ISSN 0029-3865

CNPq

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-024/93

A Note on Quantum Structure Constants

by

Leonardo Castellani and
Marco A.R-Monteiro

Rio de Janeiro
1993

ISSN 0029-3865

CBPF-NF-024/93

A Note on Quantum Structure Constants

by

*Leonardo Castellani** and
*Marco A.R.-Monteiro**†

†Centro Brasileiro de Pesquisas Físicas — CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

*Istituto Nazionale di Fisica Nucleare, Sezione di Torino and
Dipartimento di Fisica Teorica
Via P. Giuria 1, 10125 Torino, Italy

DFTT-18/93
April 1993

email addresses:
Decnet=31890::castellani
Bitnet=castellani@to.infn.it

CBPF-NF-024/93

Abstract

The Cartan-Maurer equations for any q-group of the A_{n-1}, B_n, C_n, D_n series are given in a convenient form, which allows their direct computation and clarifies their connection with the $q = 1$ case. These equations, defining the field strengths, are essential in the construction of q-deformed gauge theories. An explicit expression $\omega^i \wedge \omega^j = -Z^{ij}{}_{kl}\omega^k \wedge \omega^l$ for the q-commutations of left-invariant one-forms is found, with $Z^{ij}{}_{kl}\omega^k \wedge \omega^l \xrightarrow{q\to 1} \omega^j \wedge \omega^i$.

Key-words: Quantum gravity; Differential geometry; Quantum algebras.

Quantum groups [1]-[4] appear as a natural and consistent algebraic structure behind continuously deformed physical theories. Thus, in recent times, there have been various proposals for deformed gauge theories and gravity-like theories [5] based on q-groups.

Such deformations are interesting from different points of view, depending also on which theory we are deforming. For example, in quantized q-gravity theories space-time becomes noncommutative, a fact that does not contradict (Gedanken) experiments under the Planck length, and that could possibly provide a regularization mechanism [7,8]. On the other hand, for the q-gauge theories constructed in [6] spacetime can be taken to be the ordinary Minkowski spacetime, the q-commutativity residing on the fiber itself. As shown in [6], one can construct a q-lagrangian invariant under q-gauge variations. This could suggest a way to break the classical symmetry via a q-deformation, rather than by introducing ad hoc scalar fields. Note also that, unlike the $q = 1$ case, the q-group $U_q(N)$ is simple, thus providing a "quantum unification" of $SU(N) \otimes U(1)$.

In order to proceed from the algebraic q-structure to a dynamical q-field theory, it is essential to investigate the differential calculus on q-groups. Indeed this provides the q-analogues of the "classical" definitions of curvatures, field strengths, exterior products of forms, Bianchi identities, covariant and Lie derivatives and so on, see for ex. [9] for a review.

In this Letter we address and solve a specific problem: to find the Cartan-Maurer equations for any q-group of the A, B, C, D series in explicit form. These equations define the field strengths of the corresponding q-gauge theories [6]. The A_{n-1} case was already treated in [9], where the structure constants were given explicitly, and shown to have the correct classical limit.

To our knowledge, this problem has been tackled previously only in ref. [10]. There, however, the authors use (for the B, C, D q-groups) a definition for the exterior product different from the one introduced in ref.s [11], adopted in [12,13,9] and in the present Letter. As we will comment later, their choice leads to a more complicated scenario.

Quantum groups are characterized by their R-matrix, which controls the non-commutativity of the quantum group basic elements $T^a{}_b$ (fundamental representation):

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} \tag{1}$$

and satisfies the quantum Yang-Baxter equation

$$R^{a_1 b_1}{}_{a_2 b_2} R^{a_2 c_1}{}_{a_3 c_2} R^{b_2 c_2}{}_{b_3 c_3} = R^{b_1 c_1}{}_{b_2 c_2} R^{a_1 c_2}{}_{a_2 c_3} R^{a_2 b_2}{}_{a_3 b_3}, \tag{2}$$

a sufficient condition for the consistency of the "RTT" relations (1). Its elements depend continuously on a (in general complex) parameter q, or even on a set of parameters. For $q \to 1$ we have $R^{ab}{}_{cd} \stackrel{q\to 1}{\longrightarrow} \delta^a_c \delta^b_d$, i.e. the matrix entries $T^a{}_b$

commute and become the usual entries of the fundamental representation. The q-analogue of $\det T = 1$, unitarity and orthogonality conditions can be imposed on the elements $T^a{}_b$, consistently with the RTT relations (1), see [3].

