t)|^p\,|g(x)|^p\,dt\right)^{1/p}$$

$$\leq \|B\|_{p'}\|\omega\|_p .$$

Therefore, by (4.4), as an operator on $L^2(\mathbb{R}^+)$, V_B depends continuously on B. The same also holds for $(I+V_B)^{-1}$, and hence for $\Omega = (I+V_B)^{-1}\,B(\cdot,0)$.

Let us refine the previous estimates in order to get more information about Ω. Denoting

$$s(x) \equiv B(x,0)$$

$$\zeta(x) \equiv \exp\eta(x) ,$$

we obtain

$$|s(x)| \leq \xi(x)\ \zeta(x) ,$$

$$|(V_Bs)(x)| \leq \int_x^\infty \xi(t)\zeta(x)\xi(t)\zeta(t)dt \leq \xi(x)\zeta(x)\zeta(x) \int_x^\infty \xi(t)\,dt$$

$$= \xi(x)\zeta(x)[\eta(x)\zeta(x)],$$

$$|(V_B^n s)(x)| \leq \xi(x)\zeta(x)[\eta(x)\zeta(x)]^n/n! \;,$$

so that

$$|\Omega(x)| = |-\sum_{n=0}^{\infty} (-V_B^n s)(x)| \leq \xi(x)\zeta(x) \exp [\eta(x)\zeta(x)] \;.$$

Since the function

$$\zeta(x) \exp [\eta(x)\zeta(x)]$$

is bounded, and $t\xi(t) \in C_0(\mathbb{R}^+)$, we conclude that $t\Omega(t) \in C_0(\mathbb{R}^+)$. Since

$$\int_0^\infty |\Omega(t)| \, dt \leq \int_0^\infty dt \int_t^\infty |\Omega'(s)| \, ds = \|\Omega'\|_{(1)} \;,$$

we have

$$\Omega \in L^p(\mathbb{R}^+) \;, \qquad 1 \leq p \leq \infty.$$

Now rewrite (4.3) as

$$\Omega(x) + \int_0^\infty B(x,s)\,\Omega(s+x)\,ds + B(x,0) = 0 \;.$$

By differentiation we get

$$\begin{aligned} 0 &= \Omega'(x) + \int_0^\infty B(x,s)\Omega'(s+x)ds + \int_0^\infty \partial_1 B(x,s)\Omega(s+x)ds + \partial_1 B(x,0) \\ &= \Omega'(x) + \int_x^\infty B(x,t-x)\Omega'(t)dt + \int_x^\infty \partial_1 B(x,t-x)\Omega(t)dt + \partial_1 B(x,0), \end{aligned}$$

or, if we denote $A \equiv \partial_1 B$,

$$\Omega' = -(I+V_B)^{-1}[A(\cdot,0) + V_A\Omega] \;.$$

Since $A \in L^2(M)$ and $\Omega \in L^2(\mathbb{R}^+)$, $V_A\Omega \in L^2(\mathbb{R}^+)$. Also $A(\cdot,0) \in L^2(\mathbb{R}^+)$, so that

$$\Omega' \in L^2(\mathbb{R}^+) .$$

The proof is complete.

<u>Claim 4</u>. Let us consider differentiable functions

$$\Omega: \mathbb{R} \to \mathbb{R}$$

that vanish at ∞ such that

$$\Omega' \in L^{(1)}(\mathbb{R}) \cap L^1(\mathbb{R})$$

Then, for $1 \leq p \leq q \leq \infty$, the operators

$$G(x,\Omega): L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$$

$$\phi \mapsto \psi_x(y) \equiv \int_0^\infty \Omega(x+t+y)\ \phi(t)\ dt$$

have the following properties:

(a) For any fixed Ω and $p = q$,

$$\lim_{x\to+\infty} \| G(x,\Omega) \|_{p,q} = 0 ,$$

uniformly in p.

(b) For $q < \infty$, or $p = q = \infty$, G is compact.

(c) Consider the norm

(4.5) $$\|\Omega\| \equiv \|\Omega'\|_1 + \|\Omega'\|_{(1)} .$$

Then

(4.6) $$\|G(x,\Omega)\|_{p,q} \leq \|\Omega\| ,$$

for all p, q and x, and $G(x,\Omega)$ will depend continuously on the pair (x,Ω), in the sense of any p-q operator norm.

Proof: Let

$$\tau(x) \equiv \pm \int_x^{\pm\infty} |\Omega'(t)|\, dt , \qquad \pm x \geq 0 .$$

Then

$$|\Omega(x)| \leq \tau(x) ,$$

so that

$$\|\Omega\|_\infty \leq \|\tau\|_\infty \leq \|\Omega'\|_1 ,$$

and

$$\|\Omega\|_1 = \int_{-\infty}^0 dx \left| \int_{-\infty}^x \Omega'(t)\, dt\right| + \int_0^\infty dx \left|- \int_x^\infty \Omega'(t)\, dt\right|$$

$$\leq \int_{-\infty}^0 dx \int_{-\infty}^x |\Omega'(t)|\, dt + \int_0^\infty dx \int_x^\infty |\Omega'(t)|\, dt = \|\tau\|_1$$

$$\leq \|\Omega'\|_{(1)} .$$

Therefore

$$\|\psi_x\|_\infty \leq \|\phi\|_\infty \|C_{x+y}\Omega\|_1 \leq \|\Omega'\|_{(1)} \|\phi\|_\infty ,$$

where C_x is the characteristic function of $[x,\infty)$. Also

$$\|\psi_x\|_\infty \leq \|C_{x+y}\, \Omega\|_\infty \|\phi\|_1 \leq \|\Omega'\|_1 \|\phi\|_1 ,$$

and

$$\|\psi_x\|_1 \leq \int_0^\infty dy \int_0^\infty |\phi(t)|\ |\Omega(x+t+y)|\, dt$$

$$\leq \int_0 |\phi(t)|\, dt \int_{x+t}^\infty |\Omega(y)|\, dy \leq \|\Omega'\|_{(1)} \|\phi\|_1 .$$

