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String-localized Quantum Fields and Modular Localization

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Abstract

We study free, covariant, quantum (Bose) fields that are associated with irreducible representations of the Poincaré group and localized in semi-infinite strings extending to spacelike infinity. Among these are fields that generate the irreducible representations of mass zero and infinite spin that are known to be incompatible with point-like localized fields. For the massive representation and the massless representations of finite helicity, all string-localized free fields can be written as an integral, along the string, of point-localized tensor or spinor fields. As a special case we discuss the string-localized vector fields associated with the point-like electromagnetic field and their relation to the axial gauge condition in the usual setting.

Contents

1 Introduction

In their paper [8], Brunetti, Guido and Longo (BGL) established a general connection between positive energy representations of the Poincaré group and localization properties of state vectors in the Hilbert space of the representation. This *modular localization* is not associated with any position operators, which are known to be problematic in the relativistic case, but rather with the Lorentz boosts corresponding to wedge-like regions in Minkowski space and the PCT operator. Using these concepts, the authors of [8] show that every irreducible, positive energy representation of the Poincaré group admits dense sets of vectors that are naturally localized in space-like cones in Minkowski space with arbitrarily small opening angles.

It is well known that in the case of the irreducible representations of finite spin or helicity this localization can be sharpened to double cone localization, by making use of the concrete realization of the representation in the one particle space of a covariant Wightman field. The BGL concept, however, applies also to the Wigner representations of zero mass and infinite spin, where a localization in the sense of point-like fields is not possible [69]. This latter result excludes compact localization in the sense of Wightman fields, even with infinitely many components, and applies also to the special constructions in [1, 30, 31]. We note in passing that these representations have recently found applications in work on 'tensionless strings' in String Theory [42, 54].

The localization spaces for space-like cones of [8] are abstractly defined in terms of intersections of wedge-localized spaces without concrete formulas for their generation. In a previous letter [47] we showed that the spaces for the zero mass and infinite spin representations can be explicitly described in terms of *string-localized fields.* The strings can be depicted as the cores of the space-like cones of [8]. More precisely, the fields considered in [47] are operator valued distributions $\varphi(x, e)$ where x is a point in Minkowski space and e is in the manifold of space-like directions

$$H := \{e \in \mathbb{R}^d : e \cdot e = -1\}. \tag{1}$$

The localization region of $\varphi(x, e)$ is the space-like string (or ray) $x + \mathbb{R}_0^+ e$ in the sense that if the strings $x_1 + \mathbb{R}_0^+ e_1'$ and $x_2 + \mathbb{R}_0^+ e_2$ are space-like separated for all e_1' in some open neighborhood of e_1,[1] then

$$[\varphi(x_1, e_1), \varphi(x_2, e_2)] = 0. \tag{2}$$

The field transforms in a covariant way under a unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ according to[2]

$$U(a, \Lambda)\, \varphi(x, e)\, U(a, \Lambda)^{-1} = \varphi(\Lambda x + a, \Lambda e)\,, \quad (a, \Lambda) \in \mathcal{P}_+^\uparrow. \tag{3}$$

Thus, the space-like direction, e, substitutes for the usual the Lorentz index. The restriction of U to the translation group is assumed to satisfy the spectrum condition, i.e., the joint spectrum of its generators is a subset of the forward light cone. These

[1] That is, $x_1 + \mathbb{R}_0^+ e_1$ and $x_2 + \mathbb{R}_0^+ e_2$ are space-like separated *and* e_1 and e_2 are space-like separated, c.f. Lemma A 1.

[2] We denote elements of $\mathcal{P}_+^\uparrow$ by pairs (a, Λ) with $a \in \mathbb{R}^d$ and Λ in the Lorentz group $\mathcal{L}_+^\uparrow$.

properties essentially define what is meant by a 'string-localized field' in this paper. Further restrictions but also some generalizations will be introduced later.

The purpose of the present paper is twofold. On the one hand we will supply the mathematical details of the construction in [47]. On the other hand, we put this construction into a wider context by exploring the relation between the modular localization and string-like localization in quantum field theory (QFT). Our considerations are restricted to free fields but we expect our findings also to be of relevance in more general situations in particle physics. In particular it is our desire to find a path to a (possibly perturbative) construction of massive interacting string-localized objects whose existence and general properties are predicted on structural grounds in the setting of algebraic quantum field theory (AQFT) [10]. We hope that our analysis of string-localized free fields will turn out to be a useful step into that direction.

The quest for the understanding of string-like extended objects is almost as old as the Lagrangian quantization approach to point-like quantum field theory and it is appropriate to start by recalling some of its history. The idea that string-localized objects are useful dates back to the early days of pre-renormalization QED when P. Jordan [33] proposed to use exponential line integrals over electromagnetic vector potentials in order to arrive at gauge invariant composites involving matter fields. His completely algebraic proof of the Dirac monopole quantization was a nice application of string-like objects in QED that unfortunately has remained largely unknown up to this date.

In more recent (post-renormalization) times Mandelstam [41] and Wilson [68] made extensive use of expressions involving finitely- or infinitely-extended integrals over local gauge fields. A more recent rigorous treatment of the perturbative aspects of such objects can be found in [57]. Jordan [34, and earlier papers quoted therein] in his series of publications under the somewhat misleading title "neutrino theory of light" was also the first to use such nonlocal expressions in space-time dimension $d = 1 + 1$ for what we now call bosonization/fermionization, apparently not quite aware that this trick is limited to $d = 1 + 1$. Later this formalism was used for several purposes. In [35] exponential line integrals served to obtain an improved treatment of the Thirring model, and in [59] as well as in [26] it was used to illustrate the abstract setting of the Doplicher-Haag-Roberts theory of superselection sectors.

The first systematic structural analysis for semi-infinite strings in massive QFT is due to Buchholz and Fredenhagen (B-F) [10]. In this case the string-like objects are massive charge-carrying fields within the setting of AQFT whose localization core is a semi-infinite space-like string and whose particle and symmetry aspects are the same as for point-like interpolating fields. The B-F strings are thus dynamical objects, i.e. their string–localization is due to interaction.

There is an important case where strings appear naturally without interaction, namely the massive representations of the Poincaré group in space-time dimension $d = 1 + 2$ with non-integer (or non-half-integer) spin. The occurrence of braid group statistics in this case was first explored in [37] and a realization of anyons (particles with abelian braid group statistics) along the line of a Aharonov-Bohm effect was proposed in [67]. A description of the general case of plektonic statistics, the string-like nature of the associated operators as well as their scattering theory appeared in [24, 25, 27]. The first construction of string-localized anyon one-particle states is due to one of the present authors [45] who in a previous paper [43] also showed that a mass-shell description of the associated fields is not possible. A relativistic field theory of anyons, even in the absence of genuine interactions, does not yet exist.

The strings of String Theory have little relation to string localization in the sense of the present paper. This is not surprising in view of the different history and motivation. Whereas string-localized fields are objects which fit naturally into the conceptual framework of QFT, String Theory is an attempt to transcend QFT and whose main contemporary motivation is the incorporation of all interactions (including gravity) into a scheme which at least on a perturbative level remains ultraviolet-finite. The word "string" refers in this case to its historical connections with quantized Nambu-Goto string Lagrangians which, however, do not lead to string-localized quantum fields [16, 21]. Since some of the details behind these differences are quite interesting we will return to this issue in a separate section at the end of this paper.

After this digression on the history of string-like objects in QFT we come back to the contents of the present paper. Starting from an irreducible representation of the Poincaré group in $d = 3$ or $d = 4$ space-time dimensions our aim is to find the most general string-localized field that generates this representation when applied to the vacuum and is moreover free in the sense that it is completely determined by the two-point function. Our main findings are as follows:

- Such fields exist for all irreducible (true, i.e. bosonic) representations where the representation of the 'little group' is either faithful, or trivial. This applies to all massive representations (of integer spin), the massless scalar (helicity zero) representations and the massless infinite spin representations[3].

- For the massive and the scalar massless representations all string localized fields can be written as a line integral over point-like fields. This is not possible for the infinite spin representations and the corresponding string localized fields are thus truly elementary.

- For the massless representations in $d = 4$ with finite, nonzero helicity, string fields can be defined if the previous definition is modified and a tensor (or spinor) index is added to the field in addition to the space-like direction e. In particular, photons can be described by a string-localized field with a 4-vector index in addition to e. The requirement that this field is a vector potential for the (point-localized) electromagnetic tensor field fixes it uniquely and leads naturally to the axial gauge condition. Similarly, Fermions can be described by string-localized fields with a spinor index in addition to e.

- String localization improves the short distance behavior of propagators in such a way that the singularities do not get worse with increasing spin.

The third point above is related to the well known fact that the possibilities to intertwine the Wigner canonical representations with covariant spinorial representations is more restricted for massless finite helicity representations than for the massive ones.[4] The

[3] For $d = 3$, 'infinite spin' is really a misnomer because the representation of the little group is one-dimensional in this case. See Section 6.2.

[4] In the case of Fermions, it is related to the covering aspect of the half-integer spin representations.

group theoretical reason lies in the different stabilizer groups ('little groups') for time-like and light-like vectors. What matters is the restriction to the little group of the representation of the Lorentz group occurring in the covariant transformation law. This restriction must contain the canonical representation considered as a subrepresentation and this requirement excludes in the massless case certain covariant transformation laws. The best known case is that of free photons which, in a Hilbert space with positive definite metric, can be described by a point-like field strength tensor but not by a point-like vector potential that would have a better short distance behavior than the field strength. In Section 5 we shall discuss this case further and in particular show how the photon can be described by a 'vector string' $A_\mu(x,e)$ which in addition to Lorentz transformations of x and e that determine the localization suffers a matrix transformation of the 'internal' vector index μ. This vector string satisfies the so-called 'axial gauge condition' in conventional quantum electrodynamics, but in the latter case e is not considered as specifying a string direction and therefore is kept unchanged under Lorentz transformations. This is achieved at the prize of an (abelian) gauge transformation. We also overcome the singularity of the axial gauge at $e \cdot p = 0$ (contributing to its unpopularity) by treating the potential as a distribution in e.

We hope to return to this interesting alternative viewpoint to the gauge theoretical setting in a separate work. We will also refrain here from investigating possible links between the string localized vector potentials and the Jordan-Mandelstam-Wilson string-like objects and for the massless representations we shall limit explicit constructions to helicity 0 and 1. Since in most of the present work we will be dealing with scalar string localized fields in the sense of Eqs. (2)-(3) we will usually omit the pre-fix 'scalar'. Without loss of generality we may restrict our considerations to self-conjugate (hermitian, Majorana) Bose fields. The extension to half-integer spins and Fermi fields does not bring in fundamentally new aspects and will not be treated for reasons of space.

The organization of the subsequent sections is as follows: In the next section we discuss the concept of modular localization, emphasizing its difference to the Newton-Wigner localization of particle states. The third section presents the key concept for the modular localization of positive energy representations of the Poincaré group, namely the interwiners between the Wigner canonical form of the representation and the covariant string localized form, and discusses their uniqueness. The string-localized fields are represented in terms of these intertwiners and creation and annihilation operators for the Wigner particle states in the basic formula (39) in Theorem 3.3. In Section 4 this formula is specialized to the massive representations by calculating the intertwiner. Here it is also shown that all string-localized fields for these representations can be written as line integrals over point-fields. Section 5 contains the discussion of string-localized vector fields for photons, while Section 6 is concerned with the massless, 'infinite spin' representations in $d = 4$ and $d = 3$ where point-localized fields do not exist. In Section 7 we return briefly to the comparison of string localized fields and the strings of String Theory. The final section 8 contains a resumé of the main results and an outlook. In order not to burden the main text with too many technical details we present the proofs of several lemmas in the Appendix.

2 Modular Localization

Localization and causality are central concepts which have accompanied relativistic quantum field theory right from its beginning through all stages of its development. Since these properties first appeared in the quantum setting as a result of quantizing classical fields, it was natural to assume that the classical relativistic notions of locality and causality continue to apply in the quantum realm. However the conceptual difference between observables and states, which in QFT becomes more accentuated by the omnipresence of vacuum polarization, required a more careful adaptation of these concepts.

Historically the first step towards an intrinsic formulation of relativistic quantum physics independent of any classical analogies was undertaken by Wigner in 1939 when he identified relativistic particle states with irreducible positive energy representations of the Poincaré group. These representations come with two notions of localization: the Newton-Wigner (NW) localization [48] that was formulated some years afterward, and the more recent modular localization [8, 22, 45].

The NW localization is the result of the adaptation of Born's quantum mechanical probability density for particle positions to Wigner's relativistic representation theoretical setting. Newton and Wigner define, in the single particle space, a position operator whose spectral projectors are supposed to measure the probability of detecting a (single) particle in different space-time regions. States localized in disjoint space regions at *fixed time* in some given frame of reference are orthogonal. This localization incorporates macro-causality and the cluster property, and is perfectly well-suited for scattering theory. On the other hand it is not consistent with relativistic covariance and causality, except in an approximate sense for distances of the order of the Compton wave length or smaller. In fact, it is by now well understood that *any* notion of localization that requires the set of states localized in a space-time region $\mathcal{O}$ to be orthogonal to the states localized in the causal complement $\mathcal{O}'$ is incompatible with translational covariance and positivity of the energy [40, 49].

A localization concept for quantum systems compatible with relativistic covariance and causality is contained in the formalism of local quantum field theory. This notion of localization refers not to positions of particles, but to local measurements of observables and to charge creation. The algebra $\mathcal{A}$ of observables in quantum field theory has a natural net structure which assigns to each space time region $\mathcal{O}$ a sub-algebra $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$. Typically, the algebra $\mathcal{A}(\mathcal{O})$ is generated by smeared field operators $\Phi(f)$ (or their neutral currents in case the fields are charged) with test functions f supported in $\mathcal{O}$. A key point is that the net structure of the observables allows a *local comparison* of states: Two states are locally equal in a region $\mathcal{O}$ if and only if the expectation values of all operators in $\mathcal{A}(\mathcal{O})$ are the same in both states. Local deviations from any state, in particular the vacuum state, can be measured in this manner, and states that are indistinguishable from the vacuum in the causal complement of some region ('strictly localized states' [39]) can be defined.

Due to the unavoidable correlations in the vacuum state in relativistic quantum theory (the Reeh-Schlieder property [52]), the space $\mathcal{H}(\mathcal{O})$ obtained by applying the operators in $\mathcal{A}(\mathcal{O})$ to the vacuum is, for any open region $\mathcal{O}$, dense in the Hilbert space and thus far from being orthogonal to $\mathcal{H}(\mathcal{O}')$. This somewhat counterintuitive fact is inseparably linked with a structural difference between the local algebras and the algebras encountered in non-relativistic quantum mechanics or the global algebra of a quantum field, associated with the entire Minkowski space-time. Whereas the latter has minimal projections (corresponding to optimal observations), the local algebras are type III in the terminology of Murray and von Neumann. Some physical consequences of this difference are reviewed in [70]. The Reeh-Schlieder property also implies that the expectation value of a projection operator localized in a bounded region can not be interpreted as the probability of detecting a single particle in that region since it is necessarily nonzero in the vacuum state. This is not surprising because *strict* localization requires arbitrarily high energies which in a relativistic theory may be accompanied by the creation of particles. A direct comparison with NW localization can be made in the case of free fields which are well defined as operator valued distributions in the space variables at a fixed time. The one-particle states that are NW localized in a given space region at a fixed time are not the same as the states obtained by applying field operators smeared with test functions supported in this region to the vacuum. The difference lies in the non-local energy factor $(\mathbf{p}^2 + m^2)^{1/2}$ linking the non-covariant NW states with the states defined in terms of the covariant field operators.

Causality in relativistic quantum field theory is mathematically expressed through *local commutativity*, i.e., mutual commutativity of the algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}')$. There is an intimate connection of this property with the possibility of preparing states that exhibit no mutual correlations for a given pair of causally disjoint regions. In fact, in the recent paper [11] Buchholz and Summers show that local commutativity is a necessary condition for the existence of such uncorrelated states. Conversely, in combination with some further properties (*split property* [18], existence of scaling limits), that are physically plausible and have been verified in models, local commutativity leads to a very satisfactory picture of statistical independence and local preparabilty of states in relativistic quantum field theory. We refer to [61, 65] for thorough discussions of these matters and [70] for a brief review.

Consequent application of the above mentioned concepts avoids the defects of the NW localization and resolves spurious problems rooted in assumptions that are in conflict with basic principles of relativistic quantum physics. An example is the apparent difficulty [29] with Fermi's famous gedankenexperiment [23] which he proposed in order to show that the velocity of light is the limiting propagation velocity in quantum electrodynamics. An argument which takes into account the progress on the conceptual issues of causal localization and in mathematical rigor since the times of Fermi and confirms his conclusion can be found in [13], see also [70].

Modular localization of single particle states is a concept that is intrinsically defined within the representation theory of the Poincaré group but draws its motivation from local quantum field theory. The space $\mathcal{H}(\mathcal{O})$ obtained by applying the operators of a local algebra $\mathcal{A}(\mathcal{O})$ to the vacuum vector Ω can be regarded as the domain of the Tomita involution $S_{\mathcal{O}}$ that maps $A\Omega$ to $A^*\Omega$. In the special case where $\mathcal{O}$ is a space-like wedge, the Tomita involution has a geometrical interpretation according to the Theorem of Bisognano and Wichmann [4]: It is determined by the PCT operator combined with a rotation and the generator of the Lorentz boosts associated with the wedge. It has been realized in recent years by Brunetti, Guido and Longo [8] and by B. Schroer [22] that by appealing to this interpretation of the Tomita involution for wedges and using the spatial counterpart of Tomita-Takesaki theory [53] it is possible to partially invert the above procedure of passing from local algebras to localized states. Namely, there is a natural localization structure on the representation space for any positive energy representation of the proper Poincaré group which upon second quantization gives rise to a local net of operator algebras on the Fock space over the representation Hilbert space.

In the context of Wigner's description of elementary relativistic systems the starting point is an irreducible representation U_1 of the Poincaré group on a Hilbert space $\mathcal{H}_1$ that after second quantization becomes the single-particle subspace of the Hilbert space (Fock-space) $\mathcal{H}$ of the field. (We emphasize, however, that the construction works for arbitrary positive energy representations, not only irreducible ones.) The construction then proceeds according to the following 3 steps [8, 22, 45]. To maintain simplicity we limit our presentation to the bosonic situation and refer to [45] and [22] for the general treatment.

Step 1. Fix a reference wedge region, e.g.

$$W_0 = \left\{ x \in \mathbb{R}^d ; x^{d-1} > \left| x^0 \right| \right\}, \tag{4}$$

and consider the one-parameter group $\Lambda_{W_0}(\cdot)$ of Lorentz boosts which leave W_0 invariant, and the reflection j_{W_0} across the edge of the wedge. More specifically, $\Lambda_{W_0}(t)$ acts as

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

and j_{W_0} acts as the reflection on the coordinates x^0 and x^{d-1}, leaving the other coordinates unchanged. Then use the Wigner representation $U_1(\cdot)$ of the boosts and the reflection[5], to define

$$\Delta_{W_0}^{it} := U_1(\Lambda_{W_0}(-2\pi t)), \ J_{W_0} := U_1(j_{W_0}) \tag{5}$$

$$S_{W_0} := J_{W_0} \Delta_{W_0}^{\frac{1}{2}}. \tag{6}$$

The operator $\Delta_{W_0}^{\frac{1}{2}}$ is unbounded (in general), closed and positive, J_{W_0} is an anti-linear involution commuting with $\Delta_{W_0}^{it}$, and S_{W_0} is anti-linear and closed with $S_{W_0}^2 \subset 1$. These properties characterizes S_{W_0} as a *Tomita involution*[6] which is uniquely determined by its eigenspace to the eigenvalue $+1$, i.e.,

$$K(W_0) := \left\{ \psi \in \text{domain of } \Delta_{W_0}^{\frac{1}{2}}, \ S_{W_0}\psi = \psi \right\}. \tag{7}$$

This is a closed, real linear subspace of $\mathcal{H}_1$ satisfying

$$\overline{K(W_0) + iK(W_0)} = \mathcal{H}_1, \ K(W_0) \cap iK(W_0) = 0, \tag{8}$$

[5] In certain cases an irreducible representation of $\mathcal{P}_+^\uparrow$ has to be doubled in order to accommodate the anti-unitary (since time is inverted) reflection. This is always the case with zero mass finite helicity representations and more generally if particles are not self-conjugate.

[6] Operators with this property are the corner stones of the Tomita-Takesaki modular theory of operator algebras. Here they arise in the spatial Rieffel van Daele setting [53] of modular theory from a realization of the geometric Bisognano-Wichmann situation within the Wigner representation theory.

and

$$J_{W_0} K(W_0) = K(W_0') = K(W_0)^{\perp} \tag{9}$$

where $\perp$ refers to orthogonality in the sense of the symplectic form $\mathrm{Im}(\cdot,\cdot)$ on $\mathcal{H}_1$. Eq. (8) means that the complex subspace, spanned by $K(W_0)$ together with the eigenspace $iK(W_0)$ of S_{W_0} to eigenvalue -1 is dense[7] in $\mathcal{H}_1$. This property and the absence of nontrivial vectors in the intersection of the two real spaces means that $K(W_0)$ is a real *standard subspace* in the sense of [53]. Conversely, a real standard subspace K determines uniquely a Tomita involution S (generally not related to group representation theory) with domain $K + iK$, defined by

$S(\psi + i\varphi) := \psi - i\varphi$ for $\psi, \varphi \in K$. Its polar decomposition then leads to an antiunitary involution J with $JK = K^{\perp}$ and a unitary group Δ^{it} leaving K invariant. The application of Poincaré transformations to the reference space $K(W_0)$ generates a family of wedge spaces $K(W) = U_1(a,\Lambda)K(W_0)$ if $W = (a,\Lambda)W_0$, with corresponding Tomita involutions S_W. (The definition is consistent because every Poincaré transformation which leaves W_0 invariant commutes with $\Lambda_{W_0}(t)$ and j_{W_0}, cf. [8].) There is an equivalent view on the construction of his family, which we introduce here for later reference. Namely, one associates to a wedge $W = gW_0$, $g \in \mathcal{P}_+^{\uparrow}$, the boosts $\Lambda_W(t) := g\,\Lambda_{W_0}(t)\,g^{-1}$ and reflection $j_W := g\,j_{W_0}\,g^{-1}$. Then the operators Δ_W, J_W and S_W are defined as in eqs. (5) and (6) with W_0 replaced by W. Note that in particular, by (9),

$$K(W') = K(W)^{\perp}. \tag{10}$$

The above scheme applies also to ray representations of the Poincaré group corresponding the half-integral spin [22] but (9) generalizes to

$$K(W)^{\perp} = ZK(W') \tag{11}$$

where the "twist" operator Z satisfies $Z^2 = -1$. An interesting situation arises if the spin s is not half-integer, as it happens in $d = 1+2$ dimensions for anyons [45]. In that case the spin-statistics factor $Z^2 = e^{i2\pi s}$ is not a real number.

Step 2. A sharpening of the localization is obtained by intersecting the localization spaces for wedges, defining for any causally closed region $\mathcal{O}$ contained in some wedge

$$K(\mathcal{O}) := \cap_{W \supset \mathcal{O}} K(W). \tag{12}$$

The crucial question is whether these spaces are standard. According to an important theorem of Brunetti, Guido and Longo [8] standardness holds, for all irreducible positive energy representations of the proper Poincaré group, if $\mathcal{O}$ is a *space-like cone*, i.e. a set of the form

$$C = a + \cup_{\lambda \geq 0} \lambda D \tag{13}$$

where a (the apex of the cone) is a point in Minkowski space and D is a double cone, space-like separated from the origin. The double cone regions D are conveniently envisaged as intersections of a forward light cone with a backward cone whose apex is inside the forward cone.

[7] The complex subspace $K(W_0) + iK(W_0)$ is closed in the graph norm associated with the Tomita operator S_{W_0}. Its denseness in the Wigner norm is a one-particle version of the Reeh-Schlieder theorem.

The resulting family $C \to K(C)$ of closed real subspaces of $\mathcal{H}_1$, indexed by the set $\mathcal{C}$ of space-like cones C, has the following properties:
1. *Isotony:* If $C_1 \subset C_2$, then
$$K(C_1) \subset K(C_2)\ .$$
2. *Locality:* If C_1 is causally separated from C_2, i.e., $(x-y)^2 < 0$ for $x \in C_1$, $y \in C_2$, then
$$K(C_1)\ \subset\ K(C_2)^\perp\ . \tag{14}$$
3. *Poincaré covariance:* For all C and $g \in \mathcal{P}_+^\uparrow$
$$U_1(g)\, K(C)\ =\ K(gC)\ .$$
4. *Standardness:* $K(C)$ is standard for all space-like cones C.
In case of the finite spin/helicity representations standardness also holds if $\mathcal{O}$ is an (arbitrary small) double cone.

It is a remarkable fact that these properties, plus Haag Duality (10) for space-like cones C, uniquely characterize the family $C \to K(C)$ constructed above in the massive case. This follows from the algebraic Bisognano-Wichmann theorem [44], whose proof uses precisely the above properties.

The constructive clout of modular localization is revealed in two applications, the first of which is the basis of our construction of string-localized fields.