The (uniparametric) R-matrices for the q-groups of the A_{n-1}, B_n, C_n, D_n series can be found in ref. [3]. We recall the projector decomposition of the $\hat{R}$ matrix defined by $\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd}$, whose $q \to 1$ limit is the permutation operator $\delta^a_d \delta^b_c$:

A_n series:

$$\hat{R} = qP_+ - q^{-1}P_- \tag{3}$$

with

$$\begin{aligned} P_+ &= \tfrac{1}{q+q^{-1}}(\hat{R} + q^{-1}I) \\ P_- &= \tfrac{1}{q+q^{-1}}(-\hat{R} + qI) \\ I &= P_+ + P_- \end{aligned} \tag{4}$$

B_n, C_n, D_n series:

$$\hat{R} = qP_+ - q^{-1}P_- + \varepsilon q^{\varepsilon - N} P_0 \tag{5}$$

with

$$\begin{aligned} P_+ &= \tfrac{1}{q+q^{-1}}[\hat{R} + q^{-1}I - (q^{-1} + \varepsilon q^{\varepsilon-N})P_0] \\ P_- &= \tfrac{1}{q+q^{-1}}[-\hat{R} + qI - (q - \varepsilon q^{\varepsilon-N})P_0] \\ P_0 &= \tfrac{1-q^2}{(1-\varepsilon q^{N+1-\varepsilon})(1+\varepsilon q^{\varepsilon-N+1})}K \\ K^{ab}{}_{cd} &= C^{ab}C_{cd} \\ I &= P_+ + P_- + P_0 \end{aligned} \tag{6}$$

where $\varepsilon = 1$ for B_n, D_n, $\varepsilon = -1$ for C_n, and N is the dimension of the fundamental representation $T^a{}_b$, i.e. $N = 2n+1$ for B_n and $N = 2n$ for C_n, D_n. C_{ab} is the q-metric, and C^{ab} its inverse (cf. ref. [3]).

From (3) and (5) we read off the eigenvalues of the $\hat{R}$ matrix, and deduce the characteristic equations:

$$(\hat{R} - qI)(\hat{R} + q^{-1}) = 0 \quad for\ A_{n-1} \quad (Hecke\ condition) \tag{7}$$

$$(\hat{R} - qI)(\hat{R} + q^{-1})(\hat{R} - \varepsilon q^{\varepsilon-N} I) = 0, \quad for\ B_n, C_n, D_n \tag{8}$$

The differential calculus on q-groups, initiated in ref.s [11], can be entirely formulated in terms of the R matrix. The general constructive procedure can be found in ref. [12], or, in the notations we adopt here, in ref. [9].

As discussed in [11] and [12], we can start by introducing the (quantum) left-invariant one-forms $\omega_a{}^b$, whose exterior product

$$\omega_{a_1}{}^{a_2} \wedge \omega_{d_1}{}^{d_2} \equiv \omega_{a_1}{}^{a_2} \otimes \omega_{d_1}{}^{d_2} - \Lambda^{a_2}{}_{a_1}{}^{d_2}{}_{d_1}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} \omega_{c_1}{}^{c_2} \otimes \omega_{b_1}{}^{b_2} \tag{9}$$

is defined by the braiding matrix Λ:

$$\Lambda_{a_1}^{a_2}{}_{d_1}^{d_2}|^{c_1}_{c_2}{}^{b_1}_{b_2} \equiv d^{f_2} d^{-1}_{c_2} \hat{R}^{b_1 f_2}{}_{c_2 g_1} (\hat{R}^{-1})^{c_1 g_1}{}_{a_1 e_1} (\hat{R}^{-1})^{a_2 e_1}{}_{d_1 g_2} \hat{R}^{d_2 g_2}{}_{b_2 f_2} \tag{10}$$