Thus, $G(x,\Omega)$ is bounded by $\|\Omega\|$ as an operator from L^p to L^q, for $p \le q$, $p,q \in \{1,\infty\}$. By the Marcel Riesz Interpolation Theorem, the same holds for $p \le q$, $p,q \in [1,\infty]$.

Let us show now that

$$\lim_{h\to 0} \|G(x+h,\Omega) - G(x,\Omega)\|_{p,q} = 0 :$$

$$\|\psi_{x+h} - \psi_x\|_\infty \le \|\phi\|_\infty \int_0^\infty \left|\left\{C_{x+y+h}(t) - C_{x+y}(t)\right\}\Omega(t)\right| \, dt$$

$$= \int_y^\infty \left|\left\{C_{x+h}(t) - C_x(t)\right\}\Omega(t)\right| \, dt$$

$$\le \int_0^\infty \left|\left\{C_{x+h}(t) - C_x(t)\right\}\Omega(t)\right| \, dt \to 0$$

by the Lebesgue Dominated Convergence Theorem. For $p = 1$,

$$\|\psi_{x+h} - \psi_x\|_1 \le \int_0^\infty |\phi(t)| dt \int_{x+t}^\infty |\Omega(y+h) - \Omega(y)| \, dy$$

$$= \|\phi\|_1 \int_{-\infty}^\infty |\Omega(y+h) - \Omega(y)| \, dy \to 0 ,$$

again by the LDC Theorem. A further application of the MRI Theorem gives us the result for any $p = q \in [1,\infty]$.

Now, $G(x,\Omega)$ is linear in Ω. By the estimate (4.6) above, it is a uniformly continuous function of Ω. Since it is a continuous function of x, for fixed Ω, we conclude that it is a continuous function of the pair (x,Ω).

The compactness of $G(x,\Omega)$ for $q < \infty$ is a consequence of G being the uniform limit of integral operators on finite intervals:

$$\lim_{M\to\infty} \| G(x,\Omega) - G(x,(1-C_M)\Omega)\|_{p,q} = 0 .$$

In the case of $L^\infty(\mathbb{R}^+)$, we observe that for $\|\phi\|_\infty \le 1$

$$\|\psi_x'\|_\infty \le \|\Omega'\|_1 \le \|\Omega\|$$

and that

$$|\psi_x(y)| \le \int_y^\infty |\Omega(t)|\, dt \to 0 , \quad \text{as } y \to \infty ,$$

uniformly with respect to ϕ.

Claim 5. Consider the family of integral equations

(4.7) $$B(x,y) + \int_0^\infty \Omega(x+t+y)\, B(x,t)\, dt + \Omega(x+y) = 0 , \quad y \ge 0,$$

for real x. Assume that Ω is differentiable,

$$\Omega(x) = o(x^{-2}) , \qquad x \to +\infty ,$$

and

$$\|\Omega'\|_{(1)} + \|\Omega'\|_\infty < \infty .$$

Then:

(a) For each fixed x, there exists a unique solution $B(x,\cdot)$ of (4.7); this solution belongs to the Sobolev spaces $W^{1,p}(\mathbb{R}^+)$ for $1 \le p \le \infty$, and vanishes at $y = +\infty$.

(b) If Ω' is continuous so is $\partial_1 B(x,y)$.

(c) Let B_α be the restriction of B to $\{x \ge \alpha,\ y \ge 0\}$; then the mapping

$$H: \Omega \longrightarrow B_\alpha$$

is continuous with respect to the norms

$$\|\Omega\| \equiv \|\Omega'\|_1 + \|\Omega'\|_{(1)}$$

$$\|B_\alpha\| \equiv \sup_{x\geq\alpha} \|B(x,\cdot)\|_{1,p} \quad .$$

Proof: The operators $I + G(x,\Omega)$ are strictly positive in L^2 as is shown in Section 6. Now, assume that $\phi_x \in L^p$, for $p < \infty$ and

(4.8) $$\phi_x = - G(x,\Omega)\phi_x \quad .$$

By (4.6), $\phi_x \in L^\infty$. But if $\phi_x \in L^\infty$ satisfies (4.8), we can conclude that $\phi_x \in L^2$ and thus $\phi_x \equiv 0$.

To prove this last assertion, we only have to observe that for any $\phi \in L^\infty$,

$$|y\ \psi_x(y)| = |y \int_0^\infty \Omega(x+t+y)\phi(t)\ dt|$$

$$\leq \|\phi\|_\infty \left\{ |y+x| \int_{y+x}^\infty |\Omega(t)|\ dt + |x| \int_{y+x}^\infty |\Omega(t)|\ dt \right\}$$

and this last quantity tends to 0 as $y \to +\infty$.