Application 1: Construction of interaction-free algebraic nets [8, 22]. Given a family of real subspaces $K(\mathcal{O}) \subset \mathcal{H}_1$ as defined by (12), one can apply the CCR (Weyl) respectively CAR second quantization functor to obtain a covariant $\mathcal{O}$-indexed net of von Neumann algebras $\mathcal{A}(\mathcal{O})$ acting on the Fock space $\mathcal{H} = \mathcal{F}(\mathcal{H}_1)$ built over $\mathcal{H}_1$. For integer spin/helicity values [22] the modular localization in Wigner space implies the identification of the symplectic complement with the complement in the sense of relativistic causality, i.e. $K(\mathcal{O})^\perp = K(\mathcal{O}')$ (spatial Haag duality). The Weyl functor takes the spatial version of Haag duality into its algebraic counterpart. One proceeds as follows: For each Wigner wave function $\psi \in \mathcal{H}_1$ the associated (unitary) Weyl operator is defined as
$$\mathrm{Weyl}(\psi) := \exp i \left\{a^*(\psi) + a(\psi)\right\},\ \mathrm{Weyl}(\psi) \in \mathcal{B}(\mathcal{H}) \tag{15}$$
where $a^*(\psi)$ and $a(\psi)$ are the usual creation and annihilation operators on Fock space. We then define the von Neumann algebra corresponding to the localization region $\mathcal{O}$ in terms of the operator algebra generated by the image of the localized subspace $K(\mathcal{O})$
$$\mathcal{A}(\mathcal{O}) := \{\mathrm{Weyl}(\psi)|\ \psi \in K(\mathcal{O})\}''\ .$$
(By the von Neumann double commutant theorem, our generated operator algebra is weakly closed by definition.) The functorial relation between real subspaces and von Neumann algebras preserves the causal localization structure and commutes with the improvement of localization through intersections ($\cap$) according to

$K(\mathcal{O}) = \cap_{W\supset\mathcal{O}} K(W)$, $\mathcal{A}(\mathcal{O}) = \cap_{W\supset\mathcal{O}}\, \mathcal{A}(W)$ as expressed in the commuting diagram [55]

$$\begin{array}{ccc} \{K(W)\}_W & \longrightarrow & \{\mathcal{A}(W)\}_W \\ \downarrow\cap & & \downarrow\cap \\ K(\mathcal{O}) & \longrightarrow & \mathcal{A}(\mathcal{O}) \end{array} \tag{16}$$

where the vertical arrows denote the tightening of localization by intersection and the horizontal denote the action of the Weyl functor. The case of half-integer spin representations is analogous [22, 44]. The only significant difference is the mismatch between the causal and symplectic complements that is taken care of by the twist operator Z, cf.(11).

It is important to note that while the spaces $K(W)$ for wedges are uniquely determined in the *one-particle* space by the representation of the Poincaré group alone, also in the presence of interaction [45], this is in general no longer so for the space $\mathcal{K}(W) = \mathcal{A}(W)^{\mathrm{sa}}\Omega$ generated by the wedge algebra in the *whole* Hilbert space. In fact, the Tomita involution associated with $\mathcal{K}(W)$ involves, besides the Lorentz boosts and rotations, the PCT operator and hence the scattering matrix. The PCT operators for the in- and out-fields of an interacting field differ, despite the fact that both transform with respect to the same representation of the orthochronous, proper Poincaré group and their PCT operators coincide on the one-particle space.

The scheme of passing from particle- to field- localization described above works in particular for Wigner's infinite spin representations; one only must be aware that in this case one cannot achieve a better localization than that in space-like cones since the infinitely many degrees of freedom coming from the faithful representation of the little group do not allow a compact localization. The generating fields in this case are operator-valued distributions supported on semi-infinite strings (the cores of space-like cones) and their construction and derivation of their properties constitutes the main content of the present work.

A different mechanism which leads to string localization is that of $d = 1{+}2$ "anyons" i.e., Wigner representations with anomalous spin which activates the universal, instead of the standard two-fold, covering group of the Lorentz group. Such a generalization leads to a spin-statistics situation characterized by a complex modification of the spatial Haag duality $K(\mathcal{O})^{\perp} = ZK(\mathcal{O}')$, $Z^2 = e^{2\pi i s}$. This requires string-localization, but contrary to the previous case the passing from the spatial to the algebraic setting cannot be done in a functorial way even if no genuine physical interaction is present [43]. Since the methods of construction of localized operator algebras are significantly different from the present ones, this matter will be pursued in a separate work.

Application 2: Partial results on constructive aspects of modular localization in presence of interaction [5, 55]. In presence of interactions there do not exist any compactly localized operators which create a one-particle state without a vacuum polarization admixture when acting on the vacuum ('polarization free generators' (PFG)). It comes therefore as a pleasant surprise that the first line of the commuting square (16) remains intact in the following sense: modular theory secures the existence of wedge-localized PFG which are unbounded operators affiliated to the algebra $\mathcal{A}(W)$ [5]. In physical-intuitive terms: wedge localization is the best compromise between field states and Wigner particle states.

Wedge localized PFG that are operator-valued distributions on a translation-invariant dense domain ('tempered PFG') are in more than two space-time dimensions only compatible with trivial scattering, but in $d = 1+1$ they lead precisely to the Zamolodchikov-Faddev (Z-F) algebra setting for factorizing models [5, 55]. This observation brings a wealth of new insights: (1) It attributes a space-time interpretation to the hitherto rather abstract auxiliary Z-F algebra (which extends the creation/annihilation operators of free theories without affecting their "on-shell nature"). (2) It decouples the bootstrap-formfactor program for factorizing models from the quantization of classically integrable systems (the necessity to find a complete system of infinitely many conserved anomaly-free currents) and replaces the recipes of that program by derivations of its rules from first principles of general QFT using modular theory. (3) It strengthens the idea that there is nothing intrinsic about the ultraviolet problems of the standard approach; they are simply the unavoidable price to pay if one enters QFT via the classical parallelism referred to as quantization (which worked so well for passing from mechanics to quantum mechanics). Whereas intrinsic formulations of QFT which avoid singular generators have been known for several decades, it is only the recent progress of modular localization which is opening an avenue for new constructions. The ideal situation would be to be able to construct QFT in analogy to what has been done in factorizing models [56], namely in terms of a two-step process in which the first step consists in constructing generators of wedge-localized algebras and the second step in tightening localization by intersecting wedge algebras as it was already successfully achieved for the factorizing models. Only such an approach is capable of revealing the true frontiers of QFT beyond those generated by the use of singular field coordinatizations.

Our study of string-localized fields in this paper is a less ambitious step in this direction: Free string-localized fields $\varphi(x, e)$ are less singular than point-like fields $\varphi(x)$ because intuitively speaking they transfer part of the quantum fluctuations to fluctuation in the space-like string direction e so that the power-counting allows more possibilities. As a matter of fact the short-distance behavior of free string-localized fields does not become worse with increasing spin and there is no clash any more between quantum physics and the technical necessity to use (point-like) vector-potentials for photons since the physical photon space supports stringlike-localized vector potentials. It is expected that their use in suitably defined interactions will lead to a more complete understanding why in QFT the renormalizability requirement alone determines a unique interaction for vector particles [19] and the role of the gauge formalism is at best to facilitate its construction. This is different from classical field theory where there are many possible interactions involving classical vector potentials and one needs the gauge principle in order to select the Maxwellian one.

An interesting but difficult question is whether modular localization has directly verifiable observable consequences. Clearly, the states in $K(\mathcal{O})$ are in general not strictly localized in $\mathcal{O}$ in the sense of [39], i.e. giving the same expectation values as the vacuum state for observables localized outside the region $\mathcal{O}$. (The strictly localized states for some region do not form a linear space.) However, they should practically look like the vacuum state for observables localized outside $\mathcal{O}$. We can substantiate this quantitatively in the case of a free field. In this case, let ϕ be a single particle vector in $K(\mathcal{O}) + iK(\mathcal{O})$. Then the deviation of the corresponding state $\omega = (\phi, \cdot\,\phi)$ from the vacuum state ω_0 is dominated by the vacuum fluctuations for observables localized outside $\mathcal{O}$. More precisely, there is a constant $c > 0$ such that

$$|\omega(A) - \omega_0(A)| \leq c\,(\Delta A)_{\omega_0} \tag{17}$$

for all self-adjoint $A \in \mathcal{A}(\mathcal{O}')$. Here, $(\Delta A)_{\omega_0}$ denotes the vacuum fluctuation, $(\Delta A)^2_{\omega_0} = \omega_0(A^2) - \omega_0(A)^2$. (This can be shown by the methods used in [6, Lemma 4.1]) The important point is that the physical significance (in contrast to the mathematical definition) of the modular localized subspaces is *not* intrinsic to the single particle theory or the representation of the Poincaré group, but relates to the local observables of an underlying quantum field theory. This also holds outside the realm of free fields.

As a last point in this section we return to the comparison of the modular, or field theoretical, localization in relativistic QFT and localization in terms of projection operators as in non-relativistic many-body quantum mechanics. Characteristic for the latter is that the algebra of observables can be written as the tensor product of two type I factors, corresponding respectively to observables localized (at a fixed time) in a spatial region and in its complement[8]. In relativistic quantum field theory the local algebras $\mathcal{A}(\mathcal{O})$ are type III (physically, this can ultimately be attributed to vacuum fluctuations) and can therefore not be regarded as factors in a tensor product. On the other hand, the split property [18] mentioned earlier goes a long way towards recovering the quantum mechanical picture. By this property (which is a consequence of very reasonable bounds on phase space degrees of freedom [9,12,14]) local observable algebras separated by a positive security distance can be regarded as sub-algebras of commuting type I factors. Pictorially, each type I algebra can be thought of as the algebra of a sharply localized core region, augmented by a “halo” in the security region where the localization is “fuzzy”. The minimal projectors in the type I algebra can be regarded as QFT analogs of localizing projectors but there is no concept corresponding to an x-space probability density à la Born. Another difference is that the vacuum restricted to the split type I factor is not a pure state but rather a thermal equilibrium state at temperature $(2\pi)^{-1}$ with respect to a ‘modular Hamiltonian’ which is determined by the canonical split construction [18].

The factorization in transverse direction to light-like regions and the split property have recently played a crucial role in the QFT formulation of ”holographic projection” which repairs the loose ends of the old ”light-cone quantization” and converts some of the underlying ideas into valuable constructive instruments of rigorous local quantum physics [56]. All algebraic modular localization results mentioned in this paper (including the split property) have a spatial counterpart in the Wigner representation theoretical setting and can (for free fields) be obtained by a functorial construction from the latter.

[8] In Fock space, these are the algebras generated by creation and annihilation operators for wave functions with support in complementary regions in $\mathbb{R}^{d-1}$.

3 General Construction and Uniqueness of the Fields

We want to sketch the construction of string–localized fields, and discuss the question to what extent they are fixed by our assumptions. Exploiting the fact that free fields are fixed by the single particle states which they create (by a generalization [57] of the Jost-Schroer theorem [58]), this is reduced to the construction of certain intertwiners and the question of their uniqueness.

To set the stage, let us recall the irreducible unitary positive-energy representations U_1 of the Poincaré group $\mathcal{P}_+^\uparrow$ in d-dimensional Minkowski space, $d = 3, 4$. Namely, U_1 is determined by the mass value $m \geq 0$ and an irreducible unitary representation D of a certain subgroup G of $\mathcal{P}_+^\uparrow$, the so–called little group. It is the stabilizer subgroup in $\mathcal{L}_+^\uparrow$ of a fixed vector $\bar{p}$ in the mass shell for $m \geq 0$,

$$H_m^+ := \{p \in \mathbb{R}^d : \ p \cdot p = m^2, \, p^0 > 0\}.$$

If $m > 0$, then G is the rotation subgroup, and if $m = 0$ then G is isomorphic to the euclidean group in $d - 2$ dimensions. (If in this case D is faithful, the resulting representation U_1 is a so–called "infinite spin" representation. If D is not faithful but non-trivial, we shall speak of a "helicity representation".) The representation U_1 is induced by D as follows. The representation space is $\mathcal{H}_1 := L^2(H_m^+, d\mu; \mathfrak{h})$, where $d\mu$ be the Lorentz invariant measure on H_m^+ and $\mathfrak{h}$ is the representation space of D. On this Hilbert space, U_1 acts according to

$$\big(U_1(a,\Lambda)\psi\big)(p) = e^{ia\cdot p}\, D(R(\Lambda,p))\, \psi(\Lambda^{-1}p)\,. \tag{18}$$

Here $R(\Lambda, p) \in G$ is the so–called Wigner rotation, defined by

$$R(\Lambda,p) := B_p^{-1}\, \Lambda \; B_{\Lambda^{-1}p}, \tag{19}$$

where for almost all $p \in H_m^+$, B_p is a Lorentz transformation which maps $\bar{p}$ to p. We will denote the set of p for which B_p is defined by $\dot{H}_m^+$.

Each of the considered representations extends to a representation of the proper Poincaré group $\mathcal{P}_+$ as follows.[9] Let j_0 be the reflection at the edge of the standard wedge W_0, cf. (4). Choosing $\bar{p}$ invariant under $-j_0$, the adjoint action of j_0 leaves G invariant, and D extends to a representation of the subgroup of $\mathcal{L}_+$ generated by G and j_0 by an anti–unitary involution $D(j_0)$. (Explicit expressions for these representers will be given in the relevant cases later on.) One now defines an anti-unitary involution $U_1(j_0)$ by

$$\big(U_1(j_0)\psi\big)(p) := D(j_0)\, \psi(-j_0 p). \tag{20}$$

If the family B_p, $p \in \dot{H}_m^+$, is chosen so that

$$j_0 B_p j_0 = B_{-j_0 p}, \tag{21}$$

[9] $U_1(\mathcal{P}_+)$ acts in the same Hilbert space as $U_1(\mathcal{P}_+^\uparrow)$, except for $m = 0$ and *finite* helicity, where the Hilbert space has to be doubled: One has to take the direct sum of representations for helicity n and $-n$, and $U_1(j_0)$, defined in Eq. (20), also flips $n \leftrightarrow -n$.

then one checks that $U_1(j_0)$ extends U_1 to an (anti-) unitary representation of $\mathcal{P}_+$ (which is generated by $\mathcal{P}_+^\uparrow$ and j_0).

For later reference, we fix some notations concerning the manifold H of space-like directions, cf. (1). The Poincaré group acts on H by letting the translations act trivially, i.e.

$$g\,e := \Lambda e \qquad \text{if } g = (a, \Lambda) \in \mathcal{P}_+^\uparrow, \tag{22}$$

$$j\,e := \Lambda\, j_0\, \Lambda^{-1} e \qquad \text{if } j = (a, \Lambda)\, j_0\, (a, \Lambda)^{-1} \in \mathcal{P}_+^\downarrow. \tag{23}$$

Similarly, one gets wedge regions W_H in H arising from Minkowski wedges W as follows. For $W = (a, \Lambda)W_0$, we define

$$W_H := \Lambda W_0 \cap H. \tag{24}$$

(ΛW_0 is the wedge which arises from W by translation and contains the origin in its edge.)

3.1 The Concept of Intertwiners; Uniqueness

The concrete formula (18) for U_1 (the realization of the representation in a "Wigner base") is not suitable as it stands for the construction of covariant, local fields by second quantization. The problem is twofold: The transformation matrix $D(R(\lambda, p))$ depends on p (except in the scalar case), which leads to a nonlocal transformation law in x upon Fourier transformation, and the Wigner rotation factors have singularities which cause problems with local commutativity. In the standard setting of point-localized fields both difficulties are overcome by replacing the "Wigner bases" by "covariant bases". This is achieved with the help of so–called intertwiner functions which intertwine the representer of the Wigner rotation factor $R(\Lambda, p)$ with a representer of Λ and lead to the well-known formulas for the point-like free fields ([63], see also our Section 4.2). Here, we consider a new solution, which in contrast to the mentioned one works also for the massless infinite spin representations which remained outside the covariant spinorial formalism. Namely, our string-localized fields will be constructed with the help of intertwiner functions $u(e, \cdot)$ which depend on the points e in the set H of space-like directions, and absorb the Wigner rotation factor $R(\Lambda, p)$, trading it with a transformation $e \to \Lambda e$.

We now define these intertwiner functions in detail. Let $H^{\rm c}$ be the complexification of H,

$$H^{\rm c} := \{e \in \mathbb{C}^d,\ e \cdot e = -1\}, \tag{25}$$

where the dot denotes bilinear extension of the Minkowski metric to $\mathbb{C}^d$,

$$e \cdot e := e' \cdot e' - e'' \cdot e'' + 2i\, e' \cdot e'' \quad \text{if} \quad e = e' + ie''. \tag{26}$$

Let further $\mathcal{T}_+$ be the tuboid consisting of all $e = e' + ie'' \in H^{\rm c}$ such that e'' is in the interior of the forward light cone (in $\mathbb{R}^d$). We will consider subsets Θ of $\mathcal{T}_+$ of the form

$$\Theta = H^{\rm c} \cap \big(\Omega_1 + i\mathbb{R}^+\Omega_2\big), \tag{27}$$

where Ω_1 and Ω_2 are compact subsets of $\mathbb{R}^d$ and Ω_2 is contained in the forward light cone. Note that Θ is bounded due to the condition $e'' \cdot e'' \leq 1$ for $e = e' + ie'' \in \mathcal{T}_+$, and its compact closure is given by $\Theta \cup (\Omega_1 \cap H)$.

Definition 3.1 (Intertwiners) *A function $u : \mathcal{T}_+ \times \dot{H}_m^+ \to \mathfrak{h}$ is called an intertwiner function for D if it satisfies the following conditions. Firstly, it has the "intertwiner property"*

$$D(R(\Lambda, p))\, u(\Lambda^{-1}e, \Lambda^{-1}p) = u(e, p) \tag{28}$$

for $(e, p) \in \mathcal{T}_+ \times \dot{H}_m^+$ and $\Lambda \in \mathcal{L}_+^\uparrow$. Secondly, for almost all p the function $e \mapsto u(e, p)$ is analytic on the tuboid $\mathcal{T}_+$. Finally, the following bound is satisfied. There is a constant $N \in \mathbb{N}_0$ and a function M on $\dot{H}_m^+$ which is locally L^2 w.r.t. $d\mu$ and polynomially bounded, and for every $\Theta \subset \mathcal{T}_+$ of the form indicated in Eq. (27), there is a constant $c = c_\Theta$ such that for all $e \in \Theta$ holds

$$\|u(e, p)\| \leq c\, M(p)\, |e''|^{-N}. \tag{29}$$

Here, $|e''|$ denotes any norm in $\mathbb{R}^d$ and the norm of u refers to the little Hilbert space $\mathfrak{h}$.

Remarks. 1. If the growth order N of $e \to u(e, p)$ in (29) is zero, then the function $e \mapsto u(e, p)$ has a unique extension to the real boundary H as a weakly continuous $\mathfrak{h}$-valued function which we denote by the same symbol.

2. Given u, we define the *conjugate intertwiner*

$$u_c(e, p) := D(j_0)\, u(j_0 e, -j_0 p). \tag{30}$$

It transforms as in Eq. (28) and satisfies the bounds (29), and is anti-analytic in $-\mathcal{T}_+$. It is noteworthy that we find "self-conjugate" intertwiners in all cases.

The bound (29) is chosen so that for fixed p the function $e \mapsto u(e, p)$ is of moderate growth near the "real boundary" H in the sense of [7] and therefore admits a distributional boundary value in $\mathcal{D}'(H)$. In particular, for every $h \in \mathcal{D}(H)$, the (weak) integral

$$u(h, p) := \int d\sigma(e)\, h(e)\, u(e, p), \tag{31}$$

where σ is the Lorentz invariant measure on H, can be defined by letting the argument e of $u(e, p)$ approach H from H^{c} inside the tuboid $\mathcal{T}_+$ after the integration, cf. [7, Thm. A.2].

These smeared intertwiner functions give rise to a family of single particle vectors which behave covariantly and are modular–localized in "truncated space-like cones" (for $N > 0$), or in "space-like half-cylinders" (for $N = 0$). By a *truncated space-like cone*, we mean a region in Minkowski space of the form $\mathcal{O} + \mathbb{R}_0^+ \Omega$, where $\mathcal{O}$ and Ω[10] are bounded subsets of $\mathbb{R}^d$ and H. By a *space-like half cylinder* we mean a region in Minkowski space of the form $\mathcal{O} + \mathbb{R}_0^+ e$, with $e \in H$. These single particle vectors are constructed as follows. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $h \in \mathcal{D}(H)$, we define $\psi(f, h)$ and $\psi_c(f, h)$ as

$$\psi(f, h)(p) := Ef(p)\, u(h, p), \tag{32}$$

$$\psi_c(f, h)(p) := Ef(p)\, u_c(h, p), \tag{33}$$

[10] Ω must be small enough, namely contained in some wedge.

$p \in H_m^+$, where Ef is the restriction to the mass shell of the Fourier transform of f. If the growth order N of $e \mapsto u(e,p)$ in (29) is zero, we define $\psi(f,e)$ and $\psi_c(f,e)$ analogously without the smearing with $h(e)$. The bound (29) makes sure that the $\mathfrak{h}$-valued functions $\psi(f,h)$ etc. on H_m^+ are in L^2, hence in $\mathcal{H}_1$. The intertwiner property (28) implies that these single particle vectors behave covariantly under the Poincaré group. Most importantly, the analyticity of u, together with the bound (29), implies that $\psi(f,h)$ is modular–localized in the truncated space-like cone $\mathrm{supp}\, f + \mathbb{R}_0^+ \mathrm{supp}\, h$. The idea behind this assertion is as follows. Let $\mathcal{G}$ denote the strip

$$\mathcal{G} := \mathbb{R} + i(0,\pi) \tag{34}$$

and $\mathcal{G}^-$ its closure, $\mathcal{G}^- := \mathbb{R}+i[0,\pi]$. It is known that for $e \in W_H$, the map $z \mapsto \Lambda_W(z)e$ is analytic and has values in $\mathcal{T}_+$ for $z \in \mathcal{G}$, cf. Eq. (A.7).[11] Hence for $e \in W_H$, the function $z \mapsto u(\Lambda_W(z)e,p)$ is analytic on $\mathcal{G}$. The bound (29) ensures that the same holds after smearing with a test function h, if $\mathrm{supp}\, h \subset W_H$, and that the smeared intertwiner is polynomially bounded for large p. This implies that $\psi(f,h)$ is in the domain of the operator S_W whenever W contains $\mathrm{supp}\, f$ and W_H (or its closure) contains $\mathrm{supp}\, h$ or, equivalently, whenever W contains the truncated space-like cone $\mathrm{supp}\, f + \mathbb{R}_0^+ \mathrm{supp}\, h$. The details are spelled out in Appendix A. We get the following result.

Proposition 3.2 (Properties of single particle state vectors) *Let $u(e,p)$ be an intertwiner function as in Definition 3.1, and $\psi(f,h)$, $\psi_c(f,h)$ be defined as in* (32), (33).

0) For $f \in \mathcal{S}(\mathbb{R}^d)$ and $h \in \mathcal{D}(H)$, the state vector $\psi(f,h)$ is in $\mathcal{H}_1 = L^2(H_m^+, \mathrm{d}\mu) \otimes \mathfrak{h}$. Furthermore, a single particle version of the Reeh-Schlieder theorem holds: Let $\mathcal{O}$ and $\mathcal{U}$ be arbitrary open sets in $\mathbb{R}^d$ and H, respectively. Then the linear span of $\psi(f,h)$, $\mathrm{supp}\, f \subset \mathcal{O}$, $\mathrm{supp}\, h \subset \mathcal{U}$, *is dense in the single particle space.*

i) The family transforms covariantly under U_1:

$$U_1(g)\psi(f,h) = \psi(g_* f, g_* h)\,, \quad g \in \mathcal{P}_+^\uparrow, \tag{35}$$

$$U_1(j)\psi(f,h) = \psi_c(j_* \bar{f}, j_* \bar{h}), \quad j \in P_+^\downarrow, \tag{36}$$

where g_* denotes the push–forward, $(g_* f)(x) = f(g^{-1}x)$.[12] The same holds for ψ_c.

ii) Let f be a smooth function on $\mathbb{R}^d$ with support in a double cone $\mathcal{O}$, and let h be a smooth function on H with support in a compact set Ω.[13] Then $\psi(f,h)$ is localized in the truncated space-like cone $\mathcal{O} + \mathbb{R}_0^+\Omega$. More precisely, if W contains $\mathcal{O} + \mathbb{R}_0^+\Omega$, then it is in the domain of S_W, and

$$S_W \psi(f,h) = \psi_c(\bar{f}, \bar{h})\,. \tag{37}$$

iii) If the growth order N of $e \mapsto u(e,p)$ in (29) *is zero, then 0) and i) hold with h replaced by e and $g_* h$ replaced by ge. Moreover, if* $\mathrm{supp}\, f \in \mathcal{O}$ *then $\psi(f,e)$ is localized in the space-like half cylinder $\mathcal{O} + \mathbb{R}_0^+ e$, and $S_W \psi(f,e) = \psi_c(\bar{f}, e)$ whenever W contains $\mathcal{O} + \mathbb{R}_0^+ e$.*

[11] W_H has been defined in (24) and $\Lambda_W(z)e$ refers to the action of $\mathcal{P}_+^\uparrow$ on H defined in (22).

[12] Here, ge and je, for $e \in H$, is meant as in equations (22) and (23).

[13] Ω must be small enough, cf. footnote 10.

Note that *ii*) is equivalent to

$$\psi(f,h) + \psi_c(\bar{f},\bar{h}) \in K_1(\mathcal{O} + \mathbb{R}_0^+\Omega). \tag{38}$$

For the self-conjugate intertwiners which we find in the subsequent sections, $\psi(f,h)$ is in $K_1(\mathcal{O}+\mathbb{R}_0^+\Omega)$ for real valued f, h. The proof of the proposition is given in Appendix A.

Second quantization then yields a string-localized free quantum field. We also assert the converse, namely that every string-localized free quantum field arises this way, and we also discuss to what extent the corresponding intertwiner functions are unique. Let $a^*(\psi)$ and $a(\psi)$, $\psi \in \mathcal{H}_1$, denote the creation and annihilation operators in the bosonic Fock space over $\mathcal{H}_1$. We shall write symbolically[14]

$$a^*(\psi) =: \int_{H_m^+} d\mu(p)\, \psi(p) \cdot a^*(p) \quad \text{and} \quad a(\psi) =: \int_{H_m^+} d\mu(p)\, \overline{\psi(p)} \cdot a(p).$$

Our results are summarized in the following theorem, which is valid for all bosonic particle types except those corresponding to the helicity representations.