For $q \to 1$ the braiding matrix Λ becomes the usual permutation operator and one recovers the classical exterior product. Note that the "quantum cotangent space" Γ, i.e. the space spanned by the quantum one-forms $\omega_a{}^b$, has dimension N^2, in general bigger than its classical counterpart ($dim\Gamma = N^2$ only for the $U_q(N)$ groups). This is necessary in order to have a bicovariant bimodule structure for Γ (cf. ref. ([10].). The same phenomenon occurs for the q-Lie generators defined below. For these, however, one finds restrictions (induced by the conditions imposed on the $T^a{}_b$ elements) that in general reduce the number of independent generators. Working with N^2 generators is more convenient, since the nice quadratic relations (16) of the q-Lie algebra become of higher order if one expresses them in terms of a reduced set of independent generators. For a discussion see [13].

The relations (7) and (8) satisfied by the $\hat{R}$ matrices of the A and B, C, D series respectively reflect themselves in the relations for the matrix Λ:

$$(\Lambda + q^2 I)(\Lambda + q^{-2} I)(\Lambda - I) = 0 \tag{11}$$

for the A q-groups, and

$$\begin{aligned}(\Lambda + q^2 I)(\Lambda + q^{-2} I)(\Lambda + \varepsilon q^{\varepsilon+1-N} I)(\Lambda + \varepsilon q^{N-\varepsilon-1} I)\\ \times (\Lambda - \varepsilon q^{N+1-\varepsilon} I)(\Lambda - \varepsilon q^{-N-1+\varepsilon} I)(\Lambda - I) = 0\end{aligned} \tag{12}$$

for the B, C, D q-groups, with the same ε as in (8). We give later an easy proof of these two relations.

Besides defining the exterior product of forms, the matrix Λ contains all the the information about the quantum Lie algebra corresponding to the q-group.

The exterior differential of a quantum k-form θ is defined by means of the bi-invariant element $\tau = \sum_a \omega_a{}^a$ as follows:

$$d\theta \equiv \frac{1}{q - q^{-1}} [\tau \wedge \theta - (-1)^k \theta \wedge \tau], \tag{13}$$

The normalization $\frac{1}{q-q^{-1}}$ is necessary in order to obtain the correct classical limit (see for ex. [9]). This linear map satisfies $d^2 = 0$, the Leibniz rule and commutes with the left and right action of the q-group [12].

The exterior differentiation allows the definition of the "quantum Lie algebra generators" $\chi^{a_1}{}_{a_2}$, via the formula [11]

$$da = \frac{1}{q - q^{-1}} [\tau a - a\tau] = (\chi^{a_1}{}_{a_2} * a) \omega_{a_1}{}^{a_2}. \tag{14}$$

where

$$\chi * a \equiv (id \otimes \chi)\Delta(a), \qquad \forall a \in G_q,\ \chi \in G'_q \tag{15}$$

and Δ is the usual coproduct on the quantum group G_q, defined by $\Delta(T^a{}_b) \equiv T^a{}_c \otimes T^c{}_b$. The q-generators χ are linear functionals on G_q. By taking the exterior derivative of (14), using $d^2 = 0$ and the bi-invariance of $\tau = \omega_b{}^b$, we arrive at the q-Lie algebra relations [12], [9]:

$$\chi^{d_1}{}_{d_2}\chi^{c_1}{}_{c_2} - \Lambda_{e_1}{}^{e_2}{}_{f_1}{}^{f_2}|^{d_1}{}_{d_2}{}^{c_1}{}_{c_2}\ \chi^{e_1}{}_{e_2}\chi^{f_1}{}_{f_2} = \mathbf{C}^{d_1}{}_{d_2}{}^{c_1}{}_{c_2}|_{a_1}{}^{a_2}\chi^{a_1}{}_{a_2} \tag{16}$$

where the structure constants are explicitly given by:

$$\mathbf{C}^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}|_{c_1}{}^{c_2} = \frac{1}{q-q^{-1}}[-\delta^{b_1}_{b_2}\delta^{a_1}_{c_1}\delta^{c_2}_{a_2} + \Lambda_b{}^b{}_{c_1}{}^{c_2}|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}]. \tag{17}$$