Now we claim that if

$$F(x,\Omega) \equiv \{I + G(x,\Omega)\}^{-1}$$

then

$$\lim_{\Omega_1\to\Omega} \left\{ \sup_{x\geq\alpha} \|F(x,\Omega_1) - F(x,\Omega)\|_p \right\} = 0 \quad .$$

This is a consequence of the continuity of $G(x,\Omega)$ as a function of the pair (x,Ω) and its asymptotic behavior as $x \to \infty$. Denote

$$(T_x\Omega)(y) \equiv \Omega(x+y) \quad ;$$

then

$$B(x,\cdot) = -F(x,\Omega)\ T_x\Omega$$

and therefore

$$\lim_{\Omega_1\to\Omega} \sup_{x\geq 0} \|F(x,\Omega_1)T_x\Omega_1 - F(x,\Omega)T_x\Omega\|_p$$

$$\leq \lim_{\Omega_1\to\Omega} \Big\{ \sup_{x\geq 0} \|F(x,\Omega_1)\|_p \|T_x(\Omega_1-\Omega)\|_p$$

$$+ \sup_{x\geq 0} \|F(x,\Omega_1) - F(x,\Omega)\|_p \|T_x\Omega\|_p \Big\}$$

$$\leq \lim_{\Omega_1\to\Omega} \Big\{ (\|\Omega\|+1)\|\Omega_1-\Omega\| + \|\Omega\|_p \sup_{x\geq 0} \|F(x,\Omega)-F(x,\Omega_1)\|_p \Big\} = 0 \ .$$

By differentiating (4.7) we get

$$\partial_1 B(x,y) + \int_0^\infty \{\Omega'(x+t+y)B(x,t)+\Omega(x+t+y)\ \partial_1 B(x,t)\}\,dt$$

$$+ \Omega'(x+y) = 0$$

or equivalently

$$\partial_1 B(x,y) = -F(x,\Omega) \Big\{ T_x\Omega' + G(x,\Omega')\ B(x,\cdot) \Big\}(y) \ .$$

By using the estimates in Claim 4, we get that

$$G(x,\Omega')B(x,\cdot) \in L^p$$

and depends continuously on Ω, in the sense of (4.5), as does $T_x\Omega'$. This completes the proof.

5. A Numerical Approach

To solve numerically the one-dimensional inverse scattering problem we have to:

(a) solve either the Fredholm integral equations $(3.4)_+$ or $(3.4)_-$ for $B_{\pm}(x,y)$ as functions of y, x being taken as a parameter;

(b) differentiate $B_{\pm}(x,0)$ to obtain q(x) from formula (3.8),

$$q(x) = \mp \partial_1 B(x,0).$$

Observe that approximately solving either of equations (3.4) by the Nyström quadrature technique amounts to solving a finite linear system whose order depends on both the accuracy we require and the behavior of the functions $\Omega_{\pm}$. Also we need to solve these equations for a large number of values of x.

In this section we shall describe a simpler algorithm for this problem, using an idea originally suggested by V. Bargmann and carried out by I. Kay, Cf. [10].

From now on we shall deal with Marchenko equations $(3.5)_+$ replacing B_+ by the function K introduced in (3.3a) as

$$K(x,y) \equiv B_+(x, \frac{y-x}{2})/2 ,$$

and taking instead of Ω_+ the function

(5.1) $$\omega(t) \equiv \Omega_+(t/2)/2 \ .$$

With these changes of variables, relations $(3.1)_+$, $(3.5)_+$ and $(3.8)_+$ become:

(5.2) $$\omega(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r^+(k)\, e^{ikt}\, dk \ ,$$

(5.3) $$K(x,y) + \int_x^{\infty} \omega(t+y)\, K(x,t)\, dt + \omega(x+y) = 0 \ , \qquad x \leq y \ ,$$

and

(5.4) $$q(x) = -\frac{1}{2} \frac{d}{dx} K(x,x) \ ,$$

respectively.

The basis for the algorithm is the observation that when the reflection coefficient r^- is a rational function of k with all its poles in the lower half plane, an explicit formula for q(x) can be obtained. Observe that by Theorem 4 in Section 6, the analyticity of $r^-(k)$ for Im k > 0 implies that q vanishes on the negative axis, if no eigenvalues exist.

Since r^- is rational and analytic for Im k > 0, it has the form

(5.5) $$r^-(k) \equiv r_0 \frac{\prod^M (k - \mu_j)}{\prod^L (k - \delta_j)} \ ,$$

with M < L, and $\text{Im}\ \delta_j < 0$. By (2.9), (2.10a) and (2.10b) the relation

$$t(k)t(-k) = 1 - r^-(k)r^-(-k)$$

holds on the real axis. Then denoting the roots of

$$1 - r^-(k)r^-(-k) = 0$$

located in the upper half plane by ρ_j and assuming that $|r(0)| < 1$, we get

$$(5.6) \qquad t(k)t(-k) = \prod^{L} \frac{(k - \rho_j)}{(k - \delta_j)} \frac{(k + \rho_j)}{(k + \delta_j)}$$

for real k.

