CBPF-NF-025/81

RENORMALIZATION GROUP SPECIFIC HEAT AND MAGNETIZATION OF THE ISING FERROMAGNET IN CUBIC AND HYPERCUBIC LATTICES

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ABSTRACT

A real space renormalization group approach with two-terminal clusters is proposed for the calculation of the specific heat and spontaneous magnetization (for all temperatures) of the nearest-neighbour spin-$\frac{1}{2}$ Ising ferromagnet in simple cubic and hypercubic lattices. For arbitrary temperatures only small culsters (renormalization expansion factor b=2,3 for d=3 and b=2 for d=4) are considered: they lead to reasonable values for the critical points, exponents and amplitudes and to a thermal behaviour of the specific heat (spontaneous magnetization) which is not yet (which is already quite) very close to what is expected (from series results for example). The global results improve when b increases from 2 to 3. The drastic effect (on the specific heat) of an apparently innocuous approximation is exhibited. The discussion of the $T \to 0$ and $T \to \infty$ limits is performed and the *exact* behaviours of the free and internal energies and the specific heat are obtained for sufficiently large values of b and *all* dimensionalities.

I - INTRODUCTION

During recent years much effort has been dedicated to real space renormalization group (RG) approaches[1-20] for Ising ferromagnets. More specifically in what concerns the thermal behaviour of the specific heat and the spontaneous magnetization[1,2,7,16,18] that effort has focalized almost exclusively planar lattices (and has obtained a certain success). The purpose of the present paper is to use the same type of RG for three- and four-dimensional lattices, namely the simple cubic and hypercubic ones.

II - CALCULATION METHOD

II.1 - General remarks

Let us briefly state the main relations we shall need (see Ref. [18] for further details). The d- dimensional first-neighbour ferromagnetic Ising *dimensionless* hamiltonian we are concerned with is given by

$$\mathcal{H}(S) = K \sum_{<i,j>} S_i S_j + H \sum_i S_i \qquad (S_i = \pm 1) \qquad (1)$$

If we renormalize the lattice (through a linear expansion factor b) we obtain, besides the usual additive term (noted G(K,H)), the following renormalized hamiltonian

$$\mathcal{H}'(S') = K'(K,H) \sum_{<i',j'>} S_{i'} S_{j'} + H'(K,H) \sum_{i'} S_{i'} \qquad (S_{i'} = \pm 1) \qquad (2)$$

where only K and H have been retained to construct the parameter-space; K'(K,H) and H'(K,H) are respectively even and odd functions of H. The invariance of the partition function is expressed by

$$\sum_{\{S'\}} \exp\left[G + \mathcal{H}'(S')\right] = \sum_{\{S\}} \exp\left[\mathcal{H}(S)\right] \qquad (3)$$

where the sums run over all spin configurations. This relation immediately leads to

$$g(K,H) + b^{-d} f(K',H') = f(K,H) \qquad (4)$$

where $g \equiv G/N$ is an even function of H (N is the number of sites of the original lattice) and f is the *dimensionless* free energy per site. Relation (4) refers to the renormalization of the lattice as a whole. Let us now work at the level of the (two-terminal) clusters (like those appearing in Fig.1) renormalized into a single bond (whose terminal spins we note μ_A and μ_B ; $\mu_A, \mu_B = \pm 1$). The analogous of relation (3) is given by

$$\exp\left[K'_o + K'\mu_A\mu_B + H'(\mu_A + \mu_B)\right] =$$

$$\sum_{\{\sigma_i\}} \exp\left\{K\,\Pi_{\mu_A\mu_B} + H\left[p_A(\mu_A+\mu_B) + \sum_i p_i\sigma_i\right]\right\} \qquad (5)$$

where i runs over all internal sites of the cluster (4 in Fig. 1.a), $\{\sigma_i\}$ refers to all internal spins configurations ($\sigma_i = \pm 1$), $K'_o(K,H)$ is an even function of H, the topological weights p_A and $\{p_i\}$ (determined further on) take into account the fact that the topological neighbourhood of the *cluster* sites is different from that of the *original lattice* sites (see Ref.[18] for further details) and $\Pi_{\mu_A\mu_A}$ is associated to all possible two- spins interactions in the cluster. Let us illustrate the last point on an example: for the cluster of Fig.1.a it is

$$\Pi_{\mu_A\mu_A} = (\mu_A + \mu_B)(\sigma_1+\sigma_2+\sigma_3+\sigma_4) + \sigma_1\sigma_2 + \sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1$$

For the same example relation (5) leads to

$$K'_o = \frac{1}{4}\ln(w_1 w_2) + \frac{1}{2}\ln w_3 \quad ,$$
$$K' = \frac{1}{4}\ln(w_1 w_2) - \frac{1}{2}\ln w_3 \quad , \qquad (6)$$
$$H' = \frac{1}{4}\ln\frac{w_1}{w_2} \quad ,$$
$$w_1 \equiv e^{2p_A H}\left\{e^{12K+4pH} + 4\left[e^{4K+2pH}+1\right] + e^{-4K}\left[2 + 4\,e^{-2pH} + e^{-4pH}\right]\right\} \quad ,$$
$$w_2 \equiv e^{-2p_A H}\left\{e^{12K-4pH} + 4\left[e^{4K-2pH}+1\right] + e^{-4K}\left[2 + 4\,e^{2pH} + e^{4pH}\right]\right\}$$

and

$$w_3 \equiv 2\,e^{4K}\,\mathrm{ch}\,4pH + 4(2\,\mathrm{ch}\,2pH + 1) + 2\,e^{-4K}$$

For vanishing magnetic field (H=0) the function $K'(K,0)$ admits three fixed points, namely $K = 0$, $K = \infty$ (stable fixed points) and $K = K_c$ (unstable fixed point); the latter depends on d and b.