and $\chi^{d_1}{}_{d_2}\chi^{c_1}{}_{c_2} \equiv (\chi^{d_1}{}_{d_2} \otimes \chi^{c_1}{}_{c_2})\Delta$. Notice that

$$\Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = \delta^{b_1}_{a_1}\delta^{a_2}_{b_2}\delta^{c_1}_{d_1}\delta^{d_2}_{c_2} + O(q-q^{-1}) \tag{18}$$

because the R matrix itself has the form $R = I + (q-q^{-1})U$, with U finite in the $q \to 1$ limit, see ref. [3]). Then it is easy to see that (17) has a finite $q \to 1$ limit, since the $\frac{1}{q-q^{-1}}$ terms cancel.

The Cartan-Maurer equations are found by applying to $\omega_{c_1}{}^{c_2}$ the exterior differential as defined in (13):

$$d\omega_{c_1}{}^{c_2} = \frac{1}{q-q^{-1}}(\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} + \omega_{c_1}{}^{c_2} \wedge \omega_b{}^b). \tag{19}$$

Written as above, the Cartan-Maurer equations are not of much use for computations. The right-hand side has an undefined $\frac{0}{0}$ classical limit. We need a formula of the type $\omega_{c_1}{}^{c_2} \wedge \omega_b{}^b = -\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} + O(q-q^{-1})$ that allows to eliminate in (19) the terms with the trace $\omega_b{}^b$ (which has no classical counterpart) and obtain an explicitly $q \to 1$ finite expression.

The desired "ω -permutator" can be found as follows. We first treat the case of the A_{n-1} series. We apply relation (7) to the tensor product $\omega \otimes \omega$, i.e.:

$$(\Lambda^{ij}{}_{kl} + q^2\delta^i_k\delta^j_l)(\Lambda^{kl}{}_{mn} + q^{-2}\delta^k_m\delta^l_n)(\Lambda^{mn}{}_{rs} - \delta^m_r\delta^n_s)\ \omega^m \otimes \omega^n = 0 \tag{20}$$

where we have used the adjoint indices ${}^i \leftrightarrow {}_a{}^b$, ${}_i \leftrightarrow {}^a{}_b$. Inserting the definition of the exterior product $\omega^n \wedge \omega^n = \omega^m \otimes \omega^n - \Lambda^{mn}{}_{rs}\omega^r \otimes \omega^s$ yields

$$(\Lambda^{ij}{}_{kl} + q^2\delta^i_k\delta^j_l)(\Lambda^{kl}{}_{mn} + q^{-2}\delta^k_m\delta^l_n)\ \omega^m \wedge \omega^n = 0 \tag{21}$$

Multiplying by Λ^{-1} gives $(\Lambda + (q^2+q^{-2})I + \Lambda^{-1})\ \omega \wedge \omega$, or equivalently

$$\omega^i \wedge \omega^j = -Z^{ij}{}_{kl}\omega^k \wedge \omega^l \tag{22}$$

$$Z^{ij}{}_{kl} \equiv \frac{1}{q^2+q^{-2}}[\Lambda^{ij}{}_{kl} + (\Lambda^{-1})^{ij}{}_{kl}]. \tag{23}$$

cf. ref. [9]. The ω-permutator $Z^{ij}{}_{kl}$ has the expected $q \to 1$ limit, that is $\delta^i_l \delta^j_k$.

There is another way to deduce the permutator Z, based on projector methods, that we will use for the B, C, D series. We first illustrate it in the easier A-case. Define

$$(P_I, P_J)_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} \equiv d^{f_2} d^{-1}_{c_2} \hat{R}^{b_1 f_2}{}_{c_2 g_1} (P_I)^{c_1 g_1}{}_{a_1 e_1} (\hat{R}^{-1})^{a_2 e_1}{}_{d_1 g_2} (P_J)^{d_2 g_2}{}_{b_2 f_2} \tag{24}$$

with I,J=+, −, the projectors P_+, P_- being given in (4). The (P_I, P_J) are themselves projectors, i.e.:

$$(P_I, P_J)(P_K, P_L) = \delta_{IK}\delta_{JL}(P_I, P_J) \tag{25}$$

Moreover

$$(I, I) = I \tag{26}$$

so that

$$(I, I) = (P_+ + P_-, P_+ + P_-) = (P_+, P_+) + (P_-, P_-) + (P_+, P_-) + (P_-, P_+) = I \tag{27}$$

Eq. (25) is easy to prove by using (24) and the relation, valid for all A, B, C, D q-groups:

$$d^f d^{-1}_c \hat{R}^{bf}{}_{cg} (\hat{R}^{-1})^{ce}{}_{ba} = \delta^f_a \delta^e_g \tag{28}$$