If there are no eigenvalues, t(k) is a non-vanishing analytic function in the upper half-plane; as a consequence of (5.6) t must have the form

$$t(k) = \prod^{L} \frac{k + \rho_j}{k - \delta_j} \ .$$

From (2.9) it follows that on the real axis

$$r^+(k) = -r^-(-k)t(k)/t(-k) \ ,$$

so that

$$r^+(k) = (-1)^{L+M+1} \ r_0 \ \frac{\prod^{M} (k+\mu_j) \prod^{L} (k+\rho_j)}{\prod^{L} (k-\delta_j) \prod^{L} (k-\rho_j)}$$

Consequently

$$(5.7) \qquad r^+(k) = p(k) + \sum \tilde{\rho}_j \ / \ (k-\rho_j)$$

where p(k) is analytic in the upper half-plane.

Substitution of (5.7) into (5.2) gives

$$(5.8) \qquad \dot{\omega}(t) = i \sum^{L} \tilde{\rho}_j \ e^{i\rho_j t} \ , \qquad t \geq 0 \ .$$

Using this relation in (5.3) we get that for $0 \leq x \leq y$

$$(5.9)\quad K(x,y) + i \sum^{L} \tilde{\rho}_j \left\{ \int_x^{\infty} e^{i\rho_j t} K(x,t)\, dt \right\} e^{i\rho_j y} + i \sum^{L} \left\{ \tilde{\rho}_j\, e^{i\rho_j x} \right\} e^{i\rho_j y} = 0 .$$

Therefore, in the range $0 \leq x \leq y$, K has the form

$$(5.10)\qquad K(x,y) = \sum^{L} f_j(x)\, e^{i\rho_j y} .$$

and substituting (5.10) into (5.9) we get the L×L system

$$(5.11)\quad f_j(x) - \tilde{\rho}_j \sum_m^{L} \frac{e^{i(\rho_j+\rho_m)x}}{\rho_j+\rho_m} f_m(x) = - i\tilde{\rho}_j\, e^{i\rho_j x} .$$

By using Cramer's rule and (5.4), one obtains from (5.11) the expresssion

$$(5.12)\quad q(x) = -2 \frac{d^2}{dx^2} \log \det [I - A(x)],$$

where $A = (a_{jm})$ is the matrix that appears in the system (5.11), namely

$$a_{jm}(x) \equiv \tilde{\rho}_j \frac{e^{i(\rho_j+\rho_m)x}}{\rho_j+\rho_m} .$$

Observe that the order of this matrix equals the number of poles of $r^-(k)$.

Although (5.12) is theoretically simpler than (5.11), the latter formula has more computational interest than the former. This can be seen as follows: Setting $y = x$ in (5.10), and differentiating, we have

$$(5.13)\quad \frac{d}{dx} K(x,x) = \sum \left\{ f_j'(x) + i\rho_j \right\} e^{i\rho_j x},$$

for $x \geq 0$. To compute the values of $f_j'(x)$ we differentiate (5.3) and (5.10) with respect to x and obtain

$$(5.14)\quad \partial_1 K(x,y) + \int_x^\infty \omega(t+y)\, \partial_1 K(x,t) - \omega(x+y) K(x,x) + \omega'(x+y) = 0,$$

and

$$\partial_1 K(x,y) = \sum f_j'(x)\, e^{i\rho_j y}.$$

If we now use this last expression and (5.10) in (5.14) we get the following system for $f_j'(x)$:

$$(5.15)\quad f_j'(x) - \tilde{\rho}_j \sum_m \frac{e^{i(\rho_j+\rho_m)x}}{\rho_j+\rho_m} f_m'(x) = \tilde{\rho}_j\, e^{i\rho_j x} \left\{ \rho_j + i \sum_m f_m(x)\, e^{i\rho_m y} \right\}.$$

Notice that the coefficient matrix in (5.15) is again $I - A(x)$, so that to solve this system after having solved (5.11) is a computationally cheap task. Moreover it avoids having to numerically carry out the differentiation in (5.4).

We observe that passing from r^- to r^+ makes it possible to obtain (5:12) and also makes the derivation of (5.11) quite simple. Nevertheless, a system analogous to (5.11) can be obtained by making use of r^- only, Cf. [13].

We have solved numerically the inverse problem by implementing both (5.3) and (5.11), for some rational coefficients r^-. In dealing with Marchenko equations directly, we discretized (5.3) by using Simpson's formula and we needed a 60-point mesh to obtain an accuracy of 10^{-5} in the average. The second method, even for $L = 8$, was ten times faster than the first.

As it stands, we can use the second method only for rational reflection coefficients. When solving the inverse problem for a reflection coefficient r^- which is analytic in the upper half-plane, and if there are no eigenvalues, we can use the following numerical method:

(a) approximate r^- by a rational reflection coefficient r_ε, which is analytic in the upper half plane,

(b) solve the inverse problem for r_ε by using the algorithm described in (5.4), (5.11), (5.13) and (5.15).

Theorem 3 in Section 3 gives us the conditions on the approximation r_ε under which we can expect the potential q_ε to be close to the potential q we are seeking. The main difficulty is that the common techniques for approximating a given function by a rational one, e.g. the Rehmes algorithm, can be applied only for real functions, while the requirement that all poles of r_ε lie in the lower half plane prevents us from approximating the real and the imaginary parts of r separately.