Theorem 3.3 (Existence and uniqueness of string fields) *Let U_1 be any irreducible positive energy representation of the Poincaré group with faithful or trivial representation of the little group.*

i) Let u be an intertwiner function for D, and let u_c be defined as above. Then the field $\varphi(x,e)$ defined by

$$\varphi(x,e) := \int_{H_m^+} \mathrm{d}\mu(p) \Big\{ e^{ip\cdot x}\, u(e,p) \cdot a^*(p) + e^{-ip\cdot x}\, \overline{u_c(e,p)} \cdot a(p) \Big\} \tag{39}$$

satisfies our requirements (2) *and* (3)*. It further satisfies the Reeh-Schlieder and Bisognano-Wichmann properties (see below). Moreover, if the growth order N of $e \mapsto u(e,p)$ in* (29) *is zero, then the field is a* function *in e, and the commutativity* (2) *already holds if $x_1 + \mathbb{R}_0^+ e_1$ is space-like separated from $x_2 + \mathbb{R}_0^+ e_2$.*[15]

ii) A non–trivial intertwiner function u with these properties exists for all mentioned representations. It is unique up to multiplication with a function of $e \cdot p$, which is meromorphic in the upper half plane. (That is to say, if $\hat{u}$ is another intertwiner function, then for almost all $e \in \mathcal{T}_+$ and almost all p,

$$u(e,p) = F(e\cdot p)\, \hat{u}(e,p), \tag{40}$$

where F is a numerical function, meromorphic on the complex upper half plane.)

iii) Conversely, let φ be a string-localized field, in the sense of (2) *and* (3)*, which is free in the sense that it creates only single particle vectors from the vacuum, and which satisfies in addition the Bisognano-Wichmann property*[16]*. Then it is of the form* (39) *up to unitary equivalence, with u as in Definition 3.1 and with u_c as in* (30)*. Further, it satisfies a "PCT theorem", namely, it is covariant under reflections:*

$$U(j)\varphi(x,e)U(j)^{-1} = \varphi(jx,je)^*, \quad j \in \mathcal{P}_+^{\downarrow}. \tag{41}$$

[14] Note that $\psi(p) \in \mathfrak{h}$, and "$\cdot$" stands for the contraction over the indices of a basis of $\mathfrak{h}$

[15] By Lemma A 1, this is compatible with e_1, e_2 light-like separated or identical, in contrast to the general case, c.f. Footnote 1. In particular, this case covers the scenario envisaged by Steinmann in [57], where all string directions e coincide.

[16] In the massive case, the Bisognano-Wichmann property is not an extra assumption, cf. [4, 44].

By the *Reeh-Schlieder property* we mean that products of the fields already generate a dense set from the vacuum when smeared within arbitrary, fixed, open sets $\mathcal{O} \in \mathbb{R}^d$ and $U \in H$. The *Bisognano-Wichmann property* means that the modular group Δ_W^{it} and modular conjugation J_W of the algebra associated to a wedge W coincides with the representers of the boosts $\Lambda_W(t)$ and reflection j_W associated to W, respectively.

Remarks. 1. Clearly, one can construct new intertwiner functions from given ones via Eq. (40), with F analytic on the upper half plane, polynomially bounded at infinity and of moderate growth near the reals. The corresponding fields all belong to the same Borchers class, i.e., they are relatively string-localized with respect to each other.

2. The intertwiner functions appearing in the theorem are more precisely speaking "scalar" string-localized interwiners, i.e. functions $u(p, e)$ which have no explicit vector (tensor) index. For massless helicity representations with helicity $n \neq 0$ no such intertwiners exist, c.f. the remark after Proposition 5.1. This is the reason for our exception of these representations from the theorem. However, we show in Section 5 that one can describe photons (corresponding to the direct sum of the representations with helicity ± 1) in terms of string-localized vector potentials instead of field strengths if one admits a vector index. Similarly, string-localized intertwiners for fermions need a spinor index.

A special case, for which we give examples in Section 4, is a string-field $\varphi(x, e)$ which transforms as in (3), but is *point-like* localized, i.e. the space-like commutation property (2) depends only on x and not on e. This corresponds to the following analyticity property of its intertwiner.

Proposition 3.4 (Point-localized fields) *A string-localized field, in the sense of Eq.* (3)*, is* point-like *localized if, and only if, the function* $e \mapsto u(e, p)$ *is analytic on the entire complexified* $H^{\rm c}$ *and the bound* (29)*, with growth order* $N = 0$*, holds for all compact subsets of* $H^{\rm c}$*.*

The rest of this subsection is devoted to the proof of the proposition and of the structural part of the theorem, namely *i*), *iii*) and the uniqueness part of *ii*). The existence claim of *ii*) will be proved by explicit construction: We sketch the construction of intertwiners in the next subsection, and the proof that they have the required properties will be given in each case separately, in Sections 4 through 6.

Proof of Theorem 3.3. *i*) follows from Proposition 3.2 via second quantization. The proof of the Bisognano-Wichmann property is contained in [38, proof of Thm. I.3.2], the argument being as follows. By construction, the second quantization of the Tomita involution of $K(W)$ coincides with the Tomita involution for the closure of the real space $\mathcal{A}(W)^{\rm sa}\Omega$. But the latter is just the Tomita operator associated with the von Neumann algebra $\mathcal{A}(W)$ and Ω. Since second quantization preserves the polar decomposition, this proves the Bisognano-Wichmann property.

To show the uniqueness statement of *ii*), let u and $\hat{u}$ be functions with the stated intertwiner and analyticity properties. Considering equation (28) with $\Lambda = B_p$ and using $R(B_p, p) = 1$, one finds that u is of the form

$$u(e, p) = u_0(B_p^{-1} e), \quad \text{where} \quad u_0(e) := u(e, \bar{p}). \tag{42}$$

Then u_0 transforms under the little group G according to

$$D(\Lambda)\, u_0(e) = u_0(\Lambda e), \quad \Lambda \in G. \tag{43}$$

(Note for later reference that, conversely, $u(e,p)$ is fixed by $u_0(e)$ through the left equation in (42), and u_0 transforming as in (43) implies that $u(e,p)$ satisfies the intertwiner relation (28).) Let now u_0 and $\hat{u}_0$ be solutions to this equation. We first show that in $d=4$, they are linearly dependent (which in $d=3$ is a tautology since there the little Hilbert space is one–dimensional). This can be seen as follows. Eq. (43) implies that for fixed e the vector $u_0(e)$ must be invariant under the restriction of D to the stability subgroup, in G, of e. But this condition, as we show in Lemma B 3, fixes the $u_0(e) \in \mathfrak{h}$ up to a factor if e is in the "real boundary" H of $\mathcal{T}_+$, i.e. $u_0(e)$ and $\hat{u}_0(e)$ are linearly dependent for $e \in H$. By the edge-of-the-wedge theorem for tuboids [7, Thm. A 3], this implies that $u_0(e)$ and $\hat{u}_0(e)$ are linearly dependent for all $e \in \mathcal{T}_+$. Hence in $d=4$, as well as in $d=3$, we have

$$u_0(e) = f(e)\, \hat{u}_0(e), \quad f(e) \in \mathbb{C}, \tag{44}$$

for all $e \in \mathcal{T}_+ \setminus \mathcal{N}$, where $\mathcal{N}$ denotes the set (of measure zero) where $\hat{u}_0$ vanishes. On this domain, the function f must be analytic. Further, Eq. (43) implies that f is invariant under G. For $e \in \mathcal{T}_+$, let now G_e^c denote the set of complex Lorentz transformations which leave $\bar{p}$ invariant, map e into $\mathcal{T}_+$, and are path-connected with the unit. Analyticity of f on $\mathcal{T}_+$ then implies that for $e \in \mathcal{T}_+$ the function $\Lambda \mapsto f(\Lambda e)$ is analytic on G_e^c, and invariance of f under G implies that this function is constant not only on G, but on G_e^c. But we show in Lemma B 3 that for (almost – cf. below) each pair e, $\hat{e} \in \mathcal{T}_+$ satisfying $\bar{p} \cdot \hat{e} = \bar{p} \cdot e$, there is some $\Lambda \in G_e^c$ such that $\hat{e} = \Lambda e$. Therefore, f is constant on all $e \in \mathcal{T}_+$ with $\bar{p} \cdot e =$ constant. (In fact, in the massive case the complexification of G does *not* act transitively on the set $\bar{p} \cdot e = im$, namely here one has to exclude the point $e = (i, 0, 0, 0)$, cf. Lemma B 3. But by continuity, the conclusion also holds for the set $\bar{p} \cdot e = im$.) It follows that f can be written as $f(e) = F(\bar{p} \cdot e)$, which implies Eq. (40) of the theorem by virtue of (42). It also follows that the function F is analytic on the upper half plane except, possibly, for those points $\bar{p} \cdot e$ with $e \in \mathcal{N}$, i.e. $\hat{u}_0(e) = 0$. But these are isolated points in $\mathbb{C}$, which can be seen as follows. If $e \in \mathcal{N}$, then by the covariance condition (43) $\hat{u}_0$ vanishes on the entire orbit of e under G, and by analyticity it vanishes on the orbit of e under G_e^c. Again by Lemma B 3, this implies that $\hat{u}_0$ vanishes on the entire hyper-surface $\bar{p} \cdot e =$constant. Suppose now that the set of $\bar{p} \cdot e$, $e \in \mathcal{N}$, has an accumulation point. Then $\hat{u}_0$ vanishes on all corresponding hyper-surfaces and hence vanishes altogether. Hence all points $\bar{p} \cdot e$, $e \in \mathcal{N}$, are isolated. Further, analyticity of u_0 implies that F has no essential singularity on any of these points. It follows that F is a meromorphic function.

Ad *iii*) By a version of the Jost–Schroer theorem [57], a string–localized Wightman field $\varphi(x,e)$ creating only single particle states from the vacuum is, up to unitary equivalence, of the form $a^*(\varphi(x,e)\Omega) + a(\varphi(x,e)^*\Omega)$. Here a^* and a are the creation and annihilation operators acting on the second quantization of the single particle space. Covariance (3) under translations implies that $\varphi(x,e)\Omega$ and $\varphi(x,e)^*\Omega$ are of the form[17]

$$\varphi(x,e)\Omega\,(p) =: e^{ix\cdot p}\,u(e,p), \quad \varphi(x,e)^*\Omega\,(p) =: e^{ix\cdot p}\,u_c(e,p), \tag{45}$$

where u and u_c are $\mathfrak{h}$–valued distributions. Hence our fields are, up to equivalence, of the form (39). The covariance property (3) then implies that u must satisfy the intertwining property (28).

We now show that u must also have the analyticity property. To this end, let W be a wedge and let L_W denote the generator of the unitary one-parameter group representing the boosts $\Lambda_W(t)$. The Bisognano-Wichmann property implies that if $x + \mathbb{R}_0^+ e \in W$, then firstly $\varphi(x,e)\Omega$ is in the domain of the unbounded operator $\exp(-\pi L_W)$, and secondly

$$\varphi(x,e)^*\Omega = U(j_W)\,\exp(-\pi L_W)\,\varphi(x,e)\Omega. \tag{46}$$

But the first assertion implies that the $\mathcal{H}$-valued function $t \mapsto U(\Lambda_W(t))\varphi(x,e)\Omega$ has an analytic extension into the strip $\mathbb{R} + i(0,\pi)$, weakly continuous at the boundary. (The value at $t = i\pi$ then coincides with $\exp(-\pi L_W)\varphi(x,e)\Omega$.) It follows that for almost all p the function

$$t \mapsto u(\Lambda_W(t)_* h, p) \tag{47}$$

is the boundary value of an analytic function in the strip $\mathcal{G} := \mathbb{R} + i(0,\pi)$ if h is a test function on H with support contained in W_H. (We have for a moment restored the distribution notation in order not to miss the point.) One concludes from the above that u is the boundary value, in the sense indicated after Eq. (31), of an analytic function $e \mapsto u(e,p)$ on the tuboid $\mathcal{T}_+$, of moderate growth near the real "boundary" H. (To this end, one represents the distribution u as a first order derivative (in the sense of distributions) of a continuous function on H, and recalls that the set of $\Lambda_W(t)e$, with $t \in \mathcal{G}$ and W, $e \in W$ varying, exhausts the entire tuboid, cf. Lemma A 2.) Since $U(\Lambda_W(t))\varphi(x,e)\Omega$ must be in $\mathcal{H}_1$ for all $t \in \mathcal{G}$, $e \in W$, this also shows the bound (29).

To show that u and u_c are related as in Eq. (30), we consider first $e \in W_0$. Then equation (46) implies that $u_c(e,p)$ coincides with $D(j_0)u(\Lambda_0(t)e, -j_0 p)$ at $t = i\pi$. Using that $\Lambda_0(i\pi) = j_0$, this implies equation (30) for $e \in W_0$. Let now e be an arbitrary point in H. Then $e \in \Lambda W_0$ for some Λ. Using the intertwining property (28) of u and the identity (21), one finds that equation (30) also holds for such e.

As to the PCT theorem, we now have the two identities:

$$\big(U(j_0)\varphi(x,e)\Omega\big)(p) = D(j_0)\,e^{ix\cdot(-j_0 p)}\,u(e,-j_0 p), \tag{48}$$
$$\big(\varphi(j_0 x, j_0 e)^*\Omega\big)(p) = e^{ij_0 x\cdot p}\,u_c(j_0 e, p). \tag{49}$$

By Eq. (30) and antilinearity of $D(j_0)$, the right hand sides coincide. We therefore get equation (41) for $j = j_0$ by the Reeh–Schlieder property, and for all $j \in \mathcal{P}_+^{\downarrow}$ by covariance (3). □

We finally prove the proposition on point-localized fields.

Proof of Proposition 3.4. Suppose u is analytic on the entire complexified H^{c} and satisfies the mentioned bound. Then the function $t \mapsto u(\Lambda_W(t)e)$ is analytic in the

[17] The subsequent formulas are to be understood in the sense of distributions.

strip $\mathcal{G}$ whether or not e is contained in W. The proof of Proposition 3.2 reveals that then the single particle vectors $\psi(f,h)$, $\operatorname{supp} f \subset \mathcal{O}$, are localized in $\mathcal{O}$ in the modular sense. This proves the "if" part via second quantization. Conversely, suppose $\varphi(f,h)$ is localized, in the sense of commutators, at the support of f, independently of $\operatorname{supp} h$. Then the reasoning of the above proof, ad *iii*), shows that the function in Eq. (47) is analytic in the strip $\mathcal{G}$, independently of $\operatorname{supp} h$. But the set of $\Lambda_W(t)e$, with W, $e \in H$ and $t \in \mathcal{G}$ arbitrary, exhausts the entire H^{c}, cf. Lemma A 2. As above, one concludes that $u(e,p)$ is analytic on H^{c}. The bound (29) also follows as in the above proof, with $N = 0$ since $u(e,p)$ is analytic. □

3.2 Construction of the Intertwiners: General Recipe.

We now describe the idea of the construction of intertwiner functions u with the properties required in Theorem 3.3 for irreducible positive energy representations of the Poincaré group with a faithful (or scalar) representation D of the little group G, i.e. for massive bosons and the massless infinite spin particles. The proofs of the relevant properties will be given in Sections 4 (massive case) and 6 (massless infinite spin case).

We will exploit the fact that all of these irreducible unitary representations D of G occur in the decomposition of the pullback representation acting naturally on functions on suitable G–orbits. Namely, let Γ be the G–orbit defined by

$$\Gamma := \{q \in H_0^+ : q \cdot \bar{p} = 1\}. \tag{50}$$

(Recall that $\bar{p}$ is the base point in H_m^+ whose stabilizer group is G.) It turns out that Γ is isometric to the sphere S^{d-2} for $m > 0$, and to $\mathbb{R}^{d-2}$ for $m = 0$, cf. Lemma B 2. Since every isomorphism of Γ extends, by linearity, to a Lorentz transformation which leaves $\bar{p}$ invariant, it follows that G is precisely the isometry group of Γ. Thus, the isometry $\Gamma \cong S^{d-2}$ or $\mathbb{R}^{d-2}$ establishes the isomorphism $G \cong SO(d-1)$ or $E(d-2)$, respectively for $m > 0$ or $m = 0$. Let now $d\nu$ denote the G–invariant measure on Γ, and let $\tilde{D}$ be the unitary representation of G acting on $L^2(\Gamma, d\nu)$ as

$$\big(\tilde{D}(R)v\big)(q) := v(R^{-1}q), \quad R \in G. \tag{51}$$

It is well–known that $\tilde{D}$ decomposes into the direct sum of all irreducible representations of $G \cong SO(d-1)$ for $m > 0$ and into a direct integral of all faithful irreducible representations of $G \cong E(d-2)$ for $m = 0$. Hence, for any faithful representation D of G there exists a partial isometry V from $L^2(\Gamma, \mathrm{d}\nu)$ into $\mathfrak{h}$ which intertwines the representations $\tilde{D}$ and D:

$$D(R)\, V = V\, \tilde{D}(R), \quad R \in G. \tag{52}$$

(In the case $m = 0$, V is a generalized partial isometry defined only on a dense set in $L^2(\Gamma, \mathrm{d}\nu)$.)

We now solve the intertwiner equation (28) by projecting a corresponding $L^2(\Gamma, d\nu)$-valued solution $\tilde{u}(e,p)$ onto $\mathfrak{h}$. Namely, let F be a suitable numerical function and define

$$\tilde{u}(e,p)\,(q) := F(q \cdot B_p^{-1} e), \tag{53}$$

$$u(e,p) := V\, \tilde{u}(e,p). \tag{54}$$

Then $\tilde{u}$ solves the analogue of (28) with D replaced by $\tilde{D}$, and u solves Eq. (28) by construction. Now note that the imaginary part of $q \cdot e$ is strictly positive if $q \in H_0^+$, $e \in \mathcal{T}_+$. Hence the analyticity property can be satisfied if F has an analytic extension into the upper complex half plane. It turns out that a proper choice for F is

$$F(w) := w^\alpha, \tag{55}$$

for suitable $\alpha \in \mathbb{C}$. In case $\alpha \notin \mathbb{Z}$, the power w^α is understood via the branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_0^-$ with $\ln 1 = 0$, and by continuous extension from the upper half plane if $w \in \mathbb{R}^-$, i.e., $\lim_{\varepsilon \to 0+}(w + i\varepsilon)^\alpha$. The intertwiner u obtained this way will be denoted u^α in the sequel. In general, the function $q \mapsto (q \cdot e)^\alpha$ will be in $L^2(\Gamma, d\nu)$ only after smearing with a test function $h \in \mathcal{D}(H)$, $\int d\sigma(e)\, h(e)\, (q \cdot e)^\alpha$. The representation $\tilde{D}$ extends naturally to the Lorentz group on the (dense) set of functions of this form via push-forward. In fact, Bros et al. have shown [7] that this representation is unitary in $L^2(\Gamma, d\nu)$ if the real part of α is $-(d-2)/2$, and that in this case it is equivalent to the irreducible principal series representation with value $\alpha(\alpha + d - 2) = -|\alpha|^2$ of the Casimir operator. The choice (55) is also distinguished by the fact that the resulting intertwiner u^α, and hence the associated free field, satisfies the Klein Gordon equation in the variable $e \in H$, with squared mass $-\alpha(\alpha + d - 2)$, cf. [7]. The connection of our approach with the work of Bros et al. has been elaborated in [46].

4 Massive Bosons

We construct intertwiners for massive bosons, arriving at explicit expressions for the intertwiners and the ensuing two-point functions. We obtain fields with genuine string-like (in contrast to point-like) localization, and clarify their relation to point-like localized fields. We also show that they have better UV behavior than the point-like localized usual free fields.

Although intertwiners can be easily constructed from the known point-localized intertwiners, cf. Subsection 4.2 or Lemma 4.2, we shall construct them along the lines of the last subsection. Our main motivation is that these intertwiners have the additional interesting feature that they satisfy the Klein Gordon equation in the variable $e \in H$, as already mentioned. Let us first recall the irreducible representations D of the little group G and of j_0 in the massive case. For $m > 0$, G is isomorphic to $SO(d-1)$. Recall that the irreducible representations of $SO(2)$ are labeled by $s \in \mathbb{Z}$ and act in $\mathbb{C}$ as $R(\omega) \mapsto e^{is\omega}$, and that the irreducible representations of $SO(3)$ are labeled by $s \in \mathbb{N}_0$ and act in $\mathbb{C}^{2s+1} \cong \mathrm{span}\{Y_{s,k}, k = -s, \ldots, s\} \subset L^2(S^2)$, with $Y_{s,k}$ the spherical harmonics, according to

$$(D(R)Y_{s,k})(n) := Y_{s,k}(R^{-1}n). \tag{56}$$

for $n \in \mathbb{R}^3$, $\|n\| = 1$. Since $\bar{p} = (m, 0, 0, 0)$ is invariant under $-j_0$, G is invariant under the adjoint action of j_0, and the subgroup of $\mathcal{L}_+^\uparrow$ generated by G and j_0 is a semi-direct product. The above representations D of G extend to this group via an anti–unitary involution $D(j_0)$. Namely, in the case $d = 4$, $D(j_0)$ is the operator defined by anti–linear extension of

$$D(j_0)Y_{s,k} := (-1)^k Y_{s,-k}. \tag{57}$$

In $d = 3$, $D(j_0)$ is just complex conjugation. (We show in Lemma B 1 that $D(j_0)$ indeed satisfies the representation properties $D(j_0)D(\Lambda)D(j_0) = D(j_0\Lambda j_0)$, $\Lambda \in G$.)

4.1 Intertwiners.

We now specify the general construction of Section 3.2, arriving at completely explicit expressions, cf. Proposition 4.3. For $m > 0$, and base-point $\bar{p} := (m, 0, 0, 0)$ or $(m, 0, 0)$ in H_m^+, the set Γ of all $q \in H_0^+$ with $q \cdot \bar{p} = 1$, cf. (50), is isometric to the sphere S^{d-2} via the parametrization of Γ given by

$$q(\theta) := (1, \cos\theta, \sin\theta)/m, \qquad d = 3, \tag{58}$$

$$q(n) := (1, n_1, n_2, n_3)/m, \quad |n|^2 = 1, \qquad d = 4. \tag{59}$$

The isomorphism q from S^{d-2} onto Γ identifies the action of G in Γ with the action of $SO(d-1)$ in S^{d-2}, and $-j_0$ acts as $\theta \mapsto \pi - \theta$ or $(n_1, n_2, n_3) \mapsto (-n_1, -n_2, n_3)$, respectively. The isometric intertwiners $V = V_s$ from the representation $\tilde{D}$, cf. (51), to the irreducible representation for spin s come out as

$$V_s v := \int_{S^1} d\theta e^{is\theta} v(q(\theta)), \qquad d = 3, \tag{60}$$

$$(V_s v)_k := (Y_{s,k}, v) = \int_{S^2} d\sigma(n) \overline{Y_{s,k}(n)} v(q(n)), \qquad d = 4. \tag{61}$$

Here, $v \in L^2(\Gamma, d\nu)$, q is the parametrization of Γ defined in (58) and (59), and $d\sigma(n)$ denotes the rotation invariant measure on the sphere. Thus, our construction (53), (54) and (55) leads to the following intertwiners:

$$u^\alpha(e, p) = e^{-i\pi\alpha/2} \int_{S^1} d\theta e^{is\theta} \left(B_p q(\theta) \cdot e\right)^\alpha, \qquad d = 3, \tag{62}$$

$$u^\alpha(e, p)_k = e^{-i\pi\alpha/2} \int_{S^2} d\sigma(n) \, \overline{Y_{s,k}(n)} \left(B_p q(n) \cdot e\right)^\alpha, \qquad d = 4. \tag{63}$$

(We have introduced a factor $e^{-i\pi\alpha/2}$ so that $(u^\alpha)_c = u^{\bar{\alpha}}$. This follows from a calculation analogous to (108) below.) Let us discuss the three-dimensional case in more detail. For e in the real boundary H, the integrand in (62) turns out two have two distinct zeroes of order 1 as a function of θ. The corresponding pole, for real $e \in H$, of the integrand is therefore integrable iff Re $\alpha > -1$. Hence, for this range of α we expect u^α to be an intertwiner function with growth order zero, thus leading to localization in space-like half cylinders, and to fields which do not have to be smeared in e. This is indeed the case:

Proposition 4.1 (Intertwiners of growth order 0) *Consider the three-dimensional case and let Re $\alpha > -1$. Then $u^\alpha(e, p)$ is an intertwiner function in the sense of Definition 3.1, with growth order $N = 0$ in (29). More specifically, it is bounded, uniformly in $e \in \mathcal{T}_+$ and $p \in H_m^+$.*

Proof. For $e \in \mathcal{T}_+$ the imaginary part of the integrand $B_p^{-1} q(\theta) \cdot e$ is strictly positive. This allows one to find, for e in any given compact subset of $\mathcal{T}_+$, a dominating function for the integrand. Therefore the analyticity in e of the integrand implies that u^α is analytic. To prove the uniform boundedness, we denote $e_\pm := e_1 \pm ie_2$ and calculate

$$m\, q(\theta)\cdot e = e_0 - \frac{1}{2}\big(e_+e^{-i\theta} + e_-e^{i\theta}\big) = -\frac{1}{2}e_-e^{-i\theta}(e^{i\theta}-z_+)(e^{i\theta}-z_-) \tag{64}$$

$$= -ie^{-i\theta}\big((e^{i\theta}-z_+)^{-1} - (e^{i\theta}-z_-)^{-1}\big)^{-1}, \tag{65}$$

where $z_\pm := e_-^{-1}(e_0 \pm i)$ are the zeroes of the polynomial $z^2 - 2(e_0/e_-)z + e_+/e_-$. Therefore u_0^α (corresponding to u^α as in Eq. (42)) satisfies

$$|u_0^\alpha(e)| \le c\int d\theta\big(|e^{i\theta}-z_+|^{\alpha'} + |e^{i\theta}-z_-|^{\alpha'}\big),$$

where $\alpha' := \mathrm{Re}\,\alpha$. (We have used that for $w \in \mathbb{R} + i\mathbb{R}_0^+$, $|w^\alpha| \le c_\alpha |w|^{\alpha'}$, where $c_\alpha = \max\{1, e^{-\pi \mathrm{Im}\,\alpha}\}$.) By rotational invariance, we may assume $z_\pm \in [0,\infty)$. Then $|e^{i\theta} - z_\pm| \ge |\sin\theta| \ge \frac{2}{\pi}|\theta|$ in the interval $\theta \in (-\pi/2, \pi/2)$, and $|e^{i\theta} - z_\pm| \ge 1$ in its complement in S^1. Hence the integral has a bound independent of $z_\pm$, hence of e. This proves the claim. □

We now consider the case $\alpha = n \in \mathbb{N}_0$. Since the intertwiners u^n are analytic in all of H^c, they lead to point-localization according to Proposition 3.4. In the sequel, will refer to the 1-1 correspondence between $u(e,p)$ and $u_0(e)$ given in Eq. (42). It is clear from eqs. (62), (63) that u_0^n is an n-linear form on $\mathbb{C}^d$ and can therefore be written as

$$u_0^n(e)_k = \sum_{\mu_1,\dots,\mu_n} u_k^{\mu_1\dots\mu_n}\, e_{\mu_1}\cdots e_{\mu_n}. \tag{66}$$

By the covariance condition (43), the matrices $u_k^{\mu_1\dots\mu_n}$ build up an intertwiner from the natural representation of $SO(d-2)$ on the symmetric n-tensors to the irreducible representation with spin s. It is then clear (and also follows from eqs. (64), (70) below) that u^n vanishes unless $n \ge |s|$, s being the spin of the particle. Thus, the simplest point-like localized cases is $\alpha = |s|$. We now exhibit explicit expressions for this case in 3 and in 4 dimensions.