From the definition of Λ (10), using (3) and (24) we can write

$$\Lambda = (P_+, P_+) + (P_-, P_-) - q^{-2}(P_+, P_-) - q^2(P_-, P_+) \tag{29}$$

This decomposition shows that Λ has eigenvalues $1, q^{\pm 2}$, and proves therefore eq. (11). From the definition of the exterior product $\omega \wedge \omega = \omega \otimes \omega - \Lambda\omega \otimes \omega$ we find the action of the projectors (P_I, P_J) on $\omega \wedge \omega$:

$$(P_+, P_+)\omega \wedge \omega = (P_-, P_-)\omega \wedge \omega = 0 \tag{30}$$

$$(P_+, P_-)\omega \wedge \omega = (1 + q^{-2})(P_+, P_-)\omega \otimes \omega, \quad (P_-, P_+)\omega \wedge \omega = (1 + q^2)(P_-, P_+)\omega \otimes \omega \tag{31}$$

Using (27) and (30) we find :

$$\omega\wedge\omega = [(P_+, P_+) + (P_-, P_-) + (P_+, P_-) + (P_-, P_+)]\omega\wedge\omega = [(P_+, P_-) + (P_-, P_+)]\omega\wedge\omega \tag{32}$$

The ω-permutator is therefore $Z = -(P_+, P_-) - (P_-, P_+)$. We can express it in terms of the Λ matrix by observing that

$$(P_+, P_-) + (P_-, P_+) = -\frac{1}{q^2+q^{-2}}(\Lambda + \Lambda^{-1}) + \frac{2}{q^2+q^{-2}}((P_+, P_+) + (P_-, P_-)) \tag{33}$$

as one deduces from (29). Note that Λ^{-1} is given in terms of projectors by the same expression as in (29), with $q \to q^{-1}$. When acting on $\omega \wedge \omega$ the $(P_+, P_+), (P_-, P_-)$ terms in (33) can be dropped because of (30), so that finally we arrive at eq. (23).

Because of the expansion (18) and a similar one for Λ^{-1} we easily see that the ω-permutator (23) can be expanded as

$$Z_{c_1}^{c_2}{}_{d_1}^{d_2}|^{a_1}_{a_2}{}^{b_1}_{b_2} \omega_{a_1}{}^{a_2} \wedge \omega_{b_1}{}^{b_2} = \omega_{d_1}{}^{d_2} \wedge \omega_{c_1}{}^{c_2} + (q - q^{-1}) W_{c_1}^{c_2}{}_{d_1}^{d_2}|^{a_1}_{a_2}{}^{b_1}_{b_2} \omega_{a_1}{}^{a_2} \wedge \omega_{b_1}{}^{b_2} \tag{34}$$

where W is a finite matrix in the limit $q \to 1$.

Let us return to the Cartan-Maurer eqs. (19). Using (22) we can write:

$$d\omega_{c_1}{}^{c_2} = \frac{1}{q - q^{-1}} (\omega_b{}^b \wedge \omega_{c_1}{}^{c_2} - Z_{c_1}^{c_2}{}_b^b|^{a_1}_{a_2}{}^{b_1}_{b_2} \omega_{a_1}{}^{a_2} \wedge \omega_{b_1}{}^{b_2}) \tag{35}$$

where Z is given by $(\Lambda + \Lambda^{-1})/(q^2 + q^{-2})$, cf. (23). Because of (34) we see that the $\omega_b{}^b$ terms disappear, and (35) has a finite $q \to 1$ limit.

We now repeat the above construction for the case of q-groups belonging to the B, C, D series.