Let us now relate $K'_o(K,H)$ to $g(K,H)$ by introducing a new function $D_o(K,H)$ (which has to be found) through the following equation

$$g(K,H) = D_o(K,H)\; K'_o(K,H) \qquad (7)$$

Now if we perform in hamiltonian (1) of a d- dimensional hypercubic lattice the transformation

$$S_i S_j \rightarrow S_i S_j + \lambda \qquad (8)$$

where λ is an arbitrary constant, the free energy will transform as

$$f(K,H) \to f(K,H) + d\lambda K$$

therefore g must transform as

$$g(K,H) \to g(K,H) + d\lambda\left[K - b^{-d}K'(K,H)\right] \tag{9}$$

in order to preserve the invariance of equation (4). For completely similar reasons K'_o must transform as

$$K'_o(K,H) \to K'_o(K,H) + \lambda\left[n_b K - K'(K,H)\right] \tag{10}$$

in order to preserve the invariance of equation (5); n_b denotes the total number of bonds of the cluster and, for the present family of clusters (see Fig.1), is given by

$$n_b = b^d + (b-1)^2(d-1)b^{d-2} \tag{11}$$

$$\sim d\ b^d \quad \text{in the limit } b \to \infty \tag{11'}$$

Under transformations (9) and (10) relation (7) becomes

$$g(K,H) + d\lambda\left[K - b^{-d}K'(K,H)\right] = D_o(K,H)\left\{K'_o(K,H) + \lambda\left[n_b K - K'(K,H)\right]\right\}$$

which, together with relation (7) and the fact that λ is an *arbitrary* constant, leads to

$$D_o(K,H) = \frac{d}{n_b}\,\frac{K - b^{-d}K'(K,H)}{K - n_b^{-1}K'(K,H)} \tag{12}$$

We shall exhibit further on that, for finite values of b, slight violations of this relation induce surprisingly strong deteriorations of the thermal behaviour of the specific heat. In the limit $K \to 0$ we have that $K' \propto K^b$, therefore

$$D_o(0,H) \equiv D'' = \frac{d}{n_b} \tag{13}$$

In the limit $K \to \infty$ we have that[18,21] $K' \sim b^{d-1}K$, therefore

$$D_o(\infty,H) \equiv D = \frac{d}{n_b}\,\frac{1 - b^{-1}}{1 - \frac{b^{d-1}}{n_b}} < \frac{d}{n_b} \tag{14}$$

Furthermore it can be shown that $D_0(K,H)$ monotonically decreases from $D_0(0,H)$ to $D_0(\infty,H)$ for K increasing from zero to infinity. In the limit $b \to \infty$, D_0 becomes, for *all* values of K and H, a pure topological factor namely

$$D_o \sim b^{-d} \tag{15}$$

The notations D'' and D have been introduced to stress that expressions (13) and (14) respectively recover, for d = 2, equations (10) and (8) of Ref. [18]. The implications of these facts will be discussed in the next subsection.

II.2 - Specific heat

We are interested here in the case H=0. If we replace Eq.(7) into Eq.(4) we obtain the following recursion relation

$$D_o(K,0)\,K_o'(K,0) + b^{-d} f(K',0) = f(K,0) \tag{16}$$

hence

$$D_0 \frac{dK_0'}{dK} + K_0' \frac{dD_0}{dK} + b^{-d} \frac{df}{dK'} \frac{dK'}{dK} = \frac{df}{dK} \tag{17}$$

hence

$$D_0 \frac{d^2K_o'}{dK^2} + 2 \frac{dD_o}{dK} \frac{dK_o'}{dK} + K_o' \frac{d^2D_o}{dK^2}$$

$$+ b^{-d}\left[\frac{d^2f}{dK'^2}\left(\frac{dK'}{dK}\right)^2 + \frac{df}{dK'}\frac{d^2K'}{dK^2}\right] = \frac{d^2f}{dK^2} \tag{18}$$

Through recursion these expressions provide the thermal behaviours of the free energy, the internal energy ($\propto df/dK$) and the specific heat $C \equiv k_B K^2(d^2f/dK^2)$. If we consider a *constant* D_o, the above three relations respectively become relations (5), (6) and (7) of Ref. [18].

Let us first present a few general results: it is straightforward to obtain, in the limit $K \to \infty$,

$$K_0' \sim (n_b - b^{d-1})K \tag{19}$$

and

$$K' \sim b^{d-1}K \qquad \text{(used to obtain Eq.(14))} \tag{20}$$

and, for $K = 0$,

$$K_o' = b^{d-1}(b-1)\ \ln 2 \quad , \tag{21}$$

$$d^2K_o'/dK^2 = n_b \quad , \tag{22}$$

$$K' = dK'/dK = dK_o'/dK = 0 \tag{23}$$

$$\frac{d^2K'}{dK^2} = \begin{cases} 2^d & \text{if } b = 2 \\ 0 & \text{if } b \geq 3 \end{cases} \tag{24}$$

and

$$\frac{d^3K'}{dK^3} = \begin{cases} 2^d 3(d-1) & \text{if } b = 2 \\ 3^d 2 & \text{if } b = 3 \\ 0 & \text{if } b \geq 4 \end{cases} \tag{25}$$

These results lead (by using Eq.(12)), in the limit $K \to \infty$, to

$$\frac{dD_o}{dK} = \frac{d^2D_o}{dK^2} = 0 \tag{26}$$

and, for $K = 0$, to

$$\frac{dD_o}{dK} = \begin{cases} -\dfrac{d(d-1)}{2^{d-1}(d+3)^2} & \text{if } b = 2 \\ 0 & \text{if } b \geq 3 \end{cases} \tag{27}$$

and

$$\frac{d^2D_o}{dK^2} = \begin{cases} -\dfrac{d(d-1)(d^2+2d-1)}{2^{d-2}(d+3)^3} & \text{if } b = 2 \\ -\dfrac{8d(d-1)}{3^{d-1}(4d+5)^2} & \text{if } b = 3 \\ 0 & \text{if } b \geq 4 \end{cases} \tag{28}$$