Lemma 4.2 *The intertwiner u^s is given as $u^s(e,p) = u_0^s(B_p^{-1}e)$, with u_0^s as follows. In 3 dimensions, $u_0^{|s|}$ is given, up to a real factor, by*

$$u_0^{|s|}(e) = i^{|s|} \times \begin{cases} (e_1+ie_2)^s & \text{if } s \ge 0, \\ (e_1-ie_2)^{|s|} & \text{if } s < 0. \end{cases} \tag{67}$$

In 4 dimensions, u_0^s is given, up to a real factor, by

$$u_0^s(e)_k = i^s\sqrt{\frac{(s+k)!}{(2s)!(s-k)!}}\,\big\{(e_1+ie_2)\partial_{e_3} - (\partial_{e_1}+i\partial_{e_2})e_3\big\}^{s-k}(e_1-ie_2)^s. \tag{68}$$

For real $e \in H$, it coincides with

$$u_0^s(e)_k = i^s\,(1+e_0^2)^{s/2}\,\overline{Y_{s,k}(n(e))}, \tag{69}$$

where $Y_{s,k}$ are the spherical harmonics, and $n(e) := (1+e_0^2)^{-1/2}(e_1,e_2,e_3) \in S^2$.

Note that Eq. (68) exhibits analyticity of the intertwiner on the whole of $H^{\rm c}$, while Eq. (69) exhibits its covariance (43) under rotations.

Proof. The 3-dimensional case follows straightforwardly from

$$\big(q(\theta)\cdot e\big)^{|s|} = (-2m)^{-|s|}\,\big((e_1+ie_2)^{|s|}e^{-i|s|\theta}+(e_1-ie_2)^{|s|}e^{i|s|\theta}\big)+\sum_{\nu=-|s|+1}^{|s|-1} c_\nu e^{-i\nu\theta},$$

which is a consequence of equation (64). To prove the 4–dimensional case, define $\hat u_0^s(e)$ for real $e=(e_0,e_1,e_2,e_3)\in H$ by the r.h.s. of Eq. (69). Recalling that $Y_{s,k}$ is the restriction of a polynomial to the sphere, homogeneous of degree s, it is clear that multiplying $Y_{s,k}(n(e))$ with the factor $(1+e_0^2)^{s/2}$ amounts to restricting the same polynomial to the sphere $H\cap\{e_0=\text{const}\}$. This implies that $\hat u_0^s$ coincides with the r.h.s. of Eq. (68). It remains to show that $\hat u^s$ coincides with u^s, as defined in (63), up to a real factor. To this end, one first checks that $\overline{Y_{s,k}(n(e))}$, and hence $\hat u_0^s(e)$, is a solution to (43). Hence, in view of the uniqueness property, it suffices to show that the s-components coincide up to a real factor. To this end, we write $n\in S^2$ as $n=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$, and have

$$m\; q(n)\cdot e = e_0-\frac{1}{2}\sin\theta\big((e_1+ie_2)e^{-i\phi}+(e_1-ie_2)e^{i\phi}\big)-\cos\theta e_3. \tag{70}$$

This implies

$$(q(n)\cdot e)^s = (-2m)^{-s}\,(\sin\theta)^s e^{is\phi}\,(e_1-ie_2)^s\;+\sum_{k=-s}^{s-1} c_k(\theta)e^{ik\phi}.$$

Using $Y_{s,s}(\theta,\phi)=c_s\,(\sin\theta)^s\,e^{is\phi}$, this yields $u_0^s(e)_s=e^{-i\pi s/2}\,c\,(e_1-ie_2)^s$, But this coincides with $\hat u_0^s(e)_s$ up to a real factor. This completes the proof. □

From the uniqueness statement (ii) in Theorem 3.3, cf. equation (40), we then have the form of the most general intertwiner function:

Proposition 4.3 (The general form of massive intertwiners) *Let F be an analytic function on the upper half plane which is polynomially bounded at infinity and has moderate growth near the reals. Then*

$$u(e,p):=F(e\cdot p)\,u^{|s|}(e,p), \tag{71}$$

where $u^{|s|}$ is given by Eq. (67) or (68), is an intertwiner function, in the sense of Definition 3.1, for mass $m>0$ and spin s.

Conversely, every such intertwiner function is of this form.

Proof. Since the proof of the first statement is straightforward, we only show the "converse" statement. In view of the uniqueness assertion in Theorem 3.3, it only remains to prove the properties of F apart from being a meromorphic function on the upper half plane. We first show that it must be analytic. To this end, let us determine the zeroes of u^s in the four-dimensional case with $s>0$. Firstly, $u^s(e)_s=0$ implies, by

Eq. (68) for $k = s$, that $e_1 = ie_2$. Consider then the 0-component of $u^s(e)$. By Eq. (68), $u^s(e)_{k=0}$ is a sum with one term proportional to $(e_3)^s$, while all other summands contain a factor $(e_1 - ie_2)^n$, $0 < n \le s$. Now these terms vanish due to $e_1 = ie_2$, and therefore $u^s(e)_0 = 0$ implies that $e_3 = 0$. On the other hand, $e_1 = ie_2$ and $e_3 = 0$ obviously imply $u^s(e) = 0$. It follows that $u_0^s(e) = 0$ if and only if e is of the form $(e_0, ie_2, e_2, 0)$. Such e is in H^c if and only if $e_0 = \pm i$, and in $\mathcal{T}_+$ if and only if $e_0 = +i$ and $|e_2|^2 < 1$. In particular, for all zeroes in $\mathcal{T}_+$ holds $\bar{p} \cdot e = im$. Hence the only possible pole of F in the upper half plane is at im. But there are points $e \in \mathcal{T}_+$ with $\bar{p} \cdot e = im$ and $u^s(e) \neq 0$, for example $(i, -ie_2, e_2, 0)$. Hence F may not have a pole at im, and must therefore be analytic on the upper half plane. The same conclusion holds, of course, if $s = 0$, and a similar consideration holds in the three-dimensional case.

To show the boundedness condition on F, note that u_0^s is a polynomial in e, and that B_p is a polynomial in p. Therefore $u^s(e, p) \equiv u_0^s(B_p^{-1}e)$ does not fall off for large p, hence the bound (29) on u implies a similar bound for $F(e \cdot p)$, and it follows that F must be polynomially bounded. Similarly, one concludes that F must have moderate growth near the reals. □

Remarks. 1. The intertwiner leads to point-like localized fields if and only if F is entire, that is, analytic on the complex plane, cf. Proposition 3.4. Note that the boundedness condition then implies that F is a polynomial. This complies well with the fact that the (mass shell restriction of the) momentum space two-point function of a compactly–localized observable is an entire function of p on the complex mass hyperboloid [17] (which coincides with H^c up to a scaling factor m), and in fact a polynomial in the case of a Wightman field [60].

2. By a calculation analogous to (108), one finds that the intertwiners u^s coincide with their "conjugate" intertwiners $(u^s)_c$, as defined in (30). Hence for u as in the proposition, we have

$$u_c(e, p) = \overline{F(-e \cdot p)}\, u(e, p).$$

3. For spin 1 in 4 dimensions, we get an explicit formula for the two-point function of the field corresponding (as in (39)) to u: Namely, from Eq. (68) we have (up to an overall factor) $u_0^1(e)_{\pm 1} = \mp i(e_1 \mp ie_2)$ and $u_0^1(e)_0 = i\sqrt{2}\, e_3$. The above remark then yields

$$\big(\Omega, \varphi(x, e)\varphi(x', e')\Omega\big) = \int d\mu(p) e^{ip\cdot(x'-x)} F(-e \cdot p) F(e' \cdot p) \big\{(e \cdot p)(e' \cdot p) - e \cdot e'\big\}. \tag{72}$$

4.2 String-localized Fields from Point–Fields

In order to obtain a good vantage point for the issue of *point-like fields versus proper strings* it is necessary to remind the reader of the basic results of Wigner's particle-based representation theoretical approach to interaction-free fields and their associated algebras. In case of a massive particle there are intertwiners $v(p)$ which connect the (m, s) irreducible one-particle Wigner representation with wave functions (and their associated quantum fields) transforming under certain finite-dimensional (non-unitary) representations D' of the Lorentz group. More precisely, $v(p)$ is a linear map from the representation space of D' onto the little Hilbert space $\mathbb{C}^{2s+1}$, satisfying

$$D(R(\Lambda,p))\,v(\Lambda^{-1}p) = v(p)\,D'(\Lambda), \tag{73}$$

where $D = D^{(s)}$ denotes the spin s representation of $SO(3)$, as before. The associated quantum field then transforms covariantly under D' [63]:

$$U(a,\Lambda)\,\Phi_r(x)\,U(a,\Lambda)^* = \Phi_{r'}(a+\Lambda x)\,D'(\Lambda)_{r'r} \tag{74}$$

$$\Phi_r(x) = (2\pi)^{-\frac{3}{2}} \int d\mu(p) \sum_{k=-s}^{s} \big\{e^{ipx} v(p)_{k,r} a^*(p,k) + e^{-ipx} v_c(p)_{k,r} b(p,k)\big\}.$$

Here, $v_c(p) := D(i\sigma_2) \circ v(p)$ is the conjugate intertwiner[18], and a, b are the Wigner annihilation operators as before (the self-conjugate situation $b = a$ being always a special case). In fact, it is well-known [32, 63] that for given (m, s) there is a countably infinite number of intertwiners and corresponding covariant fields. Namely, for any two half-integers $A, \dot{B}$ satisfying the restriction

$$\left|A - \dot{B}\right| \leq s \leq A + \dot{B} \tag{75}$$

there is an intertwiner from the (m, s) representation to the representation $D' := D^{(A,\dot{B})}$, the representation in the space of $2A$ undotted and $2\dot{B}$ dotted symmetrized spinors. For given (m, s), the infinitely many different associated fields (and their derivatives) form the linear part of the *Borchers equivalence class* of point-like fields[19].

For the comparison with string-localized fields it is helpful to emphasize the following points.

- The intertwiners v above, as well as their "conjugates" v_c, are determined by the covariant transformation law for the field Φ_r without invoking the quantum requirement of localization — the latter is rather a consequence of covariance (in the free field case). This is, of course, the reason why historically it was possible for Pascual Jordan to kick-start quantum field theory by (Lagrangian) field quantization without having to wait for Wigner's more intrinsic approach that does not rely on classical concepts. The fortunate circumstance that covariance implies localization does not hold any more if the fields have a more general transformation behaviour, e.g. as in Eq. (39) but with D' infinite-dimensional, or as in Eq. (3). An important instance is the case of Wigner's infinite spin representation. In fact, the covariant field equations for these representations found by Wigner [66] have no localization properties. The argument x of his covariant wave function on which the Poincaré group acts covariantly does not admit the interpretation as a point of localization, not even as the end point of a string, if vanishing of quantum mechanical commutators is taken as a criterion for localization. On the other hand, the fields we construct in Section 6 are both covariant and string localized.

[18] The usual notation $u(p)$ would lead to confusion with our notation $u(e, p)$.
[19] The full class is formed by the (Wick-ordered) composites of these fields.

- Only some of the fields in the infinite family indexed by the pairs $A, \dot{B}$ satisfying (75) permit a description in terms of Lagrangian quantization i.e. are associated to an action principle (canonical quantization, functional integration) [64]. For the lowest spins up to $s = 4$ these Lagrangian quantization descriptions of the Wigner approach to massive particles have been explicitly computed (Dirac, Duffin-Kemmer. Rarita-Schwinger for $s = \frac{1}{2}, 1, \frac{3}{2}$) [15]. As emphasized by Weinberg [64] and formalized in the Epstein-Glaser approach [20], one does not need a Lagrangian (but only an interaction polynomial) in order to set up causal perturbation theory[20], a fact which has been confirmed in subsequent work on renormalization theory in the mathematical physics setting.

An apparently pedestrian method to construct genuine string-localized fields (which in fact turns out to be the most general one, cf. below), is to smear a point-like field over a semi-infinite space-like line

$$\Phi(x,e) = \int_0^\infty dt\, f(t) \sum_r \Phi_r(x+te) w(e)_r, \tag{76}$$

where $f(t)$ is supported in the interval $[0, \infty)$ and $w(e)$ is a tensor formed from e which is Lorentz invariant in the sense that

$$D'(\Lambda) w(\Lambda^{-1} e) = w(e), \quad \Lambda \in \mathcal{L}_+^\uparrow. \tag{77}$$

One easily verifies that $\Phi(x,e)$ is string-localized in the sense of Eq. (2) and satisfies the string-covariance condition (3). In fact it is not difficult to see that in agreement with Theorem 3.3, $\Phi(x,e)$ is of the form (39), with intertwiner given by

$$u(p,e) = \tilde{f}(e \cdot p)\, u_{\mathrm{point}}(p,e) \tag{78}$$

$$u_{\mathrm{point}}(p,e)_k = \sum_r v(p)_{k,r} w(e)_r, \quad k = -s, \ldots, s, \tag{79}$$

with $\tilde{f}$, the Fourier transform of f, being analytic in the upper half-plane but not in the whole plane (in which case one falls back to point-like localization[21]).

It turns out that also the converse holds:

Theorem 4.4 (Massive, free string fields are integrals over point fields) *In the massive case* every *string-localized free field can be written as in Eq.* (76)*, i.e. as an integral, along the string, of a point-localized tensor field.*

Proof. First note that $u^s(e,p)$, as given in Lemma 4.2, is of the same form as u_{point}, cf. (79). Namely, Eq. (66) implies that

$$u^s(e,p) = v^s\, D'(B_p^{-1})\, w(e)$$

[20]The reader should be aware that although Weinberg's book contains the broadest exposition of the Wigner representation theory, the underlying philosophy (of lending support to Lagrangian quantization) is very different from that in his previous articles [64] on higher spin fields (and certainly also different from the spirit of the present article).

[21]This follows from the first remark after Proposition 4.3, which asserts that $\tilde{f}$ entire implies that $\tilde{f}$ is a polynomial, hence f has support in a point.

with $w(e)$ the s-fold symmetric tensor power of e, D' the natural representation of the Lorentz group on the symmetric s-tensors, and v^s the intertwiner from $D'|SO(d-2)$ to the irreducible representation with spin s furnished by the matrix $u_k^{\mu_1 \dots \mu_s}$ of Eq. (66). Now $v^s(p) := v^s \circ D'(B_p^{-1})$ satisfies the intertwiner relation (73) and defines a particular point-localized field for spin s. Let now u be the intertwiner corresponding, according to Theorem 3.3, to the given string-localized field $\varphi(e,x)$. Then Proposition 4.3 implies that

$$u(e,p) = F(e \cdot p) u^s(e,p) = F(e \cdot p) \sum_{r=(\mu_1,\dots,\mu_s)} v^s(p)_r \, w(e)_r,$$

where F is analytic in the upper half plane. But this implies that $\varphi(e,x)$ is indeed of the form (76), with f being the inverse Fourier transform of the boundary value of F at $\mathbb{R}$. The properties of F asserted by Proposition 4.3, namely polynomial boundedness at infinity and moderate growth near $\mathbb{R}$, then imply that f has support in the non-negative reals [51, Thm. IX.16]. □

It is interesting to go through the details in the example $s = 1$. Here we take $D'(\Lambda) := \Lambda$ acting in $\mathbb{C}^4$, $w(e) := e$, and intertwiner function $v(p) := V \circ D'(B_p^{-1})$ with $V : \mathbb{C}^4 \to \mathbb{C}^3$ given as $(Ve)_{\pm 1} := e_1 \mp i e_2$, $(Ve)_0 := \sqrt{2} e_3$. The resulting $u_{\rm point}(e,p)$ coincides, again in agreement with our uniqueness statement, with $u^1(e,p) := u_0^1(B_p^{-1} e)$ from equation (68) up to a factor.

4.3 UV–Behavior

We show that the distributional character of our free fields is, in the massive case, less singular than that of the usual point–like free fields, even more so in the direction of the localization string. This fact should lead to a larger class of admissible interactions in a perturbative approach, as compared to taking the standard point-like localized free fields as starting point.

To this end, we determine the large p behavior of the intertwiner function $u^\alpha(e,p)$. We already know that in 3 dimensions it is bounded in p, cf. Proposition 4.1. We now show that the same holds in $d = 4$, and it even falls off in the direction of e. This is a considerable improvement to the point-localized usual free field for spin s, whose intertwiner function goes at least like $|p|^s$. We consider both the 3- and the 4-dimensional case.

Proposition 4.5 (Spin-independent bounds) *i) Let $u^\alpha(e,p)$ be the 4-d intertwiner function defined in Eq.* (61)*, with Re $\alpha =: \alpha' > -1$. Then there is a constant $c > 0$ (depending on m, s and α) such that for all $e \in H$ and $p \in H_m^+$ the following estimate holds:*

$$\|u^\alpha(p,e)\|^2 \le c \left(m^2 + (e \cdot p)^2\right)^{\alpha'}. \tag{80}$$

ii) The 3-d intertwiner function $u^\alpha(e,p)$ satisfies the same estimate (80) *as the 4-d version if $\alpha' > -1/2$.*

Proof. Ad i) By the covariance equation (43), it suffices to consider $u_0^\alpha(e)$ with e of the form $e = (e_0, 0, 0, e_3)$, $e_0^2 - e_3^2 = -1$. Then Eq. (70) implies that

$$m \, q(n) \cdot e = e_0 - \cos\theta e_3 \quad \text{ for } n = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta),$$

and for $k = -s, \ldots, s$ we have

$$|\big(u_0^\alpha(e)\big)_k| \leq c' \int \mathrm{d}\cos\theta \mathrm{d}\phi \, |e_0 - \cos\theta e_3|^{\alpha'} \leq c\, |e_3|^{\alpha'}$$

if $\alpha' > -1$. Using $e_3^2 = 1 + e_0^2$ and $u^\alpha(e,p) = u_0^\alpha(B_p^{-1}e)$ and $(B_p^{-1}e)_0 = e \cdot p/m$, this yields the claim.

Ad *ii)* We write the integrand $q(\theta) \cdot e$ as in Eq. (65) and note that $(e^{i\theta} - z_\pm)^\alpha$ is square integrable if $\alpha' > -1/2$. Then the Cauchy-Schwarz inequality implies

$$|u_0^\alpha(e)|^2 \leq c\, |e_1 - ie_2|^{2\alpha'} = c\,(1 + e_0^2)^{\alpha'}, \tag{81}$$

where $c := 1/4\, m^{-2\alpha'} c_\alpha \, (\int d\theta |e^{i\theta} - 1|^{2\alpha'})^2$. This proves the claim. □

A similar result can be achieved for a general intertwiner of the form as in Proposition 4.3, $u(e,p) = F(e \cdot p)\; u^s(e,p)$, with $u_0^s(e)$ as in equation (67) or (69) (for $d = 3$ or 4, respectively): Namely, the latter equations imply that $\|u_0^s(e)\|^2 = c\,(1 + e_0^2)^s$. With $(B_p^{-1}e)_0 = e \cdot p/m$, this proves the following

Proposition 4.6 (Norm of intertwiner) *Let $u(e,p)$ be as above in $d = 3$ or 4. Then its norm in $\mathbb{C}^{2s+1}$, or modulus in $\mathbb{C}$, respectively, is given as*

$$\|u(p,e)\|^2 = c\,|F(e \cdot p)|^2 \left(m^2 + (e \cdot p)^2\right)^s, \tag{82}$$

where $c > 0$ depends on m and s.

Note that string-localization requires F analytic (only) on the upper half plane, cf. Proposition 4.3. This is compatible with F vanishing at real infinity with any given order, and hence with a bounded norm of u (i.e., good UV behavior) — in contrast to the point-localized case where F must be a polynomial (cf. Remark 1 after Proposition 4.3).

5 String–Localized Fields for Photons

It is well-known that the free electromagnetic field $F_{\mu\nu}$ has a quantized version which complies with the requirements of (point-like) localization, covariance and Hilbert space positivity. Namely, it transforms covariantly according to

$$U(a,\Lambda)\, F_{\mu\nu}(x)\, U(a,\Lambda)^{-1} = F_{\rho\sigma}(a + \Lambda x)\, \Lambda^\rho{}_\mu\, \Lambda^\sigma{}_\nu, \tag{83}$$

and acts on the Fock space over the single particle space of the photon, which is the direct sum of helicity $\lambda = +1$ and $\lambda = -1$ spaces. In order to introduce interactions with matter fields one needs a description in terms of vector potentials. Whereas in the classical setting this is straightforward, it is well-known that the Wigner photon description does not allow a representation in terms of a covariant vector potential. This is the point of departure of the gauge theory formalism: by allowing indefinite metric (and corresponding "ghosts") one embeds the Wigner photon representation into an unphysical formalism which formally maintains the point-like local nature of a vector-potential and its milder short distance property. Of course such a construction would lead into the unphysical blue yonder if at the end of calculations in the presence of interactions one would not return to the physical setting by removing the ghosts, which is accomplished by the BRST formalism. Though this "quantum gauge formalism" has been quite successful, there are several reasons why the gauge formalism in the quantum setting should be considered as a transitory prescription of an incompletely understood physical situation. Firstly, a formulation completely in physical terms seems more desirable. The second observation is that if one invokes the renormalizability requirement as a (formally not yet completely understood) quantum principle, spin=1 interacting theories are nailed down uniquely in terms of one coupling parameter. In particular for interacting massive vector-mesons the necessary existence of additional physical degrees of freedom (usually realized as Higgs mesons [22]) within the perturbative setting is not an input, but follows from consistency [19]. Since this theory selected by the renormalization principle is unique, no further selection by a gauge principle is necessary; in fact the quasi-classical approximation reveals that the classical gauge selection principle follows from the geometrically less beautiful and less understood, but in the long run probably more fundamental quantum renormalization principle.

The present setting of string localization offers a much more mundane ghostfree and covariant description: photons can be described by string-localized vector potentials $A_\mu(x,e)$. These fields, whose construction we describe in the following, are distributions in x and in $e \in H$ transforming as

$$U(a,\Lambda)\, A_\mu(x,e)\, U(a,\Lambda)^* = A_\nu(a+\Lambda x, \Lambda e)\Lambda^\nu{}_\mu. \tag{84}$$

Actually a particular example of these fields has appeared in the literature under the heading of "axial gauge". But the direction e has been considered fixed so that their Lorentz transformation property had to be "regauged" according to

$$U(\Lambda)A_\mu(x,e)U(\Lambda)^* = A_\nu(\Lambda x, e)\Lambda^\nu{}_\mu + \text{gauge term}. \tag{85}$$

As a result of cumbersome divergences at momenta orthogonal to e, the axial gauge became unpopular in perturbative calculations. In the present setting these difficulties are overcome by considering $A_\mu(x,e)$ as a distribution in e, with the nice transformation behavior (84) which had apparently been overlooked. This opens up the possibility of a perturbative, covariant, implementation of interaction, where the weaker localization (in space-like cones) requires new techniques but promises better UV behavior. Here we only describe their construction as free fields; the issue of interactions of string-localized fields will be taken up in a separate paper.

We now define the string-localized vector potential in such a way that its physical nature within the Wigner setting is manifest, as well as the transformation property (84):

$$A_\mu(x,e) := \int_0^\infty dt\, f(t)\, F_{\mu\nu}(x+te)\, e^\nu, \tag{86}$$

where f is supported in $[0,\infty)$. By Maxwell's equations and the antisymmetry of $F_{\mu\nu}$ the vector field $A_\mu(x,e)$ satisfies the Lorentz and axial "gauge" conditions

$$\partial^\mu A_\mu(x,e) = 0, \quad e^\mu A_\mu(x,e) = 0. \tag{87}$$

[22] But if one starts the perturbation with massive vector-mesons these mesons do not possess non-vanishing vacuum expectation values.