The following is a possible strategy for solving the approximation problem:

Restricting ourselves to reflection coefficients r that die out like

$$(5.16) \qquad r(k) = o(|k|^{-1}) , \qquad |k| \sim \infty ,$$

define

$$R(k) \equiv (k+i)\, r(k)$$

and, for $\omega = e^{i\theta}$,

$$s(\omega) \equiv R\left(\frac{1}{i}\,\frac{\omega + 1}{\omega - 1}\right) .$$

By (5.16) s is continuous at $\omega = 1$. Obtain trigonometric approximations

$$s(\omega) \sim \sum a_n \omega^n = \sum a_n\, e^{in\theta}$$

for s and define

$$r_\varepsilon(k) \equiv \frac{1}{k + i} \sum a_n \left(\frac{k - i}{k + i}\right)^n$$

as the approximations we sought.

Better approximation results can probably be achieved if instead of defining r_ε with an n-th order pole, we could get its poles spread out in the lower half-plane.

6. Proofs of Some Fundamental Results

1. Existence of the Solutions $\phi_{\pm}$; Derivation of Eq. (3.4).

We recall the variation of parameters formula:

If z(t) satisfies

$$z' = A(t)z \qquad (6.1)$$
$$z(t_0) = \xi$$

and Z is a fundamental solution of (6.1), then

$$v(t) \equiv z(t) + \int_{t_0}^{t} Z(t)\, Z^{-1}(s)\, b(s)\, ds \qquad (6.2)$$

satisfies

$$v' = A(t)v + b(t)$$
$$v(t_0) = \xi$$

Here, z, ξ and b are n-vectors, while A is an n×n matrix.

Motivated by this result, which is valid for finite t_0, we write the same formula now for $t_0 = \pm\infty$, taking for (6.1) and (6.3) the equations

$$y'' = -k^2 y$$

and

$$y'' = -k^2 y + qy \qquad (6.4)$$

respectively. We impose the conditions

$$y(x,k) \sim e^{\pm ikx}, \qquad x \sim \pm\infty \qquad (6.5)$$

and regard the term qy as if it were the inhomogeneous term b(t). Instead of (6.2) we obtain the following integral equations for $\phi_{\pm}$:

$$(6.6)_{\pm} \quad \phi_{\pm}(x,k) = e^{\pm ikx} + \int_x^{\pm\infty} \frac{\sin k(t-x)}{k} q(t)\, \phi_{\pm}(t,k)\, dt$$

The proof of the existence of a unique solution for each of these equations is obtained by the successive approximations technique, cf. [1]; this proof makes it clear that these functions are defined and analytic in the upper half-plane Im k > 0, and have the asymptotic behavior (6.5) for large x. Moreover

$$|\phi_{\pm}(x,k) - e^{\pm ikx}| \leq \frac{A|e^{\mp ikx}|}{1+|k|}$$

so that

$$\left\{\phi_{\pm}(x,k) - e^{\pm ikx}\right\} e^{\mp ikx} = O(1/|k|) , \qquad |k| \sim \infty.$$

It is thus a consequence of the Paley-Wiener theorem that

$$\left\{\phi_{\pm}(x,k) - e^{\pm ikx}\right\} e^{\mp ikx} = \pm \int_0^{\pm\infty} B_{\pm}(x,y)\, e^{\pm 2iky}\, dy$$

or

$$(6.7)_{\pm} \quad \phi_{\pm}(x,k) = e^{\pm ikx} \pm e^{\pm ikx} \int_0^{\pm\infty} B_{\pm}(x,y)\, e^{\pm 2iky}\, dy .$$

By substituting $(6.7)_{+}$ into $(6.6)_{+}$ we get

$$(6.8) \quad \int_0^{\infty} B_{+}(x,y)\, e^{2iky}\, dy = e^{-ikx} \int_x^{\infty} \frac{e^{ik(t-x)} - e^{-ik(t-x)}}{2ik}\, q(t) e^{ikt} a(t)\, dt,$$

where

$$a(t) \equiv 1 + \int_0^\infty B_+(t,z)\, e^{2ikz}\, dz .$$

The right-hand side of (6.8) is thus the sum of two terms:

$$\int_x^\infty \frac{e^{2ik(t-x)}-1}{2ik}\, q(t)\,dt + \int_x^\infty \frac{e^{2ik(t-x)}-1}{2ik}\, q(t) \int_0^\infty B_+(t,z) e^{2ikz}\, dz\, dt.$$

We integrate both terms by parts. From the first we get

$$\int_x^\infty \frac{e^{2ik(t-x)}-1}{2ik}\, q(t)\, dt = \int_x^\infty e^{2ik(t-x)} \int_t^\infty q(s)\, ds$$

$$= \int_0^\infty e^{2iky}\, dy \int_{x+y}^\infty q(s)\, ds ,$$

while the second gives:

$$\int_x^\infty \frac{e^{2ik(t-x)}-1}{2ik}\, q(t) \int_0^\infty B_+(t,z)\, e^{2ikz}\, dz\, dt$$

$$= \int_x^\infty e^{2ik(t-x)}\, dt \int_t^\infty q(s) \int_0^\infty B_+(s,z)\, e^{2ikz}\, dz\, ds$$

$$= \int_0^\infty dz \int_x^\infty e^{2ik(t+z-x)}\, dt \int_t^\infty q(s)\, B_+(s,z)\, ds$$

$$= \int_0^\infty dz \int_z^\infty e^{2iky}\, dy \int_{x+y-z}^\infty q(s)\, B_+(s,z)\, ds$$

$$= \int_0^\infty e^{2iky}\, dy \int_0^y dz \int_{x+y-z}^\infty B_+(t,z)\, dt .$$

Therefore (6.8) may be rewritten as

$$0 = \int_{-\infty}^{\infty} e^{2iky}\, C_+(y)\{B_+(x,y) - \int_{x+y}^{\infty} q(t)\,dt - \int_0^y dz \int_{x+y-z}^{\infty} q(t)B_+(t,z)\,dt\}dy,$$

where C_+ denotes the characteristic function of $\mathbb{R}^+$. This gives us $(3.4)_+$; $(3.4)_-$ may be obtained in a similar way.