Equations (13,14,19-28) lead (through use of Eqs.(16-18)), in the limit $K \to \infty$, to

$$f(K,0) \sim d\,K \quad \text{(exact)} \tag{29}$$

and, for $K = 0$, to

$$f(0,0) = \frac{d\,b^{d}(1 - b^{-1})}{n_b(1 - b^{-d})} \ln 2 \xrightarrow{b \to \infty} \ln 2 \quad \text{(exact)}, \tag{30}$$

$$\left.\frac{df(K,0)}{dK}\right|_{K=0} = \begin{cases} -\dfrac{d(d-1)}{(d+3)^2} \ln 2 & \text{if } b = 2 \\ 0 \quad \text{(exact)} & \text{if } b \geqslant 3 \end{cases} \tag{31}$$

and

$$\left.\frac{d^2f(K,0)}{dK^2}\right|_{K=0} = \begin{cases} d - \dfrac{d(d-1)(2d^2+5d+1)}{(d+3)^3} \ln 2 & \text{if } b = 2 \\ d - \dfrac{16d(d-1)}{(4d+5)^2} \ln 2 & \text{if } b = 3 \\ d \quad \text{(exact)} & \text{if } b \geqslant 4 \end{cases} \tag{32}$$

We see therefore that, for large enough values of b and *all* dimensionalities, the present procedure leads, in an *unbiased* manner, to the *exact* behaviour of the free energy, internal energy and specific heat for $K = 0$ and $K \to \infty$.

The complete thermal dependence of the specific heat obtained for small values of b ($b = 2,3$ for $d = 3$ and $b = 2$ for $d = 4$) is discussed in Section III.

II.3 - Spontaneous magnetization

We are interested here in the limit $H \to 0$. Eq.(16) is extended into

$$D_o(K,H)\ K_o'(K,H) + b^{-d} f(K',H') = f(K,H) \qquad (33)$$

therefore the spontaneous magnetization $m(K) = (df/dH)_{H=0}$ satisfies the following recursive relation

$$b^{-d} m(K') \left(\frac{\partial H'}{\partial H}\right)_{H=0} = m(K) \qquad (34)$$

where the parity (with respect to H) of D_o and K_o' has been used. If we consider the configuration $\mu_A = \mu_B = 1$ in Eq.(5) and differentiate with respect to H, we obtain

$$\left\{\frac{\partial H'}{\partial H}\right\}_{H=0} = p_A + \sum_{i=1}^{(b-1)b^{d-1}} \frac{p_i}{2} \left\{ \frac{\sum_{\{\sigma_j\}} \sigma_i \exp(K\Pi_{11})}{\sum_{\{\sigma_j\}} \exp(K\Pi_{11})} \right\} \qquad (35)$$

To close the procedure let us establish the rules which provide the topological weights p_A and $\{p_i\}$. In each one of the terminal sites (A and B), b^{d-1} different original sites have been collapsed into one therefore

$$p_A = b^{d-1} \qquad (36)$$

Furthermore if we consider the whole cluster, the original lattice renormalization proportion (for the sites) b^d into 1 must be preserved, and if we we take into account that the renormalized cluster has only 2 sites, it must be

$$\sum_{i=1}^{(b-1)b^{d-1}} p_i + 2p_A = 2b^d \qquad (37)$$

Topologically equivalent sites clearly have the same weight p_i (for example sites 1,2,3 and 4 in Fig. 1.a). Once a particular choice has been established for the weights $\{p_i\}$, Eq.(35) into Eq.(34) leads to the recursive relation which provides the complete thermal dependence of the spontaneous magnetization. In particular in the limit $K \to \infty$ Eq.(35) becomes

$$\left\{\frac{\partial H'}{\partial H}\right\}_{H=0} = p_A + \frac{1}{2} \sum_{i=1}^{(b-1)b^{d-1}} p_i = b^d \tag{38}$$

where the sum rule (37) has been used. Eq.(38) transforms Eq. (34) into an identity, as it should be in order to allow a non vanishing spontaneous magnetization for vanishing temperature. On the other hand, in the limit $K \to 0$, we have

$$\left\{\frac{\partial H'}{\partial H}\right\}_{H=0} = p_A = b^{d-1} \tag{39}$$

therefore Eq.(34) becomes $m(0)b^{-1} = m(0)$ hence $m(0) = 0$ as expected.

In what concerns the choices for the weights, the simplest one (referred to as the *equal weight* choice) corresponds to

$$p_i \equiv p = 2 \qquad \forall\ i \tag{40}$$

The next simplest choice (referred to as the *different weight* one) partially takes into account the topological differences between the cluster sites. Let us first introduce the notation

$p_i \equiv q^{(o)}$ if the coordination number of the i-th site equals 2d;

$p_i \equiv q^{(1)}$ if the coordination number of the i-th site equals 2d-1;

⋮

$p_i \equiv q^{(d-1)}$ if the coordination number of the i-th site equals d+1.