It is noteworthy that these conditions are satisfied by *every* free vector field $A_\mu(x,e)$ for photons acting in the physical Hilbert space and transforming as in Eq. (84); hence they cannot be regarded as additional gauge conditions in our context. This fact will be shown in the following proposition.

A distinguished choice for the function f, which yields our "covariant" version of an axial gauge potential, is the Heaviside function. With this choice our A_μ is indeed a potential for $F_{\mu\nu}$: Namely,

$$\partial_\mu A_\nu(x,e) - \partial_\nu A_\mu(x,e) = F_{\mu\nu}(x) \tag{88}$$

holds in the sense of matrix elements between states which are locally generated from the vacuum. This choice also yields a dilatation covariant vector potential. Namely, as is well-known the field strength is covariant under an extension of the representation U of $\mathcal{P}_+^\uparrow$ to the dilatations d_λ, $\lambda > 0$: $U(d_\lambda)F_{\mu\nu}(x)\,U(d_\lambda)^{-1} = \lambda^2 F_{\mu\nu}(\lambda x)$. With $f(t)$ the Heaviside distribution, our vector potential satisfies

$$U(d_\lambda)\,A_\mu(x,e)\,U(d_\lambda)^{-1} = \lambda\,A_\mu(\lambda x, e). \tag{89}$$

However, our potential is not covariant under the entire conformal group, since special conformal transformations take space-like infinity to finite points. Thus, under such transformations the formula (86) changes its form and goes over into an integral over a finite line segment.

It is noteworthy that the scaling behavior (89) implies that the scale dimension of our $A_\mu(x,e)$ is one, whereas that of the field itself is two. Thus, our potential shares with the usual (indefinite metric) potential a better UV-behavior than the field strength.

To get more explicit expressions, let us recall the representation of the field strength in the Fock space over the single particle space of the photon. The latter is the direct sum of helicity $\lambda = +1$ and $\lambda = -1$ spaces, corresponding to the representations $D_\lambda(c, R_\phi) := e^{i\lambda\phi}$ of the little group $G \cong E(2)$. Denoting by $a^*(p,\lambda)$ the creation operator in Fock space, the field strength is given by [63]

$$F_{\mu\nu}(x) = \int_{H_0^+} \mathrm{d}\mu(p) \sum_{\lambda=\pm1} \Big\{ e^{ip\cdot x}\, u(p)_{\lambda,\mu\nu}\, a^*(p,\lambda) + e^{-ip\cdot x}\, \overline{u(p)_{\lambda,\mu\nu}}\, a(p,\lambda) \Big\}, \tag{90}$$

where $u(p)_{\lambda,\mu\nu}$ are the intertwiner functions $u(p)_\pm = i\,p \wedge \hat{e}_\mp(p)$, with $\hat{e}_\pm(p) := B_p\,\hat{e}_\pm$ and $\hat{e}_\pm := (0,1,\pm i,0)$. Then $A_\mu(x,e)$ may be written as

$$A_\mu(x,e) = \int_{H_0^+} \mathrm{d}\mu(p) \sum_{\lambda=\pm1} \Big\{ e^{ip\cdot x}\, u(e,p)_{\lambda,\mu}\, a^*(p,\lambda) + e^{-ip\cdot x}\, \overline{u(e,p)_{\lambda,\mu}}\, a(p,\lambda) \Big\}, \tag{91}$$

with "intertwiner function"

$$u(e,p)_\pm = F(e\cdot p)\, u_{\text{point}}(e,p)_\pm, \tag{92}$$

$$u_{\text{point}}(e,p)_\pm = i\left\{ (\hat{e}_\mp(p)\cdot e)\, p - (p\cdot e)\,\hat{e}_\mp(p) \right\} \quad \in \mathbb{C}^4, \tag{93}$$

where F is the Fourier transform of f. Note that $u(e,p)$ satisfies the intertwiner relation

$$D_\lambda(R(\Lambda,p))u(e,\Lambda^{-1}p)_\lambda = \Lambda^{-1}\,u(\Lambda e,p)_\lambda, \tag{94}$$

which in turn implies the transformation property (84) of $A_\mu(x,e)$ directly. Also the Lorentz and axial gauge conditions (87) follow directly from $p \cdot u(e,p)_\lambda = 0$ and $e \cdot u(e,p)_\lambda = 0$, respectively. Returning to the viewpoint of Section 3, the quantum field $A_\mu(x,e)$ is a distribution in x and $e \in H$ with certain properties, and we now make a statement on its uniqueness, analogous to the one in Theorem 3.3.

Proposition 5.1 (Uniqueness of string-localized vector potential) *Let $A_\mu(x,e)$ be a hermitian string-localized vector field for the free photon (i.e., it creates single photon states from the vacuum) transforming as in Eq.* (84) *and satisfying the Bisognano-Wichmann property. Then the field $A_\mu(x,e)$ satisfies the following.*

i) It is the form (91)*, with $u(e,p)$ as in* (92) *and where F enjoys the following properties. F is holomorphic in the upper half plane, of moderate growth near the reals, polynomially bounded at infinity, and satisfies $\overline{F(-\omega - i0)} = F(\omega + i0)$, $\omega \in \mathbb{R}$. A_μ is point-like localized if and only if F is a polynomial, i.e. its inverse Fourier transform is the delta distribution or a derivative thereof.*

ii) It is a potential for the free field strength $F_{\mu\nu}(x)$ in the sense of Eq. (88) *if, and only if, the Function F is the Fourier transform of the Heaviside distribution, i.e. $F(\omega) = i/\omega$ for $\omega \in \mathbb{R} + i\mathbb{R}^+$.*

iii) It satisfies the Lorentz and axial "gauge" conditions (87).

At this point one might wonder if there is an intertwiner function in the sense of our Definition 3.1, i.e. a *scalar* intertwiner function $u(e,p)_\lambda \in \mathbb{C}$ satisfying the above relation without Λ^{-1} appearing on the right hand side. (The corresponding field would carry no Lorentz index and transform as in (3).) But this is not the case: There is no such intertwiner function $u(e,p)_\lambda$. For the corresponding $u_0(e)_\lambda := u(e,\bar{p})_\lambda$ would be invariant under the restriction of D to the stability subgroup of e, cf. eq. (43). But, as we show in Lemma B 3, this implies $u_0(e) = 0$ unless $\bar{p} \cdot e = 0$, hence $u(e,p)_\lambda$ vanishes by analyticity. One might then wonder why the construction of Section 4.2, cf. eq. (76), does not work. The reason is that in this case there is no (non–trivial) function $w(e)$ which is invariant in the sense of eq. (77) under the representation $D'(\Lambda) := \Lambda \otimes \Lambda$ in $\mathbb{C}^4 \wedge \mathbb{C}^4$ (according to which the electromagnetic field $F_{\mu\nu}(x)$ transforms).

Proof. Ad *i*). As in the proof of *iii*) of Theorem 3.3, one concludes that $A_\mu(x,e)$ is of the form (91) for some $u(e,p)_{\lambda,\mu}$ which satisfies the intertwiner property (94). The Bisognano-Wichmann property implies that $u(e,p)_\lambda$ and is analytic in $\mathcal{T}_+$ with moderate growth near H, and satisfies the self-conjugacy condition

$$j_0\, \overline{u(j_0 e, -j_0 p)_\pm} = u(e,p)_\mp. \tag{95}$$

(Here we have used that the anti-unitary representer of j_0 on the single particle space is given by $(U_1(j_0)\phi)_\pm(p) = \overline{\phi(-j_0 p)_\mp}$.) The proof that $u(e,p)$ is as in Eq. (92) goes analogous to the proof of the uniqueness statement in *ii*) of Theorem 3.3: One first concludes that $u(e,p)$ is fixed by $u(e,\bar{p})$ via the relations

$$u(e,p)_\lambda = B_p\, u_0(B_p^{-1} e)_\lambda, \quad u_0(e)_\lambda := u(e,\bar{p})_\lambda, \tag{96}$$

and u_0 satisfies the intertwining property

$$\Lambda u_0(e)_\pm = e^{\mp i\phi}\, u_0(\Lambda e)_\pm, \quad \text{if } \Lambda = \Lambda(c, R_\phi) \in G. \tag{97}$$

But $u_{\text{point},0}(e)$ (corresponding to $u_{\text{point}}(e,p)$ from Eq. (93)) also satisfies this equation. Lemma B 4 implies that $u_0(e)_\pm$ and $u_{\text{point},0}(e)_\pm$ are linearly dependent for all $e \in H$ with $e_0 \neq e_3$, and hence, by analyticity, for all $e \in H^c$. One then concludes precisely as in the proof of Theorem 3.3 after Eq. (43) that $u_0(e)_\lambda = F(\bar{p}\cdot e)u_{\text{point},0}(e)_\lambda$, where F is analytic on the upper half plane except, possibly, at those $\bar{p}\cdot e$ with $u_{\text{point},0}(e)_\lambda = 0$. But the latter equation is satisfied if and only if e is of the form $(e_0, e_1, \pm ie_1, e_0)$. Now such e satisfies $e\cdot e = 0$ and is not in H^c. Hence $u_{\text{point},0}$ has no zeroes in H^c, and F must be analytic. Finally, one checks that $u_{\text{point}}(e,p)$ satisfies the self-conjugacy condition (95). Then $u(e,p)$ satisfies this condition if and only if $\overline{F(-\omega - i0)} = F(\omega + i0)$, $\omega \in \mathbb{R}$. The statements about moderate growth, polynomial boundedness and on point-like localization follow as in the proofs of Theorem 3.3 and Proposition 4.3.

Ad *ii*). Given any string-localized $A_\mu(x,e)$ as in *i*), define $F_{\mu\nu}(x,e)$ by

$$\partial_\mu A_\nu(x,e) - \partial_\nu A_\mu(x,e). \tag{98}$$

Suppose this field is independent of e. Then it transforms as in equation (83), and the Jost-Schroer-Pohlmeyer theorem implies that it coincides, up to unitary equivalence, with the free field strength $F_{\mu\nu}(x)$ from Eq. (90). It follows that $A_\mu(x,e)$ is a potential for the free field strength $F_{\mu\nu}(x)$ in the sense of Eq. (88) if, and only if, the above expression (98) is independent of e. The latter condition translates to e-independence of the expression

$$p_\mu u(e,p)_{\pm,\nu} - p_\nu u(e,p)_{\pm,\mu} \equiv -iF(e\cdot p)\,(e\cdot p)\big\{\hat{e}_\mp(p)_\mu p_\nu - \hat{e}_\mp(p)_\nu p_\mu\big\}.$$

Clearly, this is independent of e if and only if $F(\omega) = \text{const.}/\omega$. By hermiticity of A_μ and the correct normalization, the constant must equal the imaginary unit i. This proves the claim.

Ad *iii*). As mentioned, one checks that $u_{\text{point}}(e,p)_\pm$ from Eq. (93) is orthogonal to e and to p. Hence, by the uniqueness statement *i*), the same holds for the intertwiner corresponding to the field $A_\mu(x,e)$ at hand. This implies the "gauge" conditions. (A direct argument, without the special intertwiner u_{point}, goes as follows. Equation (97) implies that $u_0(e)$ is an eigenvector for all $\Lambda(c, R_\phi) \in G$ which leave e invariant, with eigenvalue $e^{\mp i\phi}$. Multiplying with e yields that either $\phi = 0$ or $e\cdot u_0(e) = 0$. But the proof of Lemma B 4 shows that for all $e \in H$ with $e_0 \neq e_3$ there is a $\Lambda(c, R_\phi)$ leaving e invariant and which has $\phi \neq 0$. Hence $e\cdot u_0(e) = 0$ for such e, and by analyticity for all $e \in H^c$. The same goes through for p.) □

Similarly constructed string-localized analogs of potentials for point-like "field strengths" can be incorporated into the higher helicity Wigner representations. A particularly interesting case is the string localized metric tensor as the potential for the field strength in the case of helicity 2, the latter being a tensor of rank 4. The answer to the question of whether these objects offer a useful alternative to the gauge formalism (which saves the point-like nature of potentials at the expense of introducing unphysical "ghosts" in intermediate steps) depends on whether it will be possible to extend perturbation theory to include string-like localized fields.

Our string-localized vector-potential construction has an interesting connection with the breakdown of Haag duality for non-simply connected localization regions as pointed out by Leyland, Roberts and Testard in [38]. These authors show that the flux of the electromagnetic field through a torus commutes with every observable localized in the causal complement of the torus, but is not localized in the (causal completion of the) torus. The present viewpoint helps to understand this mismatch. Namely, since $F = dA$, the flux through a torus T can be expressed by an integral of $A_\mu(x, e)$ over the torus and hence is localized in $T'' + \mathbb{R}_0^+ e$, where T'' is the causal completion of T and e can be chosen at will. Given an observable B localized in a double cone $\mathcal{O}$ causally disjoint from the torus, one can choose the direction e such that $T'' + \mathbb{R}_0^+ e$ is causally disjoint from $\mathcal{O}$, and hence the flux commutes with B. (The same can be achieved, of course, if one defines a vector potential as in the classical proof of the Poincaré Lemma via line integrals starting from a common finite base point instead of space-like infinity as in (86).)

6 Massless Infinite Spin Particles

Here we construct a family of intertwiners $u^\alpha(e, p)$ along the lines of Section 3.2 for the massless infinite spin particles, labeled by $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha < 0$. In $d = 4$ (Subsection 6.1), it turns out that for $\mathrm{Re}\,\alpha \in [-2, -\frac{1}{2})$, they have mild UV behavior, namely after smearing with a test function $h \in \mathcal{D}(H)$ they are bounded in p. We also find intertwiners which are *functions* on H, leading to localization in space-like half-cylinders. This improves the result of the abstract analysis [8] which guarantees only localization in space-like cones. In $d = 3$ (Subsection 6.2), and for $\mathrm{Re}\,\alpha \in (-1, 0)$, our intertwiners are uniformly bounded in e and p. This leads to fields which are well-behaved with respect to UV-behavior *and* to localization (in that they are localizable in space-like half-cylinders instead of cones).

We first recall the irreducible representations D of the little group G corresponding to these particle types, and of j_0. For $m = 0$, G is isomorphic to the euclidean group $E(d-2)$. Recall that the irreducible representations of $E(1) = \mathbb{R}$ are labeled by $\kappa \in \mathbb{R}$ and act in $\mathbb{C}$ as $r \mapsto e^{i\kappa r}$, and that the *faithful* irreducible unitary representations of $E(2)$ are labeled by $\kappa \in \mathbb{R}^+$, with $D = D_\kappa$ acting on $\mathfrak{h} := L^2(\mathbb{R}^2, d\nu_\kappa(k))$, where $d\nu_\kappa(k) := \delta(|k|^2 - \kappa^2) d^2k$, according to

$$(D(c, R)u)(k) := e^{ic\cdot k}\, u(R^{-1}k), \quad (c, R) \in E(2). \tag{99}$$

These representations extend to a representation of the semi-direct product of G and j_0 by the anti–unitary involution $D(j_0)$. Namely, $D(j_0)$ is complex conjugation (pointwise in the 4 dimensional case). (We show in Lemma B 1 that $D(j_0)$ indeed satisfies the representation properties $D(j_0)D(\Lambda)D(j_0) = D(j_0\Lambda j_0)$, $\Lambda \in G$.)

We now specify the general construction of Section 3.2. For $m = 0$, and base-point $\bar{p} := (1, 0, 0, 1)$ or $(1, 0, 1)$ in H_0^+, the set Γ of all $q \in H_0^+$ with $q \cdot \bar{p} = 1$, cf. (50), is isometric to the euclidean space $\mathbb{R}^{d-2}$ via the parametrization of Γ given by

$$\xi(r) := \big(\tfrac{1}{2}\big(r^2 + 1\big), r, \tfrac{1}{2}\big(r^2 - 1\big)\big), \quad r \in \mathbb{R}, \qquad d = 3, \tag{100}$$

$$\xi(z) := \big(\tfrac{1}{2}\big(z^2 + 1\big), z_1, z_2, \tfrac{1}{2}\big(z^2 - 1\big)\big), \quad z \in \mathbb{R}^2, \qquad d = 4, \tag{101}$$

where $z^2 := z_1^2 + z_2^2$ (cf. Lemma B 2). The isomorphism ξ from $\mathbb{R}^{d-2}$ onto Γ identifies the action of G in Γ with the action of $E(d-2)$ in $\mathbb{R}^{d-2}$, and $-j_0$ acts as $z \mapsto -z$.

One gets generalized intertwiners $V = V_\kappa$ from the representation $\tilde{D}$, cf. (51), to the irreducible representation $D = D_\kappa$ as

$$V_\kappa v := \tilde{v}(\kappa) = \int_{\mathbb{R}} dr e^{i\kappa r}\, v(\xi(r)), \qquad d = 3, \tag{102}$$

$$(V_\kappa v)(k) := \tilde{v}(k) = \int_{\mathbb{R}^2} d^2 z e^{ik\cdot z}\, v(\xi(z)), \quad |k|^2 = \kappa^2, \qquad d = 4. \tag{103}$$

(This is of course only defined on the dense sets where the restrictions of the Fourier transforms to a fixed value κ or to $|k|^2 = \kappa^2$, respectively, make sense.) Thus, our construction (53) and (54) leads *formally* to the following intertwiners, defined for $e \in \mathcal{T}_+$ and $p \in \dot{H}_0^+$:

$$u^\alpha(e,p) = e^{-i\pi\alpha/2} \int_{\mathbb{R}} dr\, e^{i\kappa r}\, (B_p\xi(r)\cdot e)^\alpha, \qquad d = 3, \tag{104}$$

$$u^\alpha(e,p)(k) = e^{-i\pi\alpha/2} \int_{\mathbb{R}^2} d^2 z\, e^{ik\cdot z}\, (B_p\xi(z)\cdot e)^\alpha, \quad |k| = \kappa, \qquad d = 4. \tag{105}$$

(We have again introduced a factor $e^{-i\pi\alpha/2}$ for later convenience.) Here, α is a complex number with

$$\alpha' := \mathrm{Re}\ \alpha < 0.$$

In the following subsections, we will make these expressions precise and prove that they actually have the properties they should formally have.

6.1 Intertwiners for d=4.

The function $z \mapsto B_p\xi(z)\cdot e$ is a polynomial in z without any real zeroes if $e \in \mathcal{T}_+$, cf. Eq. (A.19). It follows that the integral in Eq. (105) exists and defines a continuous function $u^\alpha(e,p)$ of k. We show in Proposition 6.1 that it has indeed the required properties and that, after smearing with a test function $h \in \mathcal{D}(H)$, it is bounded in p for $\alpha' \in [-2, -1/2)$.

Considering the limit of (105) for e approaching the real "boundary" H, one has to note that the polynomial $B_p\xi(z)\cdot e$ is linear if $e\cdot p = 0$ and quadratic if $e\cdot p \neq 0$, cf. Eq. (A.18). Hence for real $e \in H$ with $e\cdot p = 0$ the integral diverges. On the other hand, if $e\cdot p \neq 0$ then the polynomial is of the form $(z - z_0)^2 - (e\cdot p)^{-1}$, cf. Eq. (A.19). The corresponding pole, for real $e \in H$, of the integrand is therefore integrable iff $\alpha' > -1$. It follows that for $\alpha' > -1$ the singular set on H of the distribution $u^\alpha(e,p)$ consists precisely of those e with $e\cdot p = 0$. This can be cured by multiplying this distribution with a suitable power of $e\cdot p$. Then one ends up with an intertwiner which is a *function* on H, thus leading to localization in space-like half cylinders, and to fields which do not have to be smeared in e. In fact, it turns out that

$$\hat{u}^\alpha(e,p) := (e\cdot p)^2\, u^\alpha(e,p) \tag{106}$$

enjoys the mentioned properties if $-1 < \alpha' < -\frac{1}{2}$.

Proposition 6.1 (Intertwiners for infinite spin, $d = 4$) *$u^\alpha(e,p)$ is an intertwiner function in the sense of Definition 3.1. The "conjugate" intertwiner function $(u^\alpha)_c$ defined in Eq. (30) coincides with $u^{\bar\alpha}$. If $\alpha' \in [-2,-\frac{1}{2})$, then for given $h \in \mathcal{D}(H)$ the norm of $u^\alpha(h,p)$ is bounded in p.*

Further, if $\alpha' \in (-1,-\frac{1}{2})$, then $\hat u^\alpha(e,p)$ as defined in Eq. (106) is an intertwiner function with growth order $N = 0$ in (29), and whose norm is bounded by const.$\times|e\cdot p|^2$.

The rest of this subsection is concerned with the proof of the proposition.

Proof. Let $\Lambda \in G$ correspond to $(c, R_\vartheta) \in E(2)$ under the identification $G \cong E(2)$. Then, for $e \in \mathcal{T}_+$,

$$\begin{aligned} D_\kappa(\Lambda)u^\alpha(e,\bar p)(k) &= e^{ic\cdot k} u^\alpha(e,\bar p)(R_\vartheta^{-1}k) \\ &= e^{-i\pi\alpha/2}\int d^2z e^{ik\cdot z}\left(\xi\left(R_\vartheta^{-1}(z-c)\right)\cdot e\right)^\alpha \\ &= e^{-i\pi\alpha/2}\int d^2z e^{ik\cdot z}\left(\Lambda^{-1}\xi(z)\cdot e\right)^\alpha = u^\alpha(\Lambda e,p)(k). \end{aligned} \tag{107}$$

This implies Eq. (43) and thus establishes the intertwiner property (28). To prove that $(u^\alpha)_c$ coincides with $u^{\bar\alpha}$, we consider

$$\begin{aligned} u^{\bar\alpha}(e,-j_0p)(k) &= e^{-i\pi\bar\alpha/2}\int d^2z e^{ik\cdot z}\left(B_{-j_0p}\xi(z)\cdot e\right)^{\bar\alpha} \\ &= e^{-i\pi\bar\alpha/2}\int d^2z e^{ik\cdot z}\left(-B_p\xi(-z)\cdot j_0e\right)^{\bar\alpha} \\ &= e^{i\pi\bar\alpha/2}\int d^2z e^{-ik\cdot z}\left(B_p\xi(z)\cdot j_0e\right)^{\bar\alpha} = \overline{u^\alpha(j_0e,p)(k)} \\ &= \big(D(j_0)u^\alpha(j_0e,p)\big)(k). \end{aligned} \tag{108}$$

In the second line we have used the fact that $j_0\xi(z) = -\xi(-z)$ and equation (21) to conclude that $B_{-j_0p}\xi(z) = -j_0B_p\xi(-z)$. In the third line we have used the facts that $(-w)^\alpha = e^{i\pi\alpha}w^\alpha$ for $w \in \mathbb{R}+i\mathbb{R}^-$, and that $\bar w^{\bar\alpha} = \overline{w^\alpha}$ for $w \in \mathbb{C}\setminus\mathbb{R}_0^-$. This implies that $(u^\alpha)_c = u^{\bar\alpha}$, as claimed.

As to analyticity, we already know that the integrand in the definition (105) of the intertwiner, is analytic on the tuboid $\mathcal{T}_+$. It turns out that this property survives after the integration, hence $e \mapsto u^\alpha(e,p)(k)$ is analytic, point wise in p and k. This is made rigorous in Lemma A 6. Now Lemma A 5 implies that (for fixed p) the continuous functions $k \to u^\alpha(e,p)(k)$ are dominated by a suitable constant, uniformly for e in a compact set in $\mathcal{T}_+$. It follows that $u^\alpha(e,p)$ is analytic as an $L^2(\mathbb{R}^2, d\nu_\kappa)$-valued function.

The main work in establishing the bound (29) is done in Lemma A 5, where we show that for all $e = e' + ie'' \in \mathcal{T}_+$, $p \in \dot H_0^+$ and k with $|k| = \kappa$ holds

$$|u^\alpha(e,p)(k)| \le c\,|p\cdot e|^{-\alpha'+n-2} + \sum_{\nu=0}^{[n/2]} c_\nu \left(e''^2\right)^{\alpha'-n+\nu+1}\left(p\cdot e''\right)^{-\alpha'+n-\nu-1}|p\cdot e|^{\nu-1}, \tag{109}$$

where n is any natural number strictly larger than $2\alpha'+2$. This estimate implies the bound (29) as follows. Consider the canonical norm in $\mathbb{R}^d$ given by $|e|^2 := e_0^2 + \sum_{k=1}^{d-1} e_k^2$.

Let Θ be a subset of $\mathcal{T}_+$ as in (27). We claim that there are positive constants c_1 and c_2 (depending on Θ) such that for all $e = e' + ie'' \in \Theta$ the following inequalities hold:

$$c_1 |e''|^2 \leq \quad (e'')^2 \tag{110}$$
$$c_1\, p_0\, |e''| \leq \quad p \cdot e'' \quad \leq p_0\, |e''| \tag{111}$$
$$|p \cdot e'| \quad \leq c_2\, p_0. \tag{112}$$

As to the first inequality, note that e'' is contained in the cone $\mathbb{R}_0^+\Omega_2$, cf. (27), which implies that

$$(e''_0)^2 \geq (1+\varepsilon)\, |\underline{e''}|^2,$$

for some $\varepsilon > 0$ depending on Ω_2. Here we have written $|\underline{e''}|^2 := \sum_{i=1}^3 (e'')_i^2$. This implies that $e''^2 \geq \varepsilon |\underline{e''}|^2$ and hence Eq. (110), with $c_1 := (1 + 2/\varepsilon)^{-1}$. Next, note that the Cauchy Schwarz inequality implies that

$$p_0(e''_0 - |\underline{e''}|) \leq p \cdot e'' \leq p_0(e''_0 + |\underline{e''}|) \leq p_0 |e''| \tag{113}$$

holds for $p \in \dot{H}_0^+$. Now $e''^2 = (e''_0 - |\vec{e''}|)(e''_0 + |\vec{e''}|) \leq (e''_0 - |\vec{e''}|)|e''|$, hence Eq. (110) and the l.h.s. of (113) imply the l.h.s. of (111). Similarly, the Cauchy Schwarz inequality implies that $|p \cdot e'| \leq p_0(e'_0 + |\underline{e'}|)$, which proves ineq. (112) since e' has been taken from a compact set. The inequalities (110) to (112) imply that

$$(e''^2)^s \leq c\, |e''|^{2s}, \qquad s < 0, \tag{114}$$
$$(p \cdot e'')^s \leq c\, (p_0 |e''|)^s, \qquad s \in \mathbb{R}, \tag{115}$$
$$|p \cdot e|^s \leq c\, p_0^s, \qquad s \geq 0. \tag{116}$$

Using these inequalities, and $|e''| \leq c$ (which follows from (110) since $(e'')^2 \leq 1$ for $e \in \mathcal{T}_+$), one gets from ineq. (109) the bound

$$|u^\alpha(e,p)(k)| \leq c_n p_0^{-\alpha'+n-2} |e''|^{\alpha'-n} \qquad \text{for } n > 2\alpha' + 2, \tag{117}$$

and hence a similar bound for $\|u^\alpha(e,p)\|$. Choosing n large enough, one concludes that the claimed bound (29) is satisfied, with $M(p) = p_0^{-\alpha'+n-2}$ and growth order N smaller or equal to $n - \alpha'$.