2. Existence of the Solutions $\psi_{\pm}$. Derivation of Marchenko Equations (3.5).

Consider for Im $k \leq 0$ the functions

$$(6.9)_{\pm} \qquad \theta_{\pm}(x,k) \equiv \phi_{\pm}(x,-k) \ .$$

They satisfy

$$\theta_{\pm}(x,k) \sim e^{\mp ikx}, \qquad x \sim \pm\infty,$$

and are analytic for Im $k < 0$. Since for real $k \neq 0$,

$$W[\phi_+,\theta_+] = W[\theta_-,\phi_-] = -\,2ik \neq 0$$

both $\{\phi_+,\theta_+\}$ and $\{\theta_-,\phi_-\}$ are bases for the space of solutions of (6.4). Any solution of (6.4) may thus be written as

$$y = A_+\phi_+ + B_+\theta_+$$

or

$$y = A_-\theta_- + B_-\phi_- \ .$$

If

$$(6.10) \qquad \begin{aligned} \phi_+ &= \alpha\theta_- + \beta\phi_- \\ \theta_+ &= \gamma\theta_- + \delta\theta_- \ , \end{aligned}$$

Then

$$M \equiv \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

is the matrix of change of basis:

$$M\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \begin{pmatrix} A_- \\ B_- \end{pmatrix}$$

Since $\theta_{\pm} = \overline{\phi}_{\pm}$, we must have

$$\delta = \bar{\alpha} , \quad \gamma = \bar{\beta} ,$$

and this implies that

$$\det M = |\alpha|^2 - |\beta|^2 = 1 .$$

Indeed:

$$-2ik = W[\phi_+,\theta_+] = W[\alpha\theta_- + \beta\phi_-, \bar{\beta}\theta_- + \bar{\alpha}\phi_-]$$

$$= W[\alpha\theta_-, \bar{\alpha}\phi_-] - W[\bar{\beta}\theta_-,\beta\phi_-]$$

$$= (|\alpha|^2 - |\beta|^2)(-2ik) .$$

As a consequence, we have the existence of the unit incoming waves

$(6.11)_{\pm}$ $$\psi_{\pm}(x,k) \equiv \frac{1}{\alpha(k)} \phi_{\mp}(x,k)$$

for real k. By $(6.9)_-$ and (6.10) we have that

$(6.12)_-$ $$\psi_-(x,k) = \phi_-(x,-k) + \frac{\beta(k)}{\alpha(k)} \phi_-(x,k)$$

and since

$$M^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} ,$$

$$(6.12)_+ \qquad \psi_+(x,k) = \phi_+(x,-k) - \frac{\bar{\beta}(k)}{\alpha(k)}\, \phi_+(x,k) \ .$$

The existence of the S-matrix is a consequence of the fact that the solutions ψ_+ and ψ_- are independent. From (2.5), (6.11) and (6.12) we get:

$$(6.13) \qquad t(k) = 1/\alpha(k) \ ,$$

$$(6.14)_- \qquad r_-(k) = \beta(k)/\alpha(k) \ ,$$

$$(6.14)_+ \qquad r_+(k) = -\bar{\beta}(k)/\alpha(k) \ .$$

Some properties of the S-matrix may be deduced directly from these relations, but we will not carry over the proofs.

The incoming unit waves $\psi_{\pm}$ are meromorphic in the upper half plane with poles at $k = i\kappa_j$ and grow like $e^{\mp ikx}$. To derive the Marchenko equations, we introduce the functions

$$(6.15)_{\pm} \qquad g_{\pm}(x,k) \equiv \psi_{\pm}(x,k)\ e^{\pm ikx}$$

and the Fourier transforms

$$(6.16)_+ \qquad \tilde{B}(x,y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} [g_+(x,k)-1]\ e^{-2iky}\ dk$$

$$(6.16)_- \qquad \hat{B}(x,y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} [g_-(x,k)-1]\ e^{2iky}\ dk \ ,$$

so that

$$(6.17)_+ \quad g_+(x,k) = 1 + \int_{-\infty}^{\infty} \tilde{B}(x,y)\, e^{2iky}\, dy ,$$

$$(6.17)_- \quad g_-(x,k) = 1 + \int_{-\infty}^{\infty} \hat{B}(x,y)\, e^{-2iky} dy ,$$

From $(6.15)_+$, $(6.12)_+$ and $(6.9)_-$

$$g_+(x,k) = e^{ikx}\, \phi_+(x,-k) + r_+(k)\, e^{2ikx}\{\phi_+(x,k)\, e^{-ikx}\} ,$$

and thus by $(6.17)_+$, $(3.3)_+$ and $(3.1)_+$,

$$1 + \int_{-\infty}^{\infty} \tilde{B}(x,-y)\, e^{-2iky}\, dy = 1 + \int_{-\infty}^{\infty} B_+(x,y)\, e^{-2iky}\, dy$$

$$+ \int_{-\infty}^{\infty} F_+(y+x)\, e^{-2iky}\, dy \left\{1 + \int_{-\infty}^{\infty} B_+(x,-y)\, e^{-2iky}\, dy\right\},$$

or, by using the convolution theorem

$$\int_{-\infty}^{\infty} \tilde{B}(x,-y)\, e^{-2iky}\, dy = \int_{-\infty}^{\infty} B_+(x,y)\, e^{-2iky}\, dy + \int_{-\infty}^{\infty} F_+(x+y) e^{-2iky} dy$$

$$+ \int_{-\infty}^{\infty} e^{-2iky}\, dy \int_{-\infty}^{\infty} B_+(x,-(y-t))\, F_+(x+t)\, dt .$$