There are $(b-1)(b-2)^{d-1}$ sites of the first type, $2(d-1)(b-1)(b-2)^{d-2}$ sites of the second one, and $2^{d-1}(b-1)$ of the last one; in general there are $\binom{d-1}{j} 2^j (b-1)(b-2)^{d-1-j}$ sites whose weight is $q^{(j)}$. We now assume that, for $b \geqslant 3$,

$$q^{(o)}/2d = q^{(1)}/2d-1 = \ldots = q^{(d-1)}/d+1 \tag{41}$$

therefore, through use of the sum rule (37),

$$q^{(o)} = \frac{2}{1 - \frac{d-1}{db}} \xrightarrow{b \to \infty} 2 \tag{42}$$

hence

$$q^{(j)} = \frac{2d-j}{2d} \frac{2}{1 - \frac{d-1}{db}} \qquad (j = 1,2,\ldots,d-1) \tag{43}$$

It is interesting to remark that, in the limit $b \to \infty$, the number of sites associated to $q^{(o)}$ grow as b^d therefore that weight becomes the most relevant one and, as $q^{(o)} \to 2$, Eqs. (40) and (42) become equivalent therefore the *equal- and different-weight choices become indistinguishable for all values of K.* However for small values of b the second choice is expected to be better than the first one. The results are presented in Section III. Let us finally remark that the result $q^{(o)} = 2$ for $d = 1$ is exact and well known.

III - RESULTS

The results obtained for the specific heat C are presented in Table 1 and Figs.(2-4). One of the most striking facts is that, contrarily to what happens[18] for $d = 2$, C presents no maximum at $K = K_c$: although $C \propto |T-T_c|^{-\alpha}$ is satisfied, the left and right amplitudes have different signs. The maximum (or the highest maximum whenever there are more than one) occur at $K = K_M > K_c$. This unphysical "structure" has already been observed in similar treatments[16] and is expected to disappear for sufficiently high values of b; in any case we remark in Table 1 that, for $d = 3$, K_c/K_M and C_c/C_M present the good tendency towards unity when b varies from 2 to 3 ($C_c \equiv C(K_c)$ and $C_M \equiv C(K_M)$). We remark also in Table 1 that $K_c, \nu, \alpha, \beta, C_c, K_M, C_M$ (but not y_H and perhaps the spontaneous magnetization amplitude A) present the correct tendency towards the exact or series results when b varies from 2 to 3.

In order to exhibit the importance of not violating Eq.(12) we have presented in Figs.(2-4) the thermal dependences of the specific heat obtained through two different procedures: in the first of them (full line of those Figs.) we use, into Eqs. (16-18), $D_0(K,0)$ given by Eq.(12), while in the second one (dashed line of those Figs.) we approach (like in Ref.[18]) $D_0(K,0)$ by a constant D given by Eq.(14). Although $D_0(K,0)$ does not vary very much when K increases from zero to infinity (D_0 varies in the intervals [1/4; 3/16], [1/17;1/21] and [1/7 ; 1/10] for (d=3; b=2), (d=3; b=3) and (d=4; b=2) respectively), its variation is essential to avoid large unphysical negative values[18] for the specific heat.

In what concerns the spontaneous magnetization the main results are presented in Table 1 and Fig. 5, and they are quite satisfactory on the whole. The topological weights that have been used for b=2 are (see Eqs.(36) and (37)) $p_A = 2p_i = 4$ ($i = 1,2,3,4$) and $p_A = 4p_i = 8$ ($i = 1,2,\ldots,8$) for d=3 and d=4 respectively. For the case (d=3; b=3) two different choices have been used, namely the equal-weight ($p_A/9 = p_i/2 = 1$ ($i = 1,2,\ldots,18$); full line in Fig.5) and the different-weight ($p_A = 9, p_i = q^{(0)} = 18/7$ ($i=1,2$), $p_i = q^{(1)} = 15/7$ ($i=3,4,\ldots,10$) and $p_i = q^{(2)} = 12/7$ ($i=11,12,\ldots,18$); dashed line in Fig. 5) ones, the latter is slightly better.

IV - CONCLUSION

The critical points and (thermal and magnetic) exponents as well as the thermal behaviours (for all temperatures) of the specific heat and spontaneous magnetization of the Ising ferromagnet in simple cubic (d=3) and hypercubic (d=4) lattices have been approximatively calculated within a real space renormalization group framework. For renormalization expansion factors b sufficiently large, the *exact* behaviours, in both low and high temperature limits, of the free energy, the internal energy and the specific heat are recovered for *all* dimensionalities.

In what concerns the small values of b (b = 2,3 for d=3 and b=2 for d=4) most of the results are satisfactory: for example d=b=3 leads to $K_c \simeq 0.20$ (from series $K_c \simeq 0.22$), $\nu \simeq 0.82$ (from series $\nu \simeq 0.63$) and $\beta \simeq 0.35$ (from series $\beta \simeq 0.33$) and d=2b=4 leads to $K_c \simeq 0.09$ (from series $K_c \simeq 0.15$), $\nu \simeq 0.84$ (the exact value is 1/2) and $\beta \simeq 0.57$ (the exact value is 1/2). The thermal dependence of the spontaneous magnetization (calculated within two different choices, namely the equal- and different-weight ones, for the topological factors which appear in the present two-terminal cluster approach) is satisfactory as well: for example we obtain for the d=3 critical amplitude $A \simeq 1.30$ (from series $A \simeq 1.57$). Comparison of the two choices slightly favours the different-weight one for small values of b, although they become equivalent in the limit $b \to \infty$. On the other hand the thermal dependence of the specific heat (calculated within two different versions in what concerns the function D_0) is rather unsatisfactory for small values of b as the maximum (eventually more than one exists) of the curve does not occur at K_c; fortunately this unphysical result tends to disappear when b grows from 2 to 3. Comparison of the two versions illustrates the fact that small departures from the correct functional form of D_o induce large errors in the specific heat (unphysical negative values; see also Refs. [16] and [18]).

As a final comment let us say that the results presented herein as well as others[17,18,20] seem to globally support the belief that the present real space renormalization group approach (with two-terminal clusters) provides, for increasing b and for *all* temperatures, convergence towards the *exact* thermal dependences of the free energy, internal energy, specific heat, spontaneous magnetization and surface tension of d- dimensional Ising ferromagnets. The use of the full K- dependence of D_o and the different-weight choice seem to accelerate this convergence.