In order to prove boundedness of $u^\alpha(h,p)$ for $\alpha' \in [-2, -\frac{1}{2})$, we consider the best bounds contained in (117), corresponding to the smallest $n > 2\alpha' + 2$. For $\alpha' < -1$ we may take $n = 0$, hence $M(p) = p_0^{-\alpha'-2}$, $-\alpha' - 2 > -1$. For $\alpha' \in [-1, -\frac{1}{2})$, we may take $n = 1$, hence $M(p) = p_0^{-\alpha'-1}$, $-\alpha' - 1 \in (-\frac{1}{2}, 0]$. For $\alpha' \in [-\frac{1}{2}, 0)$, we may take $n = 2$, hence $M(p) = p_0^{-\alpha'}$, $-\alpha' \in (0, \frac{1}{2}]$. Hence, for $\alpha' \in [-2, -\frac{1}{2})$ one has $M(p) = p_0^r$ for some $r \in (-1, 0]$. Then Eq. (A.6) implies that the norm of $u^\alpha(h,p)$ is bounded by p_0^r (times a constant depending on h), hence it is bounded for large p. But increasing n by 1 in the above considerations, one also gets the bound p_0^{r+1}, where $r + 1 \in (0, 1]$, hence the norm is also bounded for small p. This implies that the norm of $u^\alpha(h,p)$ is bounded for given h if $\alpha' \in [-2, -\frac{1}{2})$.

The function $\hat{u}^\alpha$ inherits the intertwiner property (28) and analyticity from u^α. As to the claim on the vanishing growth order for $\alpha' \in (-1, -\frac{1}{2})$, we show in Lemma A 7 that for all $e \in \mathcal{T}_+$, $p \in \dot{H}_0^+$, $k \in \mathbb{R}^2$ with $|k| = \kappa$ the following estimate holds:

$$|u^\alpha(e,p)(k)| \leq c_1\, |p \cdot e|^{-\alpha'-1} + c_2\, |p \cdot e|^{-\alpha'-2}. \tag{118}$$

Inequality (112) then implies that $\|\hat{u}^\alpha(e,p)\| \le c_1 p_0^{-\alpha'} + c_2 p_0^{-\alpha'+1}$ for all $e \in \Theta$, where Θ is a subset of $\mathcal{T}_+$ as in (27). This proves the bound (29), with growth order $N = 0$. Ineq. (118) also implies that the norm of $\hat{u}^\alpha(e,p)$ is bounded by $|e \cdot p|^2$. This completes the proof. □

6.2 Intertwiners for d=3

Recall that we have defined intertwiners, for $e \in \mathcal{T}_+$ and $p \in \dot{H}_0^+$, by

$$u^\alpha(e,p) := e^{-i\pi\alpha/2} \int_{\mathbb{R}} \mathrm{d}r e^{i\kappa r} \, (B_p\xi(r) \cdot e)^\alpha. \tag{119}$$

Here, κ is the real number characterizing the representation of the reals (and hence of the Poincaré group) at hand. We will consider the case

$$\alpha' := \mathrm{Re}\ \alpha \in (-1, 0). \tag{120}$$

We show in Lemma A 8 that for these values of α, $u^\alpha(e,p)$ is a *bounded* function on $\mathcal{T}_+ \times \dot{H}_0^+$. With the same methods as used in the proof of Proposition 6.1, this implies the following facts:

Proposition 6.2 (Intertwiners for 'infinite spin', $d = 3$) *$u^\alpha(e,p)$ is an intertwiner function in the sense of Definition 3.1. It is uniformly bounded in e and p, in particular has growth order $N = 0$ in* (29)*. Further, the "conjugate" intertwiner function u_c defined in Eq.* (30) *coincides with $u^{\bar{\alpha}}$.*

6.3 Compactly Localized Two-Particle States

We now address the important question of the existence of compactly localized observables. If $A = A^*$ is localized in a region $\mathcal{O}$, then $A\Omega$ is modular-localized in $\mathcal{O}$, i.e.

$$A\Omega \in \mathcal{K}(\mathcal{O}),$$

where $\mathcal{K}(\mathcal{O})$ is defined as in eqs. (5) – (7) and (12), with U_1 replaced by its second quantization. This is a consequence of the Bisognano-Wichmann property which holds in our model. Thus the existence of compactly modular-localized vectors is a necessary condition for the existence of compactly localized operators. Now the spaces $\mathcal{K}(\mathcal{O})$ are, in principle, known to us and we can therefore decide whether or not this necessary condition is satisfied. The answer is positive, and we shall, as an example, exhibit two-particle state vectors which are compactly localized in the sense of modular localization. We restrict to the 4-dimensional case.

Let $F \in \mathcal{S}(\mathbb{R})$. We define, for p, q in $\dot{H}_0^+$ and k, l in $\mathbb{R}^2$, a "two–particle intertwiner function"

$$u_2(p,q)(k,l) := \int d^2z d^2w \, e^{i(k\cdot z + l\cdot w)} \, F\left(B_p\xi(z) \cdot B_q\xi(w)\right). \tag{121}$$

A straightforward calculation yields:

Lemma 6.3 *For fixed p and q, the function $(k,l) \mapsto u_2(p,q)(k,l)$ is in $L^2(\mathbb{R}^2, d\nu_\kappa)^{\otimes 2}$ and has the following intertwining property:*

$$\big(D_\kappa(R(\Lambda,p)) \otimes D_\kappa(R(\Lambda,q))\big)\, u_2(\Lambda^{-1}p, \Lambda^{-1}q) = u_2(p,q), \quad \Lambda \in \mathcal{L}_+^\uparrow. \tag{122}$$

We then define, for f_1, f_2 in $\mathcal{S}(\mathbb{R}^4)$, a covariant two–particle wave function $\psi_2(f_1,f_2)$ by

$$\psi_2(f_1,f_2)(p,q;k,l) := (Ef_1 \otimes_{\rm s} Ef_2)(p,q)\; u_2(p,q)(k,l), \tag{123}$$

where again Ef is the restriction of the Fourier transform of f to the mass shell and $\otimes_{\rm s}$ denotes that symmetrized tensor product. These wave functions have the following covariance and localization properties. (We denote by U_2 the two-fold symmetric tensor power of U_1, and define $K_2(\mathcal{O})$ as the intersection of $\mathcal{K}(\mathcal{O})$ with the two-particle space.)

Proposition 6.4 (Localization of two-particle state vectors) *Let f_1 and f_2 be in $\mathcal{S}(\mathbb{R}^4)$.*

i) $\psi_2(f_1,f_2)$ is in the two-particle space $L^2(H_0^+ \times \mathbb{R}^2, d\mu d\nu_\kappa)^{\otimes_{\rm s} 2}$.

ii) Let $g \in \mathcal{P}_+^\uparrow$ and let $j \in \mathcal{P}_+$ be the reflection at the edge of some wedge. Then

$$U_2(g)\,\psi_2(f_1,f_2) = \psi_2(g_*f_1, g_*f_2), \tag{124}$$
$$U_2(j)\,\psi_2(f_1,f_2) = \psi_2(j_*\bar{f}_1, j_*\bar{f}_2). \tag{125}$$

iii) Let f_1 and f_2 be real valued test functions with support in a compact set $\mathcal{O} \subset \mathbb{R}^4$. Then the vector $\psi_2(f_1,f_2)$ is in $K_2(\mathcal{O})$.

Proof. Equation (124) of *ii*) follows from Lemma 6.3. This equation in turn implies that the analyticity properties of $t \mapsto U_{(2)}(\Lambda_W(t))\,\psi_2(f_1,f_2)$ depend entirely on those of $t \mapsto E\Lambda_W(t)_*f_i$, $i=1,2$, and hence of the scalar representation. This implies that $\psi_2(f_1,f_2)$ is in the domain of $\Delta_W^{1/2}$ whenever W contains $\mathcal{O}$, and that

$$\Delta_W^{\frac{1}{2}}\,\psi_2(f_1,f_2) = \psi_2((j_W)_*f_1, (j_W)_*f_2)\,. \tag{126}$$

Together with Eq. (125), this implies that $S_W\psi_2(f_1,f_2) = \psi_2(\bar{f}_1,\bar{f}_2)$ whenever W contains $\mathcal{O}$. This shows *iii*) and completes the proof. □

Since our results on compact localization are not yet conclusive, it may be interesting to comment on how one would proceed to settle this problem. The compact localized wave function in the two-fold tensor product of the indecomposable infinite spin representation suggests to look for an operator of the form

$$B(x,y) = \int d\nu(k)d\nu(l)d\mu(p)d\mu(q)\, e^{ipx+iqy} u_2(p,q)(k,l)a^*(p,k)a^*(q,l) + ... \tag{127}$$
$$\omega_0\left(\left[B(x,y), B(x',y')\right]\right) = 0, \ \text{ if } \ x,y\,\text{space like to } \ x',y', \tag{128}$$

where the pure creation component has to be complemented by its hermitian adjoint and a mixed (normal ordered) a^*a component. The previous construction then guarantees that the bilinear $B(x,y)$ is local within the vacuum state if, as written in the second line, the pair x,y is space like with respect to x',y'. This is a slight generalization of the well-known statement that locality within the vacuum state is an automatic consequence of covariance within the vacuum state. The locality of the mixed contributions on the other hand is equivalent to the vanishing of the matrix elements of the commutator between particle states. The latter property is not guaranteed by covariance and the appropriate tool to decide whether it can be achieved is the validity of the Jost-Lehmann-Dyson representation. As a result of the Fock space structure the problem of locality can be systematically investigated by starting with degree two monomials in Wigner creation/annihilation operators and, in case there is no solution, to extend the calculation to include higher degree polynomials. Although a clarification is important for a physical use of these Wigner representations we will defer a systematic search for local observable sub-algebras to future work.

A negative result would mean that these string-localized fields do not admit local observable sub-algebras, which would lend theoretical support to the idea that "nature cannot make direct use" of such representation. However even if they do not appear as indecomposable asymptotic states in collision theory they might play a more hidden role; unlike tachyons they are positive energy objects and share the stability and localization properties which are common to all positive energy representations.

7 String-Localized Fields and String (Field) Theory

In this section an attempt will be made to compare the concept of string-localized fields with String Theory [50]. Despite the shared use of the word "string" this is not an easy task since the conceptual position of string theory in particle physics, its historical roots in the dual model and its ability in catalyzing new mathematical ideas notwithstanding, is not anywhere close to the firm embedding of string-localization (in the sense of the present paper) in the general principles underlying QFT. Our conclusion will be that contrary to the intuitive appeal of the common word "string", the two concepts have little in common. This said, the reasons behind this negative conclusion are actually quite interesting and worthwhile to be presented. In the present case it turns out that they reveal a lot about the role of classical versus quantum localization and the limits of Lagrangian quantization in particle physics.

An appropriate conceptual understanding of the aims of String (Field) Theory is difficult to gain by looking only at the highly technical actual computations, a glance at its history on the other hand is less confusing. String Theory started with the observation that Veneziano's proposal for a crossing symmetric (but not yet unitary) S-matrix permits an auxiliary description in terms of a classical string Lagrangian. After a classically permitted reformulation and a subsequent canonical quantization this Nambu-Goto Lagrangian reproduces the Veneziano amplitude, including the underlying mass-tower spectrum. However the operator formulation of this auxiliary string theory led to a Poincaré Lie algebra only for very special values of the string's ambient space-time dimension[23]. This result is surprising and goes somewhat against common sense since one would not expect the answer to such a deep problem (why nature selects

[23] There is however apparently no proof in the existing literature that the joint domain properties of the unbounded Lie algebra generators in the light-front quantization allow an exponentiation to a unitary representation of the Poincaré group .

a particular dimension for the space-time vessel for quantum matter) to be obtained simply from the quantization of a classical relativistic string. It was certainly not a restriction in the logic of Veneziano's dual S-matrix theory (crossing and unitarity are not capable to select a preferred space-time dimension). There is of course the alternative to treat the quantization problem of the Nambu-Goto string in compliance with its nature as an integrable system, which would not lead to a limitation in space-time dimension. But it has been shown by Bahns [3] that the result would be inequivalent to that of the canonical quantization (which is the one needed to reproduce the dual model). As the quantization compatible with the integrability, the N-G string and the string-localized fields of the present work share the property to be consistent with any space-time dimension $d \geq 3$. However, as mentioned before there are good reasons to believe that quantized classical relativistic strings do not lead to string-localization in the sense of the present article [16, 21]. The main point of our brief excursion into the history of string theory was to make clear that there is also no reason why a string-S-matrix theory (where the terminology "string" has an auxiliary meaning) should comply with the weakening of compact localization for certain positive energy representations of the Poincaré group. Whereas the auxiliary role of the canonically quantized N-G string in the the dual S-matrix prescription does not impose any physical localization properties (such properties only apply to interpolating fields of an S-matrix), the conjectured absence of string localization in the more intrinsic quantization of the N-G Lagrangian based on conserved charges would be more surprising.

It is a remarkable fact that the solutions to the bootstrap program for two-dimensional purely elastic S-matrices (factorizing models) permit a classification. The existence of an abundance of solutions shows that, contrary to the original expectations, the S-matrix bootstrap principle is in no way more restrictive than the axioms of QFT. In fact the bootstrap-formfactor program associates a unique QFT (in terms of its generalized formfactors) with a given factorizing crossing symmetric unitary S-matrix, and modular localization permits to understand the associated computational recipes on a fundamental level of local quantum physics [36, 56, and earlier papers cited therein]. The bootstrap formfactor approach is an example of a field theoretic construction outside the Lagrangian quantization scheme [2]. In contrast to the dual model and contemporary string theory the bootstrap S-matrices do not contain infinite particle towers, rather the implementation of crossing is achieved through a delicate interplay of one-particle poles with the scattering continuum. The formulation of the bootstrap-formfactor constructions in the setting of modular wedge localization and their relation to the Zamolodchikov-Faddeev algebra structure suggests to view these constructions as a special case of an extension of the ideas underlying the Wigner particle-based representation theory to interacting particles and their local fields.

There are some similarities between string-localized fields and string theory which have attracted our attention at the beginning of our research. The helicity tower of the infinite spin representation but also the Lorentz-spin tower (for fixed physical spin) of massive strings suggests an analogy with the infinite mass tower. Also the improvement of short distance behavior and in particular the somewhat surprising fact that the short distance behavior of e.g. massive string-localized fields is independent of the physical spin (for point-like free fields it gets worse with higher spin) resembles the improvement of the ultraviolet behavior in string theory. But in view of the before-mentioned significant conceptual differences concerning localization we think that these similarities are superficial.

8 Concluding Remarks

In this paper we have analyzed in detail the concept of string localized quantum fields within the setting of free fields. There are two main motives for studying such objects. One reason is their natural occurrance in concrete realizations of the B-G-L theorem [8] which assures the existence of quantum fields localized in space-like cones for all positive energy representations of the Poincaré group. This incorporates Wigner's infinite spin representations as well as the description of photons in terms of a covariant, string-localized vector potential that operates on the physical particle space.

Another motive is the theorem of Buchholz and Fredenhagen [10] which states that in a theory of local observables and massive particles separated by a mass gap, the charge-carrying fields are not worse than string-localized. Understanding the interaction free situation is a necessary preparatory step towards (possibly perturbative) constructions of interacting string-localized objects. Since our free string-fields have milder short-distance behavior of their two-point functions (independent of spin!) than point-fields they potentially widen the framework of perturbatively admissible interactions.

The modular setting and in particular the distinguished role of wedge-localized algebras as the starting objects of an algebraic approach suggest to aim for generators of wedge algebras (even if a simple algebraic characterization in terms of PFGs as in the case of factorizable models is not possible). It is not unreasonable to expect that by specifying an interaction through its lowest order (tree graph) S-matrix one obtains a first order deviation of the modular conjugation J from its free value J_0. The imbalance between the new commutant formed from J-transformed free field generators and the original free field wedge generators would then require a first order correction such that the relation between the modified generators and the commutant is correct up to first oder but violated in the next order. In this way one may arrive at an iterative scheme (for the wedge generators as well as for the S-matrix) not unlike those existing perturbative schemes for the iterative determination of local fields. The fact that the S-matrix enters in the definition of the commutant of the wedge algebra is certainly a restriction on the would-be generators of wedge algebras and whether this can be explored in an iterative scheme and how many iterative solutions one obtains from a lowest order S-matrix input are important unexplored problems. Unlike the standard approach based on the computational use of point-like fields, a construction of wedge generators remains "on-shell" (no short distance correlations) and hence free of ultraviolet problems. It is therefore expected to reveal the true frontiers created by the physical principles of QFT beyond those which are generated by the computational use of unavoidably singular point-like fields (non-/renormalizable).

The proposal to permit string-like interactions is conceptually somewhere between the standard approach and the radical idea of aiming at wedge generators and obtaining improved localizations and their possible string- or point- like field generators via intersections of algebras.

A Proofs

A.1 Proofs for Section 3.

We first establish two geometrical facts which we have used in the proofs of Proposition 3.2 and Theorem 3.3. The first one states a necessary and sufficient condition for a string

$$S_{x,e} := x + \mathbb{R}_0^+ \, e \tag{A.1}$$

to be contained in a wedge (Lemma A 1), and the second one concerns the complexified boosts (Lemma A 2).

Lemma A 1 *i) Let $x \in \mathbb{R}^d$, $e \in H$ and W be a wedge region. The string $S_{x,e}$ is contained in W if and only if $x \in W$ and e is in the closure of W_H.*
ii) If $S_{x,e}$ is causally disjoint from $S_{x',e'}$ then $(x-x')^2 < 0$ and $(e-e')^2 \leq 0$.

(W_H has been defined in Eq. (24).)

Proof. Ad *i*). It suffices to consider $W = W_0$ as defined in Eq. (4), and we suppose $d = 4$. Then $S_{x,e} \in W_0$ if and only if $|x_0 + te_0| < x_3 + te_3$ for all $t \geq 0$. This condition implies that $x \in W_0$ and $e_3 \geq 0$. We may hence assume in the following that $x \in W_0$ and $e_3 \geq 0$, since these are consequences of both conditions whose equivalence we want to establish. We have to show that then $S_{x,e} \subset W_0$ iff e is in the closure of W_0. (Note that $(W_0)_H = W_0 \cap H$.) Under our assumptions, $S_{x,e} \subset W_0$ if and only if for all $t \geq 0$ holds $f(t) > 0$ with

$$f(t) := (x_3 + te_3)^2 - (x_0 + te_0)^2 = (e_3^2 - e_0^2)t^2 + 2(x_3e_3 - x_0e_0)t + x_3^2 - x_0^2. \tag{A.2}$$

Suppose first that $e_3^2 - e_0^2 =: a \neq 0$. Then $f(t)$ is a quadratic polynomial with zeroes $t_\pm = -a^{-1}(x_3 \pm x_0)(e_3 \mp e_0)$. Thus $f(t) > 0$ for all $t \geq 0$ iff $a > 0$ and both zeroes are strictly negative. Since $x \in W_0$ (hence $x_3 \pm x_0 > 0$) by assumption, this is equivalent to $e_3 + e_0 > 0$ and $e_3 - e_0 > 0$, hence to $e \in W_0$. Suppose now that $e_3^2 - e_0^2 = 0$. We show that this implies both $f(t) > 0$ and $e \in W_0^-$. Namely, $e_3^2 - e_0^2 = 0$ implies that $f(t) = 2e_3(x_3 \pm x_0)t + x_3^2 - x_0^2$ is strictly positive since $x \in W_0$ and $e_3 \geq 0$. But $e_3^2 - e_0^2 = 0$ (together with the hypothesis $e_3 \geq 0$) also implies $e_3 = |e_0|$, i.e. $e \in \partial W_0$. This completes the proof of *i*). Ad *ii*). The hypothesis implies [8] that there is a wedge W such that $S_{x,e} \subset W$ and $S_{x',e'} \subset W'$, where W' denotes the causal complement of W. By *i*), it follows that $x \in W$, $x' \in W'$, $e \in W_H^-$ and $e' \in (W_H')^-$. This implies that claim. □

Lemma A 2 *i) Every point in the complexified H^c is of the form $z = \Lambda_W(i\theta)e$, where W is some wedge, $e \in H$ and $\theta \in [0, \pi)$.*
ii) Every point in $\mathcal{T}_+$ is of the same form, but with $e \in W$ and $\theta \in (0, \pi)$.

Proof. Let us first recall that $z = x + iy \in H^c$ if and only if $x^2 - y^2 = -1$ and $x \cdot y = 0$. Note that, by the latter condition, $y^2 > 0$ implies that x is space-like or zero and hence $y^2 \leq 1$.

Ad *i*). Clearly, $z = \Lambda_W(i\theta)\hat{e}$ iff $\Lambda z = \Lambda_{W_0}(i\theta)e$, where $\Lambda \in \mathcal{L}_+^\uparrow$ is such that $\Lambda W = W_0$, the standard wedge (4), and where $e = \Lambda\hat{e}$. One calculates

$$\Lambda_{W_0}(i\theta)e \equiv \big(\cos(\theta)e_0, e_1, e_2, \cos(\theta)e_3\big) + i\sin(\theta)\,(e_3, 0, 0, e_0). \tag{A.3}$$

We have to show that for every $z \in H^c$ there are $\Lambda \in \mathcal{L}_+^\uparrow$, $e \in H$ and $\theta \in [0,\pi)$ such that Λz coincides with the above vector. We denote this vector by $\bar{x} + i\bar{y}$. We first claim that for our given $z = x + iy \in H^c$ on can choose $\theta \in [0,\pi)$ and e so that $\bar{x}$ is in the same $\mathcal{L}_+^\uparrow$-orbit as x, and $\bar{y}$ is in the same $\mathcal{L}_+^\uparrow$-orbit as y.

This can be achieved as follows. Case 1: $y^2 > 0$, $y_0 \gtrless 0$. Then $0 < y^2 \le 1$ (see above), hence $y^2 = \sin^2\theta$ and $x^2 = -\cos^2\theta$ for some $\theta \in (0,\pi)$. (Note that $y^2 = 1$ implies $x \equiv 0$.) Putting $e := (0,0,0,\pm1)$ yields $\bar{x} = \cos(\theta)(0,0,0,\pm1)$ and $\bar{y} = \sin(\theta)(\pm1,0,0,0)$, hence does the job. Case 2: $y^2 < 0$. Then $y^2 = -\sinh^2\chi$ and $x^2 = -\cosh^2\chi$ for some $\chi \in \mathbb{R}$. Putting $e := (\sin\chi, \cosh\chi, 0, 0)$ and $\theta := \pi/2$ yields $\bar{x} = (0, \cosh\chi, 0, 0)$ and $\bar{y} = (0,0,0,\sinh\chi)$, hence does the job. Case 3: $y^2 = 0$, $y_0 \gtrless 0$ and $x^2 = -1$. Putting $e := (1,1,0,\pm1)$ and $\theta := \pi/2$ yields $\bar{x} = (0,1,0,0)$ and $\bar{y} = (\pm1,0,0,1)$, hence does the job. In the remaining case $y \equiv 0$ nothing has to be shown.

With this choice of θ and e there is, in particular, some Λ_1 such that $\Lambda_1 x = \bar{x}$. Suppose we can find some Λ_2 which leaves $\bar{x}$ invariant and maps $\Lambda_1 y$ to $\bar{y}$. Then $\Lambda := \Lambda_2\Lambda_1$ satisfies $\Lambda z = \bar{x} + i\bar{y} \equiv \Lambda_{W_0}(i\theta)e$, as claimed. It remains to prove the existence of such Λ_2. Since y is orthogonal to x, $\Lambda_1 y$ is orthogonal to $\Lambda_1 x \equiv \bar{x}$. Suppose first that $\bar{x}^2 < 0$. Then its orthogonal complement $\bar{x}^\perp$ is a three-dimensional Minkowski space, and the stability group, in $\mathcal{L}_+^\uparrow$, of $\bar{x}$ is the corresponding Lorentz group. It acts transitively on the intersection of $\bar{x}^\perp$ with the $\mathcal{L}_+^\uparrow$-orbit of (any given) $\bar{y}$. Hence there is a Λ_2 with the mentioned properties. The only other case is $x = \bar{x} = 0$ (see above), which is trivial.

Ad *ii*). Note that $\hat{e}$ from above Eq. (A.3) is in W iff $e \in W_0$. Thus we only have to show that in the above argument we can choose $e \in W_0$ and $\theta \in (0,\pi)$. But this has been achieved above, cf. case 1. This completes the proof. $\square$

The rest of this subsection is devoted to the proof of Proposition 3.2, starting with two lemmas. They concern the properties of $u(h,p)$, defined in Eq. (31).