This gives:

$$(6.18) \quad \tilde{B}(x,-y) = B_+(x,y) + F_+(x+y) + \int_{0}^{\infty} B_+(x,z)\, F_+(x+y+z)\, dz ,$$

since $B_+(x,y) = 0$ for $y < 0$. Now for $y > 0$,

$$\tilde{B}(x,-y) = 2i \sum_{\operatorname{Im} k > 0} \operatorname{Res}\left\{g_+(x,k)\, e^{2iky}\right\} .$$

Since by $(6.15)_+$, $(6.11)_+$ and (6.13)

$$g_+(x,k) = e^{ikx}\, t(k)\, \phi_-(x,k) \;,$$

we have

$$\tilde{B}(x,-y) = 2i \sum_j e^{-\kappa_j x} e^{-2\kappa_j y} \phi_-(x,i\kappa_j) \operatorname{Res} t(k)\Big|_{k=i\kappa_j}$$

It can be shown, Cf. [6], that

$$\operatorname{Res} t(k)\Big|_{k=i\kappa_j} = i\left\{ \int_{-\infty}^{\infty} \phi_+(x,i\kappa_j)\, \phi_-(x,i\kappa_j)\, dx\right\}^{-1} .$$

Setting $\phi_-(x,i\kappa_j) = c_j\phi_+(x,i\kappa_j)$, we obtain for $y > 0$

$$\tilde{B}(x,-y) = -2 \sum_j e^{-2\kappa_j(x+y)} \left\{e^{\kappa_j x} \phi_+(x,i\kappa_j)\right\} \Big/ \int_{-\infty}^{\infty} |\phi_+(x,i\kappa_j)|^2 dx,$$

where we have used the fact that the eigenfunctions of E are real. Recalling the definition of the normalizing constants m_j^+ given in Section 2, and using $(3.3)_+$ we get from (6.18):

$$-2 \sum_j m_j^+ e^{-2\kappa_j(x+y)} \left(1 + \int_0^{\infty} B_+(x,z)\, e^{-2i\kappa_j z}\, dz\right)$$

$$= B_+(x,y) + F_+(x+y) + \int_0^{\infty} B_+(x,z)\, F_+(x+y+z)\, dz \;,$$

or

$$0 = B_+(x+y) + \left\{F_+(x+y) + 2 \sum_j m_j^+ e^{-2\kappa_j(x+y)}\right\}$$

$$+ \int_0^{\infty} B_+(x,z) \left\{F_+(x+y+z) + 2 \sum_j m_j^+ e^{-2\kappa_j(x+y+z)}\right\} dz \;.$$

From this equation, by defining Ω_+ as in $(3.6)_+$, we get precisely $(3.5)_+$, and $(3.5)_-$ may be obtained in the same way. The one-dimensional version of the Marchenko equations first appeared in [9] and [11].

3. Positiveness of the Marchenko Operators

We consider the Marchenko operators M_x as defined by

$$M_x: h \to h(y) + \int_0^\infty \Omega(x+t+y)\, h(t)\, dt ,$$

for $h \in L^2(\mathbb{R}^+)$ and Ω as given by (3.6) and (3.1).

Suppose that

$$(M_x h, h) = 0 . \tag{6.19}$$

Since Ω is real, there is no loss of generality in assuming that h is real. Thus (6.19) is equivalent to

$$0 = \int_0^\infty h^2(y)\, dy + \int_0^\infty h(y)\, dy \int_0^\infty \left\{ \frac{1}{\pi} \int_{-\infty}^\infty r(k) e^{-2ik(x+y+t)} dk + 2 \sum m_j^+ e^{-2\kappa_j(x+y+t)} \right\} h(t)\, dt . \tag{6.20}$$

Now take the Fourier transform of h:

$$H(k) \equiv \int_{-\infty}^\infty h(y)\, e^{2iky}\, dy ,$$

where we extended $h \equiv 0$ for negative y. Notice that H is a bounded holomorphic function on the upper half plane.

Using Parseval's identity in the first term of (6.20), and changing the order of integration in the second term, we get

$$0 = \frac{1}{\pi} \int_{-\infty}^{\infty} |\psi(k)|^2 \, dk + \frac{1}{\pi} \int_{-\infty}^{\infty} r(k) \, e^{-2ikx} \, dk \int_0^{\infty} h(y) \, e^{-2iky} \, dy$$

$$\cdot \int_0^{\infty} h(t) \, e^{-2ikt} \, dt + 2 \sum m_j^+ \, e^{-2\kappa_j x} \int_0^{\infty} h(y) \, e^{-2\kappa_j y} dy$$

$$\cdot \int_0^{\infty} h(t) \, e^{-2\kappa_j t} \, dt \, ,$$

or using $H(-k) = \overline{H(k)}$,

$$(6.21) \quad 0 = \int_{-\infty}^{\infty} [H(k)H(-k) + r(k)e^{-2ikx} \, H^2(-k)] \, dk$$