We are happy to acknowledge useful remarks from H.Vucetich, A.C.N.Magalhães and E.M.F.Curado. One of us (H.O.M.) would also like to thank the Theoretical Physics Group of the Centro Brasileiro de Pesquisas Físicas/CNPq for financial support and hospitality during the visit when this work was done.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - Two-terminal clusters associated to: (a) d=3 and b=2; (b) d=3 and b=3; (c) d=4 and b=2.

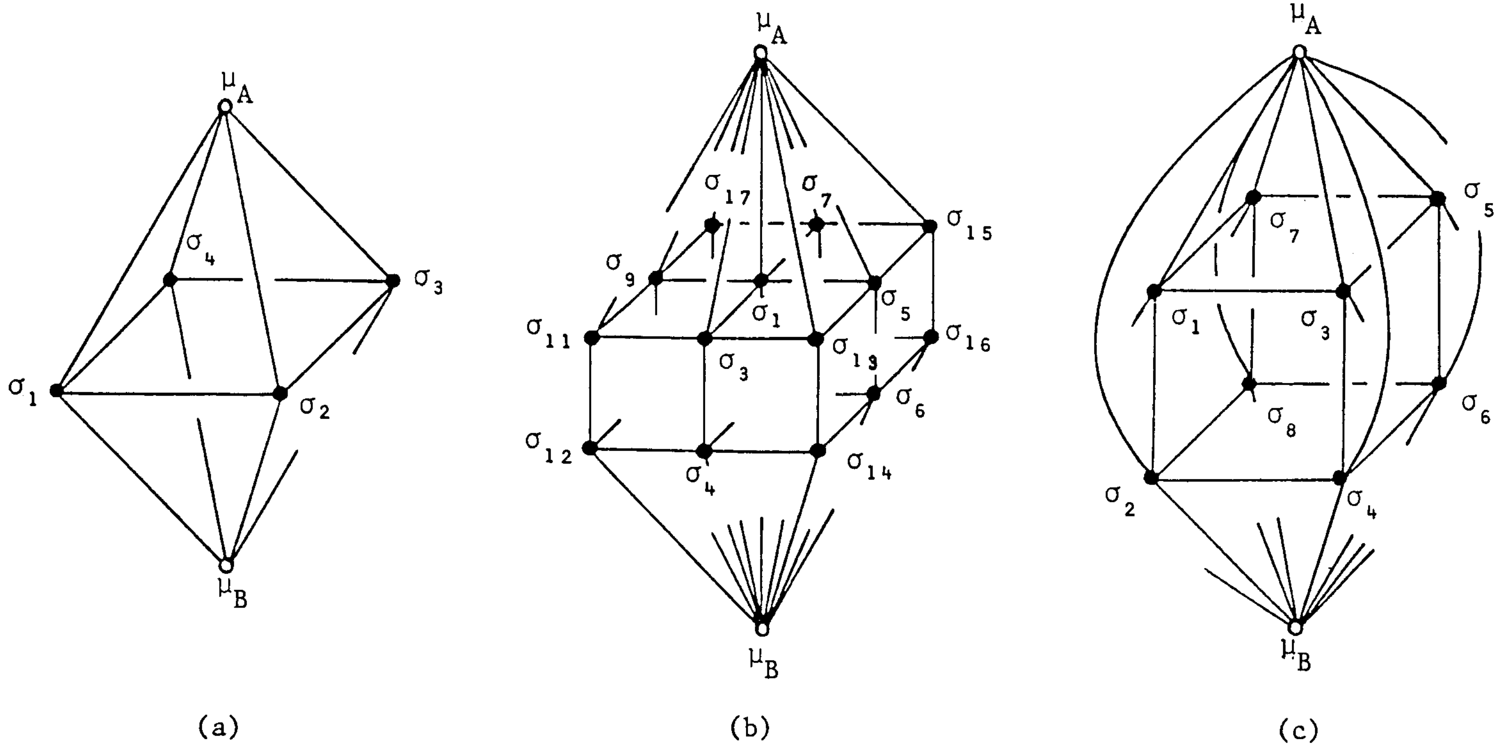
Fig. 2 - Thermal dependence of the specific heat associated to d=3 and b=2 obtained through use of the function D_o (full line) or the constant D (dashed line).

Fig. 3 - Thermal dependence of the specific heat associated to d=3 and b=3 obtained through use of the function D_o (full line) or the constant D (dashed line).

Fig. 4 - Thermal dependence of the specific heat associated to d=4 and b=2 obtained through use of the function D_o (full line) or the constant D (dashed line).

Fig. 5 - Thermal dependence of the spontaneous magnetization associated to (d=3,4; b=2) and to d=b=3 through use of the equal-weight choice (full line) or the different--weight one (dashed line).

Table 1 - Relevant critical quantities: K_c and K_M are respectively associated to the critical point and the location of the highest maximum of the specific heat C; $C_c \equiv C(K_c)$ and $C_M \equiv (K_M)$; $\nu = \ln b/\ln(\partial K'/\partial K)_{K=K_c;\ H=0}$ and $y_H^{-1} = \ln b/\ln(\partial H'/\partial H)_{K=K_c;\ H=0}$ within the RG framework; the RG values quoted for α and β have been calculated through the indicated scaling laws; A is defined through $m \sim A(b)[1-K_c(b)/K]^{\beta(b)}$; the up (down) value appearing in the rows K_M, K_c/K_M, k_B/C_c, k_B/C_M and C_c/C_M has been obtained through use of the function D_o (the constant D) given by Eq.(12) (Eq.(14)). (a) calculated from data of Ref.[23]; (b) calculated through $y_H = d-\beta/\nu$; (c) calculated through $\alpha = 2 - d\nu$; (d) for d = 4 a logarithmic factor appears as well (see Ref.[26]).