Lemma A 3 *i) Let h be a smooth function with compact support in some "wedge region" W_H.*[24] *Then for almost all fixed p the $\mathfrak{h}$-valued function*

$$t \mapsto u(\Lambda_W(t)_* h, p) \tag{A.4}$$

is the boundary value of an analytic function on the strip $\mathcal{G}$, which is weakly continuous on the closure $\mathcal{G}^-$ and satisfies the boundary condition

$$u(\Lambda_W(i\pi)_* h, p) = u((j_W)_* h, p). \tag{A.5}$$

Further, for given compact subsets Ω of H and $\mathcal{G}_0$ of $\mathcal{G}^-$, there is some $c > 0$ such that for all h with $\operatorname{supp} h \subset \Omega$ *and $z \in \mathcal{G}_0$ holds*

$$\|u(\Lambda_W(z)_* h, p)\| \le c\, M(p)\, \mathrm{p}_\Omega(h), \tag{A.6}$$

[24] W_H has been defined in (24).

where M is the dominating function from (29)*, and* p_Ω *is the semi-norm on* $\mathcal{D}(\Omega)$ *defined by* $\mathrm{p}_\Omega(h) = \sum_{|\alpha|\le N+1} \|\partial^\alpha h\|_\infty$*, N as in* (29)*.*

ii) If the growth order N of $e \to u(e,p)$ in (29) *is zero, then the analogous statements hold with the appropriate replacements $h \to e$, $g_* h \to ge$.*

(Note that the estimate (A.6) is claimed to hold also for $z = 0$, i.e. for $u(h,p)$.)

Proof. We shall make use of some details on the entire matrix-valued function $z \mapsto \Lambda_W(z)$, $z \in \mathbb{C}$. Namely, it satisfies

$$\Lambda_W(t+it') = \Lambda_W(t)\big(j_W(t') + i\sin(t')\,\sigma_W\big)\,, \tag{A.7}$$

where $j_W(t') = \frac{1}{2}\cos t'(1-j_W) + \frac{1}{2}(1+j_W)$ continuously deforms the unit to j_W when t' runs through $[0,\pi]$, and σ_W maps the wedge W continuously into the interior of the forward light cone, cf. [28]. This implies that for $e \in W_H$, the function $z \to \Lambda_W(z)e$ is analytic on $\mathcal{G}$.[25] Moreover, if e and z are in some compact subsets $\Omega \subset W_H$ and $\mathcal{G}_0 \subset \mathcal{G}^-$, respectively, then $\Lambda_W(z)e$ is in some Θ of the form (27). Hence, the bound (29) implies that there is a constant $N \ge 0$, a function $M(p)$ (locally L^2 and polynomially bounded) and $c = c_{\Omega,\mathcal{G}_0}$ such that for all $e \in \Omega$ and $z \in \mathcal{G}_0$ holds

$$\|u(\Lambda_W(z)e,p)\| \le c\,M(p)\,\sin(t')^{-N}, \tag{A.8}$$

where $t' := \mathrm{Im}\ z$.

Let now $h \in \mathcal{D}(H)$ be as in the Proposition, and fix p. For $z \in \mathcal{G}$, denote

$$F(z) := u(\Lambda_W(z)_* h, p) \tag{A.9}$$

$$= \int_W d\sigma(e)h(e)f(z,e) \quad \text{ with } f(z,e) = u(\Lambda_W(z)e,p). \tag{A.10}$$

Note that the integration variable e is in W_H, hence $f(\cdot,e)$ is analytic on the strip $\mathcal{G}$, as noted after (34). Eq. (A.8) guarantees the existence of a majorizing function for all z in a given compact subset of $\mathcal{G}$. This shows that $F(z)$ is analytic on $\mathcal{G}$. Note that Eq. (A.7) implies that $\Lambda_W(t+it')e$ approaches $\Lambda_W(t)e$ from inside the tuboid $\mathcal{T}_+$ if $t' \to 0^+$, hence $F(t+it')$ approaches $F(t)$ *by definition*, cf. the remark after equation (31). This implies that F is continuous on the lower boundary $\mathbb{R}$ of the strip. We consider now the limit of $F(t+it')$ for $t' \to \pi^-$. Note that equation (A.7) implies that $\Lambda_W(t+i\pi) = \Lambda_W(t)j_W$. Hence, for $e \in W$, $\Lambda_W(t+it')e$ approaches $\Lambda_W(t)j_We$ from $\mathcal{T}_+$. Again, it follows that by definition $F(t+it')$ approaches $u((\Lambda_W(t)j_W)_*h,p)$ as $t' \to \pi^-$. This implies equation (A.5) and continuity of F on the upper boundary $\mathbb{R}+i\pi$ of the strip. It remains to prove the bound (A.6). If $\mathcal{G}_0$ is in the interior of the strip $\mathcal{G}$, then the estimate (A.8) immediately implies that $|F(t+it')| \le c\,M(p)\,(\sin t')^{-N}\int_\Omega |h|$. This implies (A.6), since $\sin t'$ is bounded away from zero. We now discuss the boundaries of $\mathcal{G}$, considering first the lower boundary $\mathbb{R}$, namely $t' \in [0,1]$. To this end, we control $|F(t+it')|$ in the limit $t' \to 0^+$, following standard arguments, cf. [58, Thm. 2-10] and [51, Thm. IX.16].

We first introduce Lagrangian coordinates on W_H as follows. The flow of $\Lambda_W(t)$ on W_H is time-like and complete. Hence, $\Sigma := \{0\} \times S^{d-2} \cap W_H$ is a Cauchy surface for

[25] $\Lambda_W(z)e$ refers to the action of $\mathcal{P}_+^\uparrow$ on H defined in (22).

W_H and every point in W_H is of the form $e = \Lambda_W(\tau)\hat{e}$ for some unique $\tau \in \mathbb{R}$, $\hat{e} \in \Sigma$. Putting $\phi : e \mapsto (\tau, \hat{e})$ establishes a diffeomorphism $\phi : W \to \mathbb{R} \times \Sigma$. We now observe that $f(z, \Lambda_W(\tau)\hat{e}) = f(z + \tau, \hat{e})$ for $z \in \mathcal{G}, \tau \in \mathbb{R}$, and get

$$F(z) = \int_{\mathbb{R}\times\Sigma} d\tau d\Sigma(\hat{e})\, f(z + \tau, \hat{e})\, \hat{h}(\tau, \hat{e}).$$

Here $d\Sigma$ denotes the canonical volume form on $\Sigma \cong S^{d-2}$, and $\hat{h}(\phi(e)) = h(e)\frac{\phi_* d\sigma}{d\tau d\Sigma}$ (where $\frac{\phi_* d\sigma}{d\tau d\Sigma}$ denotes the Radon Nikodym derivative). With the same method as in [51, Thm. IX.16], one now shows that for $t \in \mathbb{R}$, $t' \in (0, 1]$, $\nu \geq 2$ the bound (A.8) implies the following estimate:

$$|F(t + it_1)| \leq c'\, M(p) \Big\{ \|(\partial_\tau)^{\nu-1}\hat{h}\|_\infty \int_{t_1}^{1} dt_2 \ldots \int_{t_{\nu-1}}^{1} dt_\nu\, t_\nu^{-N} + \sum_{j=0}^{\nu-2} \|(\partial_\tau)^j\, \hat{h}\|_\infty\, |P_j(t_1)| \Big\}.$$

Here $\|\cdot\|_\infty$ denotes the supremum norm, and P_j is a polynomial (of degree j). Let now ν be strictly larger than $N + 1$. Then the multiple integral over t_ν^{-N} has a finite limit for $t_1 \to 0$, and hence extends to a continuous function of t_1 on $[0, 1]$, which we denote by $P_{\nu-1}$. Then the above inequality reads

$$|F(t + it_1)| \leq c'\, \hat{c}(t_1)\, M(p) \sum_{j=0}^{\nu-1} \|(\partial_\tau)^j\, \hat{h}\|_\infty, \quad t_1 \in [0, 1],$$

where $\hat{c}(t_1) := \max_{j=0\ldots\nu-1} |P_j(t_1)|$.

This proves the claimed bound (A.6) near the lower boundary $z = t \in \mathbb{R}$.

For z near the upper boundary, $\mathbb{R} + i\pi$, one may write $\lim_{t'\to\pi^-} \Lambda_W(t + it') = \lim_{\varepsilon\to 0^+} \Lambda_W(t - i\varepsilon) j_W$ and apply an analogous argument. This completes the proof of *i*). *ii*) is shown analogously. $\square$

We now show that the distribution $u(h, p)$ inherits the covariance properties (28) from its defining analytic function $u(e, p)$.

Lemma A 4 *The family $u(h, p)$ has the following intertwining properties:*

$$D(R(\Lambda, p))\, u(h, \Lambda^{-1}p) = u(\Lambda_* h, p), \quad \Lambda \in \mathcal{L}_+^\uparrow, \tag{A.11}$$

$$u_c(h, p) = D(j_0)\, u((j_0)_*\bar{h}, -j_0 p), \tag{A.12}$$

where $(\Lambda_* h)(e) := h(\Lambda^{-1}e)$.

Proof. We choose a continuous map $\phi : H \times [0, \varepsilon) \to H^c$ such that $\phi(e, t) \in \pm\mathcal{T}_+$ for $t \gtrless 0$, respectively, and $\phi(e, 0) = e$ for each $e \in H$. Then, as remarked after Eq. (31),

$$u(h, p) = \lim_{t\to 0^+} \int d\sigma(e)\, h(e)\, u(\phi(e, t), p).$$

Equation (A.11) is a straightforward consequence of the covariance (28) and the fact that $(e, t) \mapsto \Lambda\phi(\Lambda^{-1}e, t)$ satisfies the same conditions as ϕ. Further, by definition (30) of u_c we have

$$
\begin{aligned}
u_c(h,p) &= \lim_{t\to 0^-} \int d\sigma(e)\, h(e)\, u_c(\phi(e,t),p) \\
&= \lim_{t\to 0^-} \int d\sigma(e)\, h(e)\, D(j_0)\, u(j_0\phi(e,t),-j_0p) \\
&= \lim_{t\to 0^+} \int d\sigma(e)\, h(j_0e)\, D(j_0)\, u(j_0\phi(j_0e,-t),-j_0p) = D(j_0)\, u((j_0)_*\bar{h},-j_0p).
\end{aligned}
$$

In the last equation we have used that $(e,t)\mapsto j_0\phi(j_0e,-t)$ has the same properties as ϕ. This completes the proof of the lemma. □

We are now prepared to prove Proposition 3.2.

Proof of Proposition 3.2. Ad 0). $\psi(f,h)\in\mathcal{H}_1$ follows from the bound (A.6) for $t=0$. As to the "single particle Reeh-Schlieder" property, note that the span of $\{u_0(e), e\in H\}$ carries a representation of the little group due to Eq. (43). Since D is irreducible, this set spans the little Hilbert space $\mathfrak{h}$. By going over to the Lie group (or using analyticity of u_0), the same holds if one restricts e to some open neighborhood U. This implies the Reeh-Schlieder property straightforwardly.

i) is a straightforward consequence of Lemma A 4, the equations $e^{ia\cdot p}(Ef)(\Lambda^{-1}p) = (E(a,\Lambda)_*f)(p)$ and $\overline{(Ef)(-j_0p)} = (E(j_0)_*\bar{f})(p)$, and the fact that every $j\in\mathcal{P}_+^{\downarrow}$ is of the form gj_0g^{-1}.

Ad *ii*). By Lemma A 1, the condition $\mathcal{O}+\mathbb{R}_0^+\Omega\subset W$ holds if and only if $\mathcal{O}\subset W$ and Ω is contained in the *closure* of W_H. Suppose first that ($\mathcal{O}\subset W$ and) Ω is contained in W_H. We shall consider the $\mathcal{H}_1$-valued function

$$
t\mapsto\psi_t := U_1(\Lambda_W(t))\,\psi(f,h)\,,\quad t\in\mathbb{R}. \tag{A.13}
$$

It follows from the covariance equation (35) that

$$
\psi_t(p) = \psi_t^0(p)\, u(\Lambda_W(t)_*h,p)\,,\quad \psi_t^0 := E\Lambda_W(t)_*f. \tag{A.14}
$$

It is well-known that for almost all p, $t\mapsto\psi_t^0(p)$ extends to an analytic function on the strip $\mathcal{G}$. The analyticity statement of Lemma A 3 then implies that for almost all p the $\mathfrak{h}$-valued function $z\mapsto\psi_z(p)$ is analytic on $\mathcal{G}$ and weakly continuous on $\mathcal{G}^-$. Further, it is well-known [45, Eq. (4.58)] that for any given compact subset $\mathcal{G}_0$ of $\mathcal{G}^-$, one can find a dominating function Ψ^0 of fast decrease in p such that

$$
|\psi_z^0(p)| < \Psi^0(p),\quad z\in\mathcal{G}_0,\, p\in\dot{H}_0^+. \tag{A.15}
$$

The bound (A.6) then implies that $\Psi^0(p)M(p)$ is a dominating function for ψ_z. These facts imply that $z\mapsto\psi_z$ is analytic on $\mathcal{G}$ as a $\mathcal{H}_1$-valued function, and weakly continuous on $\mathcal{G}^-$.

It follows from these facts that ψ_0 is in the domain of $\Delta_W^{\frac{1}{2}}$, and that $\Delta_W^{\frac{1}{2}}\psi_0=\psi_{i\pi}$, cf. Lemma A 9. But equation (A.5) and the equation $\psi_{i\pi}^0 = E(j)_*f$, which holds as a consequence of $\Lambda_W(i\pi)=j_W$, imply that

$$
\psi_{i\pi} = \psi((j_W)_*f,(j_W)_*h)\,. \tag{A.16}
$$

Hence we have shown that $\Delta_W^{\frac{1}{2}}\psi(f,h)$ coincides with the r.h.s. of the above equation. This implies, by Eq. (36), that S_W acts as in equation (37) of the Proposition. It remains to show that this equation holds also if Ω is only contained in the closure of W_H. But then there is a sequence of vectors ψ_n of the above form (i.e. for which (37) holds), which converges to $\psi(f,h)$. (ψ_n may be constructed via a suitable curve in the Poincaré group, or from functions h_n with support bounded away from the boundary of W_H.) Since S_W is a closed operator, Eq. (37) is also valid for $\psi(f,h)$. This completes the proof of *ii*).

iii) is shown in complete analogy. □

A.2 Proofs for Section 6.

The proof of Proposition 6.1 makes use of the Lemmas A 5 through A 7, which we now state and prove.

Lemma A 5 *Let $n \in \mathbb{N}_0$ be strictly larger than $2\alpha' + 2$. Then there are constants a_ν, b_ν, c_ν, $\nu = 0 \ldots [n/2]$, such that for all $e = e' + ie'' \in \mathcal{T}_+$, $p \in \dot{H}_0^+$, and k with $|k| = \kappa$ the following estimate holds:*

$$|u^\alpha(e,p)(k)| \le c\,|p\cdot e|^{-\alpha'+n-2} + \sum_{\nu=0}^{[n/2]} c_\nu\,(e''^2)^{\alpha'-n+\nu+1}\,(p\cdot e'')^{-\alpha'+n-\nu-1}\,|p\cdot e|^{\nu-1}. \tag{A.17}$$

(We have written $e''^2 := e''\cdot e''$.)

Proof. We write the scalar product in Minkowski space as $x\cdot p = \frac{1}{2}(x_+p_- + x_-p_+) - x_1p_1 - x_2p_2$, where $x_\pm \doteq x_0 \pm x_3$. Now for $z \in \mathbb{R}^2$, the components of $\xi(z)$ are $\xi(z)_+ = z^2$, $\xi(z)_- = 1$, $\xi(z)_1 = z_1$ and $\xi(z)_2 = z_2$. Further, $(B_p^{-1}e)_- = \bar{p}\cdot B_p^{-1}e = p\cdot e$. We therefore have

$$\begin{aligned} B_p\xi(z)\cdot e &= az^2 + b\cdot z + c, \quad \text{with} \\ a &= \frac{1}{2}(p\cdot e), \\ b &= -\big((B_p^{-1}e)_1, (B_p^{-1}e)_2\big) \in \mathbb{C}^2, \\ c &= \frac{1}{2}(B_p^{-1}e)_+ . \end{aligned} \tag{A.18}$$

Here, $b\cdot z$ denotes the standard scalar product in $\mathbb{R}^2$, and $z^2 := z\cdot z$. (Below we shall adapt the same notation for vectors in $\mathbb{C}^2$ by *bilinear* extension: $(z_1,z_2)\cdot(w_1,w_2) := z_1w_1 + z_2w_2$.) Taking account of $4ac - b^2 = e^2 = -1$, and of $2a = p\cdot e > 0$, we have

$$B_p\xi(z)\cdot e = a(z + b/(2a))^2 - 1/(4a). \tag{A.19}$$

We denote the real and imaginary parts of $b/(2a)$ by w' and w'', respectively. Then we have, after substituting $z + w' \to z$,

$$u^\alpha(e,p)(k) = e^{-i(\pi\alpha/2 + k\cdot w')} \int_{\mathbb{R}^2} d^2z e^{ik\cdot z}\big(P(z)\big)^\alpha, \quad P(z) := a(z + iw'')^2 - \frac{1}{4a}. \tag{A.20}$$

To evaluate this integral, we shall assume that the vector k points in 1-direction, so that $k \cdot z = \kappa z_1$. (The general case is obtained by replacing b in Eq. (A.18) Rb, where $R \in SO(2)$ rotates $\kappa(1,0)$ into k.) n-fold partial integration then yields

$$u^\alpha(e,p)(k) = c \int d^2 z e^{ik\cdot z} (\frac{\partial}{\partial z_1})^n \, (P(z))^\alpha \,, \tag{A.21}$$

and we shall use that

$$\partial_{z_1}^n P(z)^\alpha = \sum_{\nu=0}^{[n/2]} c_\nu \, a^{n-\nu} P(z)^{\alpha-n+\nu} (z_1 + i w_1'')^{n-2\nu}, \tag{A.22}$$

where c_ν is independent of a, b, c and z.

We now establish bounds on $P(z)$. First, note that the imaginary parts of $B_p\xi(z)\cdot e$ and of a are strictly positive, since $e \in \mathcal{T}_+$, i.e. e'' is in the interior of the forward light cone. In particular, we have from equation (A.18)

$$\text{Im } (B_p\xi(z)\cdot e) = a''(z + \frac{b''}{2a''})^2 + d \,\geq\, d, \quad d := \frac{e''\cdot e''}{4a''} > 0, \tag{A.23}$$

where $b'' \in \mathbb{R}^2$ and c'' denote the real parts of b and c, respectively. (We have used that $4a''c'' - (b'')^2 = e''\cdot e''$.) Since $P(z)$ is by definition just $B_p\xi(z - w')\cdot e$, it follows that $\text{Im } P(z) \geq d$. Secondly, we observe that

$$|P(z)| \geq |a| \left|\text{Re } \big((z + iw'')^2 - \frac{1}{4a^2}\big)\right| = |a|\,|z^2 - \rho|, \quad \rho := w''^2 + \text{Re } \big(\frac{1}{4a^2}\big). \tag{A.24}$$

It follows that

$$|P(z)| \geq \frac{1}{2}\big(|a|\,|z^2 - \rho| + d\big). \tag{A.25}$$

Let now $s := \alpha' - n + \nu$ and $m := n - 2\nu$. Note that $s < -1$ and $m \geq 0$. By Eq. (A.25), we have

$$\int d^2 z |P(z)|^s |z_1 + iw_1''|^m \leq c \int_0^\infty dr\, r\, \big(|a|\,|r^2 - \rho| + d\big)^s \, \big(r^m + \|w''\|^m\big), \tag{A.26}$$

where $\|w''\| := (w''\cdot w'')^{1/2}$ denotes the euclidean norm of $w'' \in \mathbb{R}^2$. We split the integral into $r < |a|^{-1}$ and $r > |a|^{-1}$. We show, at the end of this proof, that

$$|\rho| \leq \frac{1}{2}|a|^{-2} \;\text{ and }\; \|w''\| \leq \frac{1}{2}|a|^{-1}. \tag{A.27}$$

Hence for $r > |a|^{-1}$ holds $r^2 - \rho \geq \frac{1}{2}\, r^2$ and $\|w''\| < r$, and hence the integral in (A.26) over $r > |a|^{-1}$ is bounded by

$$2^{-s}\, |a|^s \int_{|a|^{-1}}^\infty dr\, r^{2s+m+1} = c'\, |a|^{-s-m-2}. \tag{A.28}$$

(Note that $2s + m + 1 = 2\alpha' - n + 1 < -1$.) The integral in (A.26) over $r < |a|^{-1}$ is bounded by

$$(|a|^{-m} + \|w''\|^m) \int_0^{|a|^{-1}} dr\, r\, \big(|a|\,|r^2 - \rho| + d\big)^s \leq 2(|a|^{-m} + \|w''\|^m)\, |a|^{-1}\, d^{s+1}. \tag{A.29}$$

Putting together eqs. (A.26), (A.28) and (A.29), and using the fact that

$$\|w''\| \leq \frac{a''}{2|a|^2}, \tag{A.30}$$

which we show at the end of this proof, we have

$$\int d^2z|P(z)|^s|z_1 + iw_1''|^m \leq c_1\,|a|^{-s-m-2} \,+\, c_2\,|a|^{-m-1}\,d^{s+1} \,+\, c_3\,(a'')^m|a|^{-2m-1}\,d^{s+1}.$$

Putting this inequality into Eq.s (A.21) and (A.22), one gets the claimed estimate (A.17), if one recalls that $a = p\cdot e/2$ and $d = (e'')^2(4a'')^{-1}$, and that $|p\cdot e|^{-n+2\nu} \leq (p\cdot e'')^{-n+2\nu}$.

It remains to prove equations (A.27) and (A.30). Denoting $B_p^{-1}e =: x+iy \in (\mathbb{R}^4 + i\mathbb{R}^4)\cap\mathcal{T}_+$, we have $2a = x_- + iy_-$ and $-b = \vec{x} + i\vec{y}$, where we have written e.g. $\vec{x} := (x_1, x_2) \in \mathbb{R}^2$. With this notation, one finds Im $(\bar{a}b) = x_-\vec{y} - y_-\vec{x}$, and

$$\|\text{Im } (\bar{a}b)\|^2 = -y^2x_-^2 - x^2y_-^2 \;\leq\; y_-^2 \equiv (a'')^2,$$

where the inequality holds because $y^2 = 1 + x^2 > 0$ for $e \in \mathcal{T}_+$. Since $\|\text{Im } b/a\| = |a|^{-2}\|\text{Im } \bar{a}b\|$, it follows that

$$\|w''\| \equiv \|\text{Im } \frac{b}{2a}\| \leq \frac{a''}{2|a|^2}, \tag{A.31}$$

which is (A.30). Now the r.h.s. of that inequality is smaller than $(2|a|)^{-1}$ which shows the second equality in (A.27), but also the first one because

$$|\rho| \leq \|w''\|^2 + |\text{Re } \frac{1}{4a}| \leq \|w''\|^2 + \frac{1}{4|a|^2} \leq \frac{1}{2|a|^2}.$$

This completes the proof. □

Lemma A 6 *For fixed $p \in \dot{H}_0^+$ and $k \in \mathbb{R}^2$ with $|k| = \kappa$, the function $e \mapsto u^\alpha(e,p)(k)$ is analytic on the tuboid $\mathcal{T}_+$.*

Proof. We shall show that $u^\alpha(e,p)(k)$ is analytic on $\mathbb{R}^4 + iV_+$, where V_+ denotes the forward light-cone. As in the proof of Lemma A 5, we pick an integer $n > 2\alpha' + 2$ and write $u^\alpha(e,p)(k)$ as a sum of terms of the form

$$c\int d^2z e^{ikz} f(e,z)\,, \quad f(e,z) := P_e(z)^{\alpha-n+\nu}\,Q_e(z)^{n-2\nu}, \quad \nu = 0,\ldots,[n/2],$$

where P_e, respectively Q_e, is a quadratic, respectively linear, polynomial with coefficients depending differentiably on e, cf. equations (A.21) and (A.22). If e varies in a compact set, each of these polynomials is bounded, uniformly in e, by some continuous function with quadratic, respectively linear, behavior for large $|z|$. In addition, $P_e(z)$ has no real zeroes and, as a consequence of Eq. (A.23), is uniformly bounded below by a strictly positive function with quadratic behavior. Hence for each compact subset Ω of $\mathbb{R}^4 + iV_+$ there is a continuous dominating function for $f(e,z)$, $e \in \Omega$, which goes like $|z|^{2\alpha'-n}$ for large z and is therefore integrable w.r.t. $d^2z \sim |z|d|z|$ for $2\alpha' - n < -2$. It follows that the analyticity of the integrand, for fixed $z \in \mathbb{R}^2$, survives after the integration, completing the proof. □

Lemma A 7 *Let $\alpha' \in (-1, -\frac{1}{2})$. Then there are constants c_1, c_2 such that for all $e \in \mathcal{T}_+$, $p \in \dot{H}_0^+$, $k \in \mathbb{R}^2$ with $|k| = \kappa$ the following estimate holds:*

$$|u^\alpha(e,p)(k)| \le c_1 \, |p \cdot e|^{-\alpha' - 1} + c_2 \, |p \cdot e|^{-\alpha' - 2}. \tag{A.32}$$

Proof. We use the same notations as in the proof of Lemma A 5, and consider the intertwiner function as given by the integral (A.20). Here, we partially integrate only over $\|z\| > |a|^{-1}$. We obtain

$$\int_{\|z\| > |a|^{-1}} d^2 z e^{ik \cdot z} \big(P(z)\big)^\alpha = \frac{i\alpha}{\kappa} \int_{\|z\| > |a|^{-1}} d^2 z e^{i\kappa z_1} P(z)^{\alpha - 1} 2a(z_1 + i w_1'') + (\text{bd. terms}). \tag{A.33}$$

From Eq. (A.28), with $s = \alpha' - 1$ and $m = 1$, $\nu = 0$, we know that the integral on the r.h.s. is bounded by $c|a|^{-\alpha' - 1}$ if $\alpha' < -1/2$. The boundary terms in Eq. (A.33) are given by

$$\frac{1}{i\kappa} \int_{-|a|^{-1}}^{|a|^{-1}} dz_2 e^{i\kappa\delta(z_2)} P(\delta(z_2), z_2)^\alpha$$

where $\delta(z_2) := \sqrt{|a|^{-2} - z_2^2}$, plus a similar term for $z_1 = -\delta(z_2)$. By Eq.s (A.24) and (A.27), they are bounded by

$$c|a|^{\alpha'} \int_{-|a|^{-1}}^{|a|^{-1}} dz_2 (|a|^{-2} - \rho)^{\alpha'} \le 2^{-\alpha'} c|a|^{\alpha'} \int_{-|a|^{-1}}^{|a|^{-1}} dz_2 |a|^{-2\alpha'} = c'|a|^{-\alpha' - 1}.$$

Thus, the integral over $\|z\| > |a|^{-1}$ in Eq. (A.20) is bounded by $c|a|^{-\alpha' - 1}$.