$$+ 2\pi \sum m_j^+ \, e^{-2\kappa_j x} \, H^2(i\kappa_j) \, .$$

Since h is real, so are the numbers $H(i\kappa_j)$, and therefore the summand in (6.21) is nonnegative. The integral in (6.21) is also a nonnegative number, as can be seen from the identity

$$\frac{1}{2} \int_{-\infty}^{\infty} [H(k) + r(k) \, e^{-2ikx} \, H(-k)] \, [H(-k)+r(-k)e^{2ikx} \, H(k)]$$

$$+ [|t(k)|H(-k)] \, [|t(-k)| \, H(k)] \, dk$$

$$= \int_{-\infty}^{\infty} [H(k)H(-k) + r(k)e^{-2ikx} \, H^2(-k)] \, dk \, ,$$

where we have used that $|t|^2 + |r|^2 = 1$. The integrand on the left-hand side is of the form $f(k)\overline{f(k)} + g(k)\overline{g(k)}$ and by (6.21) it must vanish. But since $t(k) \sim 1$ for $k \sim \pm \infty$, this implies that H must vanish outside a finite interval. Therefore, its Fourier transform is an entire function and since

$$h(y) \equiv 0 \quad \text{for} \quad y < 0 ,$$

h must be identically zero. This gives the positiveness of M_x.

4. Finiteness of the Point Spectrum of E

Let us prove now that E has a finite point spectrum if

$$\int_{-\infty}^{\infty} (1 + |x|)\ |q(x)|\ dx < \infty .$$

We will show this by proving that for $y(x) \not\equiv 0$ in a subspace of finite codimension contained in the domain of E,

$$(Ey,y) > 0.$$

Consequently, the eigenfunctions of E can at most fill up a subspace of finite dimension, since we know that all eigenvalues of E are negative.

Integration by parts gives

$$(6.22) \qquad (Ey,y) = \int_{-\infty}^{\infty} |y'|^2 + \int_{-\infty}^{\infty} q\ |y|^2 .$$

We need to bound the second integral in terms of the first one.

Observe that if $y(\bar{x}) = 0$, Schwartz' inequality gives for $x>\bar{x}$:

$$|y(x)|^2 \le |x-\bar{x}| \int_{\bar{x}}^{x} |y'(s)|^2 \, ds .$$

Let $y(x)$ vanish at the equally spaced points

$$x_N = R , \; x_{N-1} = R-h , \; \ldots , \; x_{-N} = -R ,$$

with R and N to be determined. Then

$$\int_{-\infty}^{\infty} |y^2 q| = \left\{ \int_{-\infty}^{-R} + \int_{R}^{\infty} + \sum_{j=-N}^{N-1} \int_{x_j}^{x_{j+1}} \right\} |y^2 q|$$

$$\le -\int_{-\infty}^{-R} |(x+R)\; q(x)| \, dx \int_{-R}^{x} |y'|^2 \, dt + \int_{R}^{\infty} |(x-R)\; q(x)| \, dx \int_{R}^{x} |y'|^2 dt$$

$$+ \sum \int_{x_j}^{x_{j+1}} |(x-x_j)\; q(x)| \, dx \int_{x_j}^{x} |y'|^2 \, dt$$

$$\le \int_{-\infty}^{\infty} |y'|^2 \, dt \left\{ \int_{|x|>R} |x\, q(x)| \, dx + h \int_{|x|<R} |q(x)| \, dx \right\} .$$

To make the bracketed quantity less than 1, and thus (6.22) positive, we have only to choose convenient values for R and $N \equiv R/h$. For example, require that R and N satisfy:

$$\int_{|x|\ge R} |x\, q(x)| \, dx < \tfrac{1}{2}$$

and

$$N > 2R\|q\|_1 .$$

5. Relation Between Properties of q and r

Equations (3.4), (3.5) and (3.7) permit us to establish a relationship between some properties of the potential q and those of the reflection coefficient r^+, more exactly, of the derivative of its Fourier transform F_+'. The proof is based on the following inequalities that hold for $x \geq 0$, Cf. [6]:

$$|\Omega_+'(x) - q(x)| \leq A\,\xi^2(x)$$

and

$$|\Omega_+'(x) - q(x)| \leq A\,\tau^2(x)\ ,$$

where

$$\xi(x) \equiv \int_x^\infty |q(s)|\,ds\ ,$$

$$\tau(x) \equiv \int_x^\infty |\Omega_+'(s)|\,ds\ .$$

Since

$$\Omega_+(t) \equiv F_+(t) + 2\sum m_j\, e^{-\kappa_j t}\ ,$$

these inequalities imply

Theorem 4. The potential q(x) satisfies

$$q(x) = O(x^{-n})\ , \qquad x \sim +\infty$$

if and only if

$$F_+'(t) = O(t^{-n})\ , \qquad t \sim +\infty.$$

If E has no point spectrum, then

$$q(x) = 0 \qquad \text{for} \quad x > A$$

if and only if

$$F_+(t) = 0 \quad \text{for} \quad t > A.$$

There exist analogous relations between the behavior of $q(x)$ and $F_-'(t)$ at $-\infty$, between the derivatives $q^{(j)}(x)$ and $F^{(j+1)}(t)$, and a similar result holds for the radial problem.

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