μ_A
σ_4
σ_3
σ_1
σ_2
μ_B
(a)
μ_A
σ_{17}
σ_7
σ_{15}
σ_9
σ_1
σ_5
σ_{11}
σ_3
σ_{13}
σ_{16}
σ_6
σ_{12}
σ_4
σ_{14}
μ_B
(b)
μ_A
σ_5
σ_7
σ_1
σ_3
σ_8
σ_6
σ_2
σ_4
μ_B
(c)

FIG.1

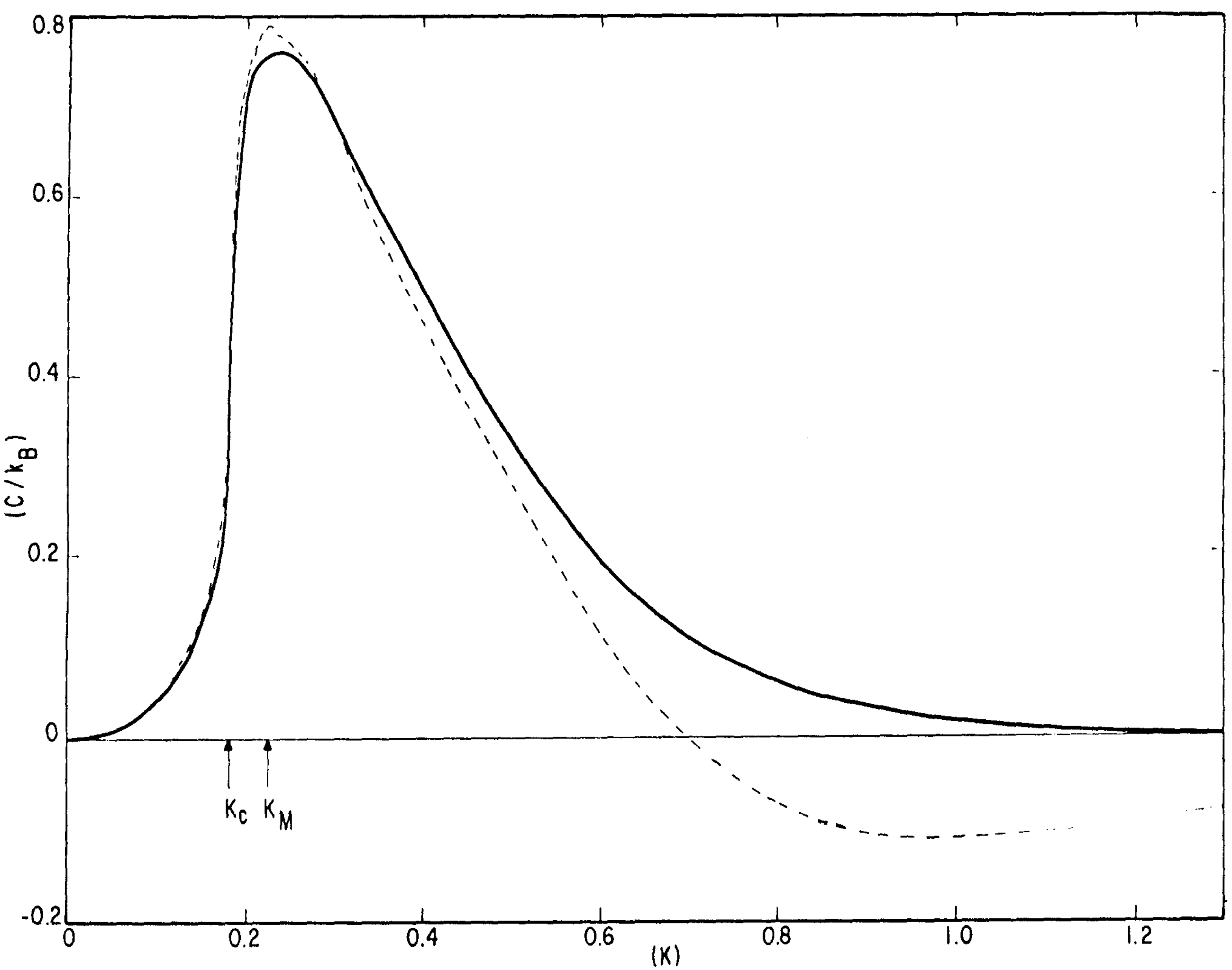


FIG. 2

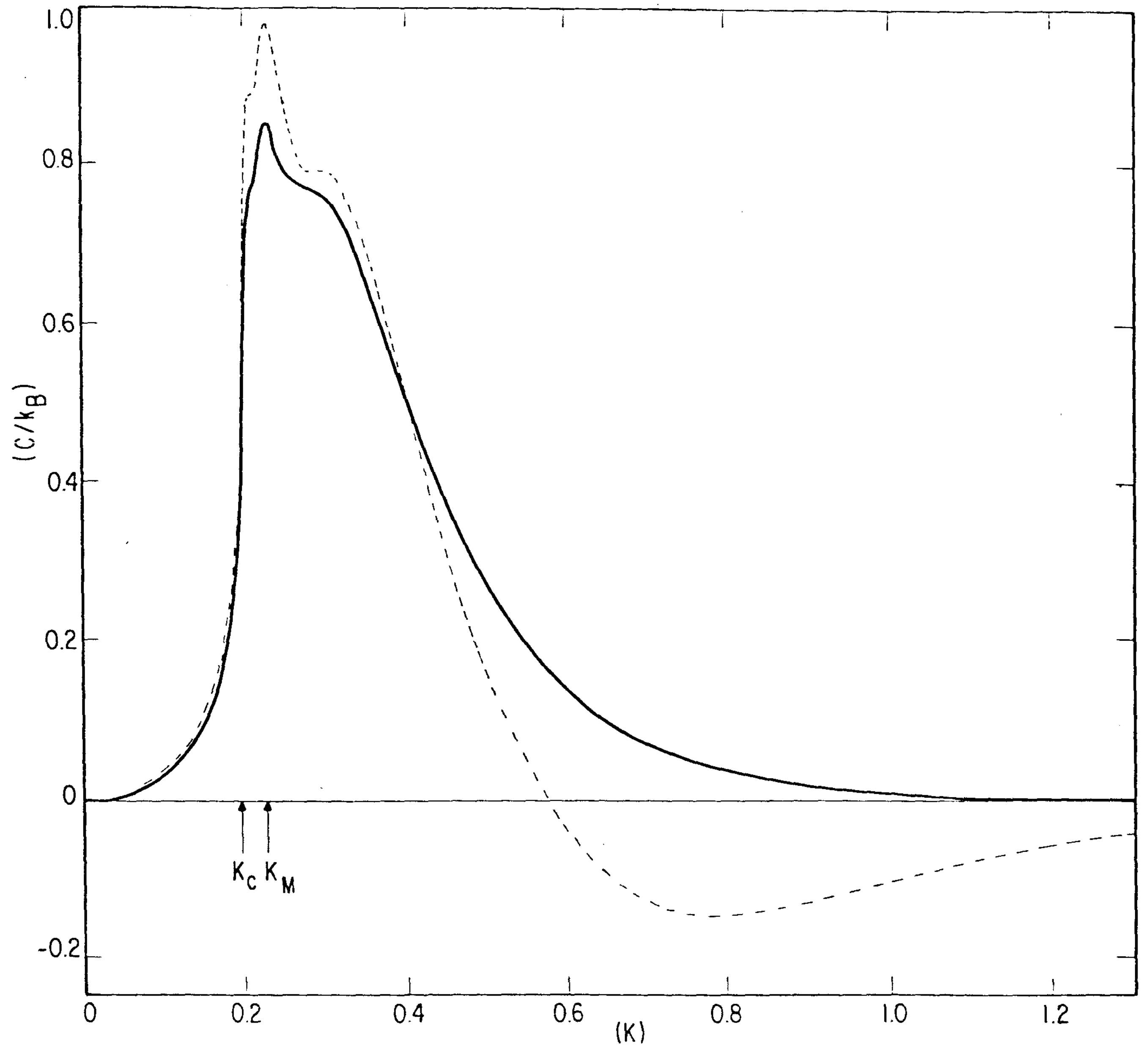


FIG. 3

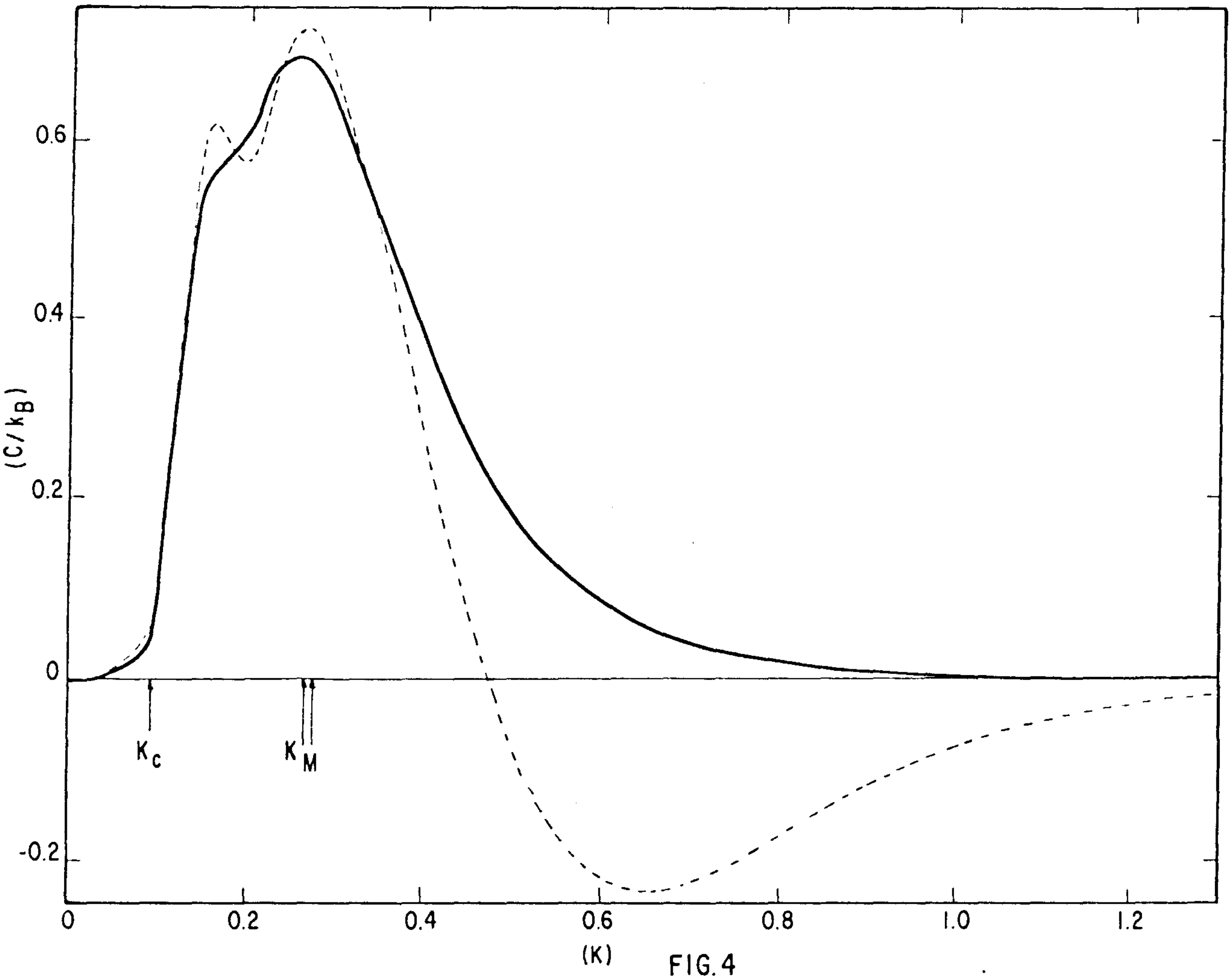


FIG. 4

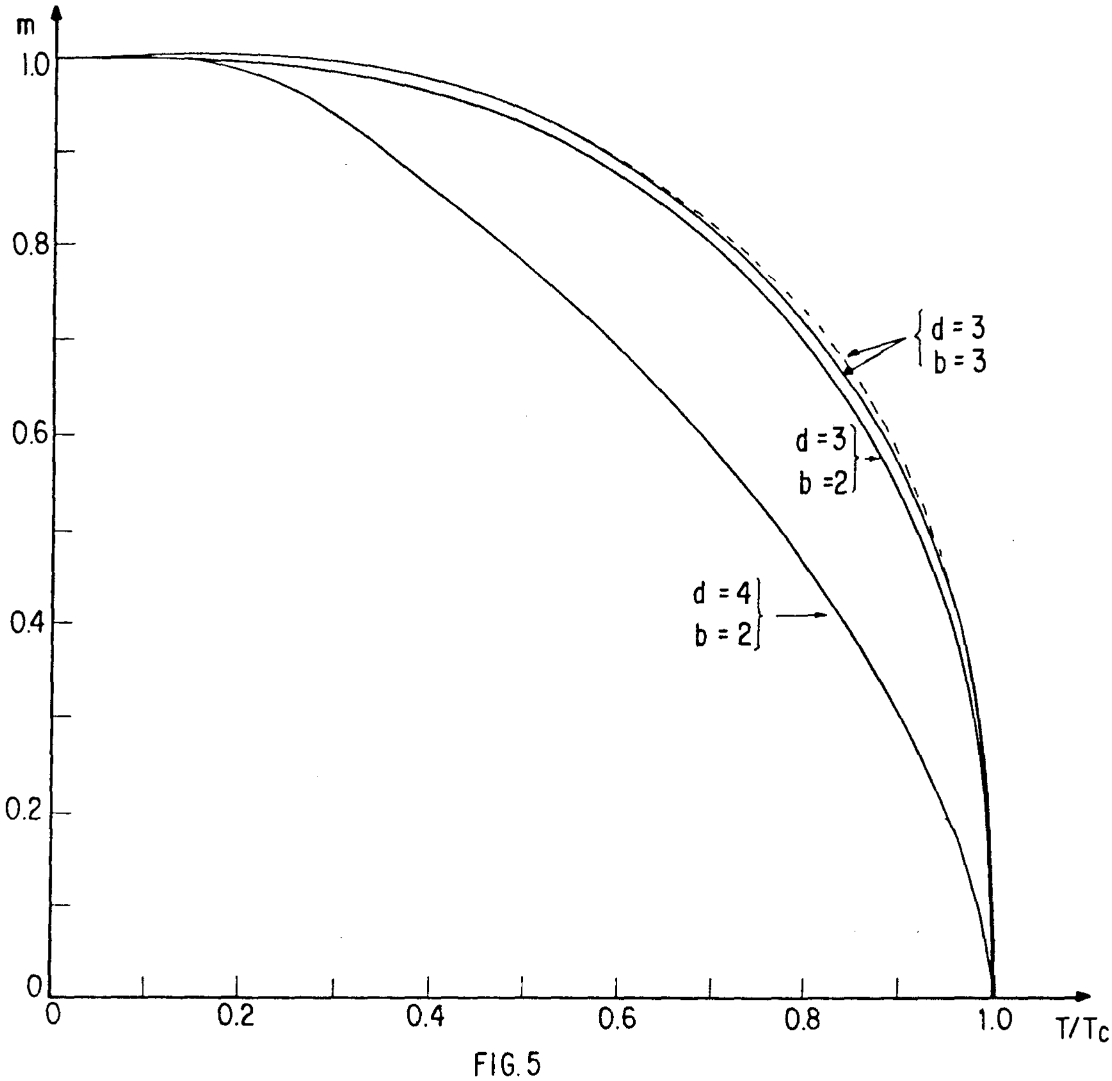


FIG. 5

	d=3				d=4	