To evaluate the integral over $\|z\| < |a|^{-1}$, we note that, by (A.24),

$$\Big| \int_{\|z\| < |a|^{-1}} d^2 z e^{ik \cdot z} \big(P(z)\big)^\alpha \Big| \le |a|^{\alpha'} \pi \int_0^{|a|^{-2}} d(r^2) |r^2 - \rho|^{\alpha'} \le c'|a|^{-\alpha' - 2} \tag{A.34}$$

where in the last inequality, valid for $\alpha' > -1$, we have used (A.27). This completes the proof. □

We now prove boundedness of the 3-d intertwiner, used in Proposition 6.2.

Lemma A 8 *Consider the three-dimensional case, with $\alpha' \in (-1, 0)$. The function $(e, p) \mapsto u^\alpha(e, p)$ is bounded on $\mathcal{T}_+ \times \dot{H}_0^+$.*

Proof. In analogy with the 4-dimensional case, cf. the proof of Lemma A 5, we have

$$P(r) := B_p \xi(r) \cdot e = ar^2 + br + c, \quad \text{with} \tag{A.35}$$
$$a = \frac{1}{2}(p \cdot e),$$
$$b = -(B_p^{-1} e)_1,$$
$$c = \frac{1}{2}(B_p^{-1} e)_+ .$$

Note that the imaginary parts of $B_p\xi(r)\cdot e$ and a are strictly positive since $e''\in V_+$.

The zeroes of the quadratic polynomial (A.35) are $r_\pm \doteq (-b\pm 1)/2a$, and we write the polynomial as

$$P(r) = a(r-r_+)(r-r_-) = \big((r-r_+)^{-1}-(r-r_-)^{-1}\big)^{-1}. \tag{A.36}$$

(We have used that $r_+ - r_- = a^{-1}$.) For r close to the zeroes $r_\pm$, the modulus $|P(r)|^{\alpha'}\sim |r-r_\pm|^{\alpha'}$ is integrable since $\alpha'>-1$, but for large $|r|$ the modulus $|P(r)|^{\alpha'}\sim |r|^{2\alpha'}$, hence it is only integrable for $\alpha'<-\frac{1}{2}$. To treat all $\alpha'\in(-1,0)$ simultaneously, we hence need to keep the oscillating factor for large r and partially integrate except for an ε-neighborhood around each of the real parts of the poles $r_\pm$. To this end, denote the real parts of r_+ and r_- by $r'_\pm$, labeled in such a way that $r'_-\le r'_+$, and denote by $I_\pm$ the interval $(r'_\pm-\varepsilon, r'_\pm+\varepsilon)$. Here, $\varepsilon>0$ is fixed and independent of $r'_\pm$, so it may happen that the intervals overlap. In any case, we get

$$\begin{aligned} e^{i\pi\alpha/2}\,u^\alpha(e,p) = \int_{I_+\cup I_-} \mathrm{d}r e^{i\kappa r}\,P(r)^\alpha + \frac{1}{i\kappa}\sum_{r_l\in\partial(I_+\cup I_-)} (\pm)e^{i\kappa r_l}\,P(r_l)^\alpha \\ - \frac{1}{i\kappa}\int_{\mathbb{R}\setminus(I_+\cup I_-)} dr\, e^{i\kappa r}\,\partial_r P(r)^\alpha. \end{aligned} \tag{A.37}$$

We will use the estimate

$$\begin{aligned} |P(r)^\alpha| &\le c|(r-r_+)^{-1}-(r-r_-)^{-1}|^{-\alpha'} \\ &\le c\big(|r-r'_+|^{\alpha'}+|r-r'_-|^{\alpha'}\big) \\ &\le 2c|r-r'_\pm|^{\alpha'} \qquad \text{if } r \gtrless r_0, \end{aligned} \tag{A.38}$$

where we have denoted

$$r_0 := (r'_+ + r'_-)/2. \tag{A.39}$$

To estimate $(P(r)^\alpha)'$, note that equation (A.36) implies

$$\begin{aligned} (P(r)^\alpha)'(r) &= \alpha\big((r-r_+)^{-1}-(r-r_-)^{-1}\big)^{-\alpha-1}\,\big((r-r_+)^{-2}-(r-r_-)^{-2}\big) \\ &= \alpha\big((r-r_+)^{-1}-(r-r_-)^{-1}\big)^{-\alpha}\big((r-r_+)^{-1}+(r-r_-)^{-1}\big), \end{aligned}$$

hence

$$\begin{aligned} |(P(r)^\alpha)'(r)| &\le c|\alpha|\,\big|(r-r_+)^{-1}-(r-r_-)^{-1}\big|^{-\alpha'}\,\big|(r-r_+)^{-1}+(r-r_-)^{-1}\big| \\ &\le c|\alpha|\,\big(|r-r'_+|^{\alpha'}+|r-r'_-|^{\alpha'}\big)\,\big(|r-r'_+|^{-1}+|r-r'_-|^{-1}\big) \\ &\le 4c|\alpha|\,|r-r'_\pm|^{\alpha'-1} \quad \text{if } r\gtrless r_0. \end{aligned} \tag{A.40}$$

We shall first discuss the boundary terms, i.e. the second term in equation (A.37). From (A.38) we get

$$|P(r'_+ + \varepsilon)^\alpha| \le 2c\varepsilon^{\alpha'} \tag{A.41}$$

since $r'_+ + \varepsilon > r_0$, and similarly $|P(r'_- - \varepsilon)^\alpha| \le 2c\varepsilon^{\alpha'}$. If the intervals overlap, then these are the only boundary terms. If they do not overlap, then $r'_- + \varepsilon \le r'_+ - \varepsilon$ which implies $|P(r'_\pm \mp \varepsilon)^\alpha| \le 2c\varepsilon^{\alpha'}$ by a similar consideration. Hence in either case we have for the boundary terms the estimate

$$\sum_{r_l \in \partial(I_+ \cup I_-)} |P(r_l)^\alpha| \le 8c\varepsilon^{\alpha'}. \tag{A.42}$$

Let us discuss the first term in equation (A.37), first assuming that the intervals $I_\pm$ are disjoint. Then $r \in I_-(I_+)$ implies $r < r_0 (r > r_0)$ respectively, and (A.38) implies:

$$\begin{aligned} \int_{I_+ \cup I_-} |P(r)|^{\alpha'} &= \int_{I_-} \mathrm{d}r \, |P(r)|^{\alpha'} + \int_{I_+} \mathrm{d}r \, |P(r)|^{\alpha'} & \text{(A.43)} \\ &\le c' \int_{I_-} \mathrm{d}r \, |r - r'_-|^{\alpha'} + c' \int_{I_+} \mathrm{d}r \, |r - r'_+|^{\alpha'} = c'' \varepsilon^{\alpha'+1}. & \text{(A.44)} \end{aligned}$$

If the intervals $I_\pm$ do overlap, then a moment's thought shows that we get a “$\le$“ instead of “$=$“ in (A.43), yielding the same estimate (A.44). As to the third term in equation (A.37), let us again first assume that the intervals $I_\pm$ are disjoint. Then we write

$$\mathbb{R} \setminus (I_+ \cup I_-) = (-\infty, r'_- - \varepsilon) \cup (r'_- + \varepsilon, r_0) \cup (r_0, r'_+ - \varepsilon) \cup (r'_+ + \varepsilon, \infty), \tag{A.45}$$

where in the first two intervals $|\partial_r P(r)|^{\alpha'} \le c|r - r'_-|^{\alpha'-1}$ and in the last two intervals $|\partial_r P(r)|^{\alpha'} \le c|r - r'_+|^{\alpha'-1}$ by (A.40). Hence

$$\int_{\mathbb{R}\setminus(I_+ \cup I_-)} |\partial_r P(r)|^{\alpha'} \le c' \int_{\mathbb{R}\setminus I_-} \mathrm{d}r |r - r'_-|^{\alpha'-1} + c' \int_{\mathbb{R}\setminus I_+} \mathrm{d}r |r - r'_+|^{\alpha'-1} \le c'' \varepsilon^{\alpha'}. \tag{A.46}$$

If the intervals $I_\pm$ overlap, then the second and third intervals in (A.45) are absent, and we have the same estimate (A.46). Since ε was arbitrary, we have thus shown that $|u^\alpha(e,p)|$ is uniformly bounded, as claimed. □

A.3 A Folklore Lemma.

Lemma A 9 *Let U_t be a unitary one-parameter group, with generator K. Then ψ is in the domain of* $\exp(-\pi K)$ *if, and only if, the vector-valued map*

$$t \mapsto U_t \psi$$

is analytic in the strip $\mathbb{R} + i(0, \pi)$ *and weakly continuous on the closure of that strip. In this case,* $\exp(-\pi K)\psi$ *coincides with the analytic continuation of* $U_t\psi$ *into* $t = i\pi$.

Proof. The “only if” part is a standard result from functional calculus, and we prove here the “if” part for the convenience of the reader. Denote $U_t \psi =: \psi_t$. Let c be a smooth function with compact support. Recall that the bounded operator $c(K)$ may be written as $c(K) = \int dt \, \tilde{c}(t) \, U_t$, where $(2\pi)^{1/2} \tilde{c}$ is the Fourier transform of c, and the integral is understood in the weak sense. Then we have for any $\phi \in \mathcal{H}$, in view of the equation $\psi_{t+z} = U_t\, \psi_z$, that

$$\big(\exp(-\pi K)\bar{c}(K)\phi\,,\, \psi_{t=0} \big) = \big(\phi\,,\, c_\pi(K)\psi_0 \big) = \int dt\, \widetilde{c_\pi}(t) \big(\phi\,,\, \psi_t \big)\,, \tag{A.47}$$

where we have written $c_\pi(k) := e^{-\pi k} c(k)$. Now the last integral is the limit, for $t' \to 0$, of $\int dt\, \widetilde{c_\pi}(t+it')\big(\phi\,,\, \psi_{t+it'}\big)$. Since the integrand is analytic and tends to zero for $|t| \to \infty$, we may further translate the contour of the integral by letting t' approach π. Hence the r.h.s. of (A.47) is equal to

$$\int dt\, \widetilde{c_\pi}(t+i\pi)\big(\phi\,,\, \psi_{t+i\pi}\big) = \int dt\, \tilde{c}(t)\big(\phi\,,\, \psi_{t+i\pi}\big) = \big(\bar{c}(K)\phi\,,\, \psi_{i\pi}\big), \tag{A.48}$$

where we have used that $\widetilde{c_\pi}(t+i\pi) = \tilde{c}(t)$. We have thus established that

$$\big(\exp(-\pi K)c(K)\phi\,,\, \psi_0 \big) = \big(c(K)\phi\,,\, \psi_{i\pi} \big) \quad \phi \in, \, c \in C_0^\infty(\mathbb{R}).$$

But the span of vectors of the form $c(K)\phi$ is a core for the self-adjoint operator $\exp(-\pi K)$, hence the above equation implies that ψ_0 is in the domain of $\exp(-\pi K)$ and that $\exp(-\pi K)\psi_0 = \psi_{i\pi}$, as claimed. □

B Results on the Little Groups and the Reflections

B.1 Representation of the Reflections.

We prove that the anti–unitary operators $D(j_0)$, defined in sections 4 and 6, extend the respective representations of the little groups G to representations of the semi-direct product of G with j_0. Namely, in $d = 4$, $D(j_0)$ is defined as in Eq. (57) in the massive case or as

$$(D(j_0)u)(k) := \overline{u(k)} \tag{B.49}$$

in the massless case, respectively. In $d = 3$, $D(j_0)$ is just complex conjugation for both $m > 0$ and $m = 0$.

Lemma B 1 *In all cases ($m \geq 0$, $d = 3, 4$), $D(j_0)$ satisfies the representation properties $D(j_0)^2 = 1$ and*

$$D(j_0)D(\Lambda)D(j_0) = D(j_0\Lambda j_0), \quad \Lambda \in G. \tag{B.50}$$

Proof. We first treat the case $m > 0$. In $d = 4$, one checks that $-j_0 q(n) = q(I_3 n)$, where I_3 is the inversion $(n_1, n_2, n_3) \mapsto (-n_1, -n_2, n_3)$. Hence j_0 has the same commutation relations with the rotations as I_3, and a representer of j_0 is given by $\big(D(j_0)Y_{s,k}\big)(n) := \overline{Y_{s,k}(I_3 n)}$. But the right hand side coincides with $(-1)^k Y_{s,-k}(n)$, as in (57). In $d = 3$, the claim follows from the fact that $j_0 R(\omega) j_0 = R(-\omega)$.

In the case $m = 0$, note that $-j_0\xi(z) = \xi(-z)$, $z \in \mathbb{R}^{d-2}$, which implies that the adjoint action of j_0 on G corresponds, via the isometry $\xi(\cdot)$, to the automorphism $(c, R) \mapsto (-c, R)$ of $E(d-2)$ (put $R = \mathbb{1}$ in case $d = 3$). This implies the claim. □

B.2 The Orbits of the Little Groups.

One checks that the maps q and ξ, defined in equations (58), (59) and (100), (101), respectively, are diffeomorphisms from the sphere S^{d-2} and $\mathbb{R}^{d-2}$, respectively, onto the orbit Γ defined in equation (50). Now Γ is a space-like sub-manifold, hence a Riemannian space with the metric $g_\Gamma := -g|_\Gamma$. Then q is clearly isometric. To check that the same holds for ξ, denote by ∂_i the derivatives w.r.t. the natural coordinates z_i on $\mathbb{R}^2$, $i = 1, 2$. Then

$$g_\Gamma(\xi_*\partial_i, \xi_*\partial_j) = \sum_{k=1}^{3}(\partial_i\xi^k)(\partial_j\xi^k) - (\partial_i\xi^0)(\partial_j\xi^0) = \delta_{ij},$$

i.e., ξ is isometric. We therefore have:

Lemma B 2 *The map ξ is an isometry from $\mathbb{R}^{d-2}$ onto the orbit Γ.*

Let again G denote the stability subgroup of a fixed point $\bar{p} \in H_m^+$. In the massive case, $\bar{p} = (m, 0, 0, 0)$ and $G \cong SO(d-1)$, while in the massless case $\bar{p} = (1, 0, 0, 1)$ and $G \cong E(d-2)$.

Lemma B 3 *i) Let $\hat{e}$, $e \in H^{\mathrm{c}}$ or $H^{\mathrm{c}} \setminus \{\pm(i, 0, 0, 0)\}$, respectively in the massless or massive case, satisfy $\bar{p} \cdot \hat{e} = \bar{p} \cdot e$. Then there is a complex Lorentz transformation[26] in the connected component of the unit, which leaves $\bar{p}$ invariant and maps $\hat{e}$ to e.*

ii) In $d = 4$, consider the stability group, in G, of an arbitrary point $e \in H$, and a faithful, or scalar, representation D of G. The restriction of D to this group contains the trivial representation at most once.

iii) Consider the massless case in $d = 4$, and let D be a non–faithful (but non–trivial) "helicity" representation of the little group $G \cong E(2)$. That is to say, D acts as a direct sum of irreducible representations of the form

$$D(\Lambda(c, R_\phi))\, v = e^{in\phi}\, v \tag{B.51}$$

for some integer $n \neq 0$. Then the restriction of D to the group mentioned in ii) does not contain the trivial representation if $\bar{p} \cdot e \neq 0$.

Proof. Ad *i*) We discuss the case $d = 4$. To this end, we recall a well-known 2:1 correspondence $(A, B) \mapsto \Lambda(A, B)$ between $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and the group $\mathcal{L}_+^c$ of complex Lorentz transformations (cf. footnote 26) path-connected with the unit. Let $z \mapsto \underline{z}$ be the isomorphism from $\mathbb{C}^4$ onto $\mathrm{Mat}(2, \mathbb{C})$ given by $\underline{z} := z_0 + \sum_{i=1}^{3} z_i\sigma_i$, σ_i the Pauli matrices. This map satisfies

$$\det \underline{z} = z \cdot z \quad \text{and} \tag{B.52}$$

$$\mathrm{tr}\ \underline{z} = 2z_0. \tag{B.53}$$

Then a pair $(A, B) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ defines a transformation $\Lambda(A, B)$ of $\mathbb{C}^4$ via

$$\underline{\Lambda(A, B)z} = A\underline{z}B^t. \tag{B.54}$$

[26] That is, a complex linear transformation of $\mathbb{C}^d$ leaving the bilinear form (26) invariant.

By virtue of (B.52), $\Lambda(A,B)$ is in $\mathcal{L}_+^{\mathrm{c}}$.

We first discuss the case $m>0$. Then $\bar p=(m,0,0,0)$ and $\underline{\bar p}=m\mathbb{1}$. Hence $\Lambda(A,B)$ leaves $\bar p$ invariant iff $B^t=A^{-1}$. Then $\underline{\Lambda(A,B)z}=A\underline{z}A^{-1}$. Let now e and $\hat e$ be in $\mathcal{T}_+$ such that $\bar p\cdot e=\bar p\cdot\hat e$, i.e. $e_0=\hat e_0$. We have to show that $\underline{e}$ and $\underline{\hat e}$ are related by a similarity transformation (similar). By eqs. (B.52) and (B.53), $\underline{e}$ and $\underline{\hat e}$ have the same determinant, -1, and trace, $2e_0$. Hence they have the same characteristic polynomial, namely x^2-2e_0x-1, and the same eigenvalues, $\lambda_\pm=e_0\pm\sqrt{1+e_0^2}$. If $e_0\neq\pm i$, these eigenvalues are different, and therefore $\underline{e}$ and $\underline{\hat e}$ are both similar to $\mathrm{diag}(\lambda_+,\lambda_-)$, hence related by a similarity transformation. If $e_0=i$ or $-i$, then e_0 is a two-fold root of the characteristic polynomial. Such matrix is either equal to $e_0\mathbb{1}$ or similar to the elementary Jordan matrix with diagonal (e_0,e_0). But the first case has been excluded, for $\underline{e}=\pm i\mathbb{1}$ iff $e=(\pm i,0,0,0)$. (These points have been excluded because they are orbits by their own.) Hence e and $\hat e$ are both similar to the same Jordan matrix, and therefore similar. This proves the claim for $m>0$.

In the case $m=0$, $\bar p=\frac{1}{2}(1,0,0,1)$ and $\underline{\bar p}=\mathrm{diag}(1,0)$. One checks that $\Lambda(A,B)$ leaves $\bar p$ invariant iff

$$A=\begin{pmatrix} c & a\\ 0 & c^{-1}\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix} c^{-1} & b\\ 0 & c\end{pmatrix},\quad a,b,c\in\mathbb{C}, c\neq 0. \tag{B.55}$$

Let now e and $\hat e$ be in H^{c} such that $\bar p\cdot e=\bar p\cdot\hat e$. We have to show that there is a Λ of the above form which maps e to $\hat e$. But $\bar p\cdot e=e_0-e_3=(\underline{e})_{2,2}$. Hence we have to show that for any two matrices $\underline{e},\underline{\hat e}$ with the same determinant and $2,2$ component, there are A,B as in (B.55) such that $\underline{\hat e}=A\underline{e}B^t$. One checks that the choice

$$a:=\frac{(\underline{\hat e})_{1,2}\,c^{-1}-(\underline{e})_{1,2}\,c}{(\underline{e})_{2,2}},\quad b:=\frac{(\underline{\hat e})_{2,1}\,c-(\underline{e})_{2,1}\,c^{-1}}{(\underline{e})_{2,2}},$$

$c\neq 0$ arbitrary, does it. This proves the claim for $m=0$.

The three-dimensional case follows along similar lines.

Ad *ii*) Since G–related points have conjugate stability groups, it suffices to consider one point in each G-orbit of H. In the case $m>0$, G is isomorphic to $SO(3)$ which acts transitively on the spheres $e_0=$constant, and we consider for each $e_0\in\mathbb{R}$ the point $(e_0,0,0,\sqrt{1+e_0^2})$. Clearly, the stability subgroup, in $SO(3)$, of these points are the rotations around the 3–axis, which are represented by $D(R_3(\omega))_{kk'}=e^{ik\omega}\delta_{k,k'}$. Hence $v\in\mathfrak{h}=\mathbb{C}^{2s+1}$ is invariant if and only if $v_k=c\delta_{k,0}$. This proves the claim for $m>0$.

In order to conveniently discuss the case $m=0$, we give an explicit formula for the action of the little group $G\cong E(2)$ on $\mathbb{C}^4$. To this end, we use coordinates $z_\pm:=z_0\pm z_3$, and identify points z in complexified Minkowski space $\mathbb{C}^4$ with tuples $(z_+,z_-,\underline{z})$ with $z_\pm\in\mathbb{C}$ and $\underline{z}\in\mathbb{C}^2$, the metric being written as $z\cdot z=z_+z_--\underline{z}\cdot\underline{z}$. In these coordinates, the action of $G\cong E(d-2)$ reads

$$\Lambda(\underline{c},R_\phi)\,(z_+,z_-,\underline{z})=(z_++2\underline{c}\cdot R_\phi\underline{z}+|\underline{c}|^2z_-\,,\,z_-\,,\,R_\phi\underline{z}+z_-\underline{c}). \tag{B.56}$$

(This follows from the identification of G with $E(d-2)$ acting in Γ=$\{z:z_-=1\}\cap H_0^+$ by linear extension.) For $t\in\mathbb{R}$, consider now the sub-manifold H_t of $e\in H$ with $e_-=t$. For $t\neq 0$, it is isomophic to $\mathbb{R}^2$ via $e\mapsto(e_1,e_2)$, a nd the action of G on H_t can be identified, by virtue of Eq. (B.56), with the natural action of the

Euclidean group. It is therefore transitive. Hence every $e \in H$ with $e_- \neq 0$ is G-related to some $e(t) := (-1/t, t, \underline{0})$. Equation (B.56) shows that the stability subgroup, in G, of $e(t)$ are the rotations $\Lambda(0, R_\phi)$. These are represented in $\mathfrak{h} = L^2(\mathbb{R}^2, d\nu_\kappa)$ as $\big(D(\Lambda(0, R_\phi))v\big)(k) = v(R_\phi^{-1}k)$. Hence $v \in \mathfrak{h}$ is invariant if and only if it is the constant function, proving the claim for $t \neq 0$. For $t = 0$, $H_{t=0}$ is isomorphic to $\mathbb{R} \times S^1$, and from Eq. (B.56), one sees that G acts transitively. Hence every $e \in H_{t=0}$ is G-related to $e(0) := (0, 0, \underline{e}_0)$ with $\underline{e}_0 = (1, 0) \in S^1$. Eq. (B.56) shows that the stability subgroup, in G, of $e(0)$ are "translations" of the form $\Lambda(\underline{c}, 1)$, with $\underline{c} = (0, c_2) \perp \underline{e}_0$. It follows that $v \in \mathfrak{h}$ is invariant iff for all $k \in \mathbb{R}^2$ with $|k| = \kappa$ there holds $e^{ic_2k_2}v(k) = v(k)$. Such v must vanish except at the points $k = (\pm\kappa, 0)$, hence almost everywhere. Thus, in the case $e_- = 0$ the invariant subspace is trivial.

Ad *iii)* As we have seen in the proof of *ii)*, the stability group of e are the rotations $\Lambda(0, R_\phi)$ if $\bar{p} \cdot e \neq 0$. But the representation (B.51) does not contain any invariant vector. This proves the claim. □

Lemma B 4 *Let G_e be the stability group, in G, of a fixed point $e \in H$ satisfying $e_0 \neq e_3$. Then there is precisely one vector in $\mathbb{C}^4$, up to a constant, satisfying the eigenvalue condition*

$$\Lambda(c, R_\phi)v = e^{i\lambda\phi}v, \quad \Lambda(c, R_\phi) \in G_e, \tag{B.57}$$

where $\lambda \in \{1, -1\}$.

Proof. As in the proof of Lemma B 3, we use coordinates (z_+, z_-, z_1, z_2) in which the action of $G \cong E(2)$ is given by Eq. (B.56). Again, it suffices to consider one point e in each G-orbit, the latter being characterized by the value of e_-. We consider the point $e = (e_+, e_-, 0, 0)$, with $e_+e_- = -1$. We know that then G_e consists of the rotations $\Lambda(0, R_\phi)$. Then the eigenvalue equation (B.57) reads

$$(v_+, v_-, \cos(\phi)v_1 + \sin(\phi)v_2, \cos(\phi)v_2 - \sin(\phi)) = e^{\pm i\phi}(v_+, v_-, v_1, v_2)$$

and implies that $v_+ = 0 = v_-$ and $v_2 = \pm iv_1$, hence $v \sim (0, 0, 1, \pm i) =: \hat{e}_\pm$. □

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