	b=2	b=3 equal weight	b=3 different weight	exact or series	b=2	exact or series
K_c	0.18324	0.19808	0.19808	0.22167 [22]	0.09411	0.14966 ±0.00004 (a)
K_M	0.230 (0.226)	0.229 (0.228)	0.229 (0.228)	0.22167 [22]	0.256 (0.264)	0.14966 ±0.00004 (a)
K_c/K_M	0.80 (0.81)	0.86 (0.87)	0.86 (0.87)	1	0.37 (0.36)	1
k_B/C_c	2.505 (2.338)	2.046 (1.643)	2.046 (1.643)	0	16.95 (14.07)	—
k_B/C_M	1.31 (1.26)	1.17 (1.02)	1.17 (1.02)	0	1.44 (1.37)	—
C_c/C_M	0.52 (0.54)	0.57 (0.62)	0.57 (0.62)	1	0.09 (0.10)	1
ν	0.8705	0.8189	0.8189	0.630 ±0.0015 [24]	0.8426	1/2
y_H	2.5608	2.5714	2.5770	2.484 ± 0.004 (b)	3.3182	3
$\alpha=2-d\nu$	-0.611	-0.457	-0.457	0.110 ±0.0045 (c)	-1.37	0
$\beta=(d-y_H)\nu$	0.382	0.351	0.346	0.325 ±0.0015 [24]	0.575	1/2
A	1.30	1.29	1.30	1.569 ±0.003 [25]	1.14	(d)

TABLE 1