Using (5) and (24) we find the following projector decomposition for the Λ matrix :

$$\begin{aligned}\Lambda = \; & (P_+, P_+) + (P_-, P_-) + (P_0, P_0) + \epsilon q^{\epsilon - 1 - N} (P_+, P_0) + \epsilon q^{-(\epsilon - 1 - N)} (P_0, P_+) \\ & - q^{-2} (P_+, P_-) - q^2 (P_-, P_+) - \epsilon q^{N - \epsilon - 1} (P_0, P_-) - \epsilon q^{-(N - \epsilon - 1)} (P_-, P_0)\end{aligned} \tag{36}$$

from which we read off the eigenvalues of Λ, and prove eq. (12). Proceeding as in the A case, we find the action of the projectors on $\omega \wedge \omega$:

$$(P_+, P_+) \omega \wedge \omega = (P_-, P_-) \omega \wedge \omega = (P_0, P_0) \omega \wedge \omega = 0 \tag{37}$$

$$\begin{aligned}(P_+, P_-) \omega \wedge \omega &= (1 + q^{-2})(P_+, P_-) \omega \otimes \omega, \\ (P_-, P_+) \omega \wedge \omega &= (1 + q^2)(P_-, P_+) \omega \otimes \omega\end{aligned} \tag{38}$$

$$\begin{aligned}(P_-, P_0) \omega \wedge \omega &= (1 + \epsilon q^{-(N - \epsilon - 1)})(P_-, P_0) \omega \otimes \omega, \\ (P_0, P_-) \omega \wedge \omega &= (1 + \epsilon q^{N - \epsilon - 1})(P_0, P_-) \omega \otimes \omega\end{aligned} \tag{39}$$

$$\begin{aligned}(P_+, P_0) \omega \wedge \omega &= (1 - \epsilon q^{\epsilon - 1 - N})(P_+, P_0) \omega \otimes \omega, \\ (P_0, P_+) \omega \wedge \omega &= (1 - \epsilon q^{-(\epsilon - 1 - N)})(P_0, P_+) \omega \otimes \omega\end{aligned} \tag{40}$$

Again the sum of the projectors (P_I, P_J) yields the identity, so that we can write:

$$\omega \wedge \omega = [(P_+, P_-) + (P_-, P_+) + (P_-, P_0) + (P_0, P_-) + (P_+, P_0) + (P_0, P_+)] \omega \wedge \omega \tag{41}$$

where we have taken (37) into account. The ω-permutator Z is therefore given by

$$Z = -[(P_+, P_-) + (P_-, P_+) + (P_-, P_0) + (P_0, P_-) + (P_+, P_0) + (P_0, P_+)] \quad (42)$$

Can we express it in terms of odd powers of the Λ matrix, as in the case of the A groups ? The answer is: only partially. In fact, by elementary algebra we find that

$$Z = -\alpha(\Lambda + \Lambda^{-1}) - \beta(\Lambda^3 + \Lambda^{-3}) - (1 - \alpha q_{-\epsilon N} - \beta q_{-3\epsilon N})[(P_\sigma, P_0) + (P_0, P_\sigma)] \quad (43)$$

with $\sigma \equiv sgn(\epsilon)$ and

$$\alpha = -\frac{1 + \beta q_6}{q_2} \quad (44)$$

$$\beta = \frac{q_2 - q_{\epsilon N-2}}{q_6 q_{\epsilon N-2} - q_2 q_{3(\epsilon N-2)}} \quad (45)$$

$$q_n \equiv q^n + q^{-n} \quad (46)$$

Note: Λ^r is given by

$$\begin{aligned}\Lambda^r = (P_+, P_+) + (P_-, P_-) + (P_0, P_0) + \epsilon^r[q^{r(\epsilon-1-N)}(P_+, P_0) + q^{-r(\epsilon-1-N)}(P_0, P_+)] \\ +(-1)^r[q^{-2r}(P_+, P_-) + q^{2r}(P_-, P_+)] \\ +(-\epsilon)^r[q^{r(N-\epsilon-1)}(P_0, P_-) + q^{-r(N-\epsilon-1)}(P_-, P_0)]\end{aligned} \quad (47)$$

Let us check that Z in (43) has a correct classical limit. We have $\alpha \stackrel{q\to1}{\longrightarrow} -\frac{9}{16}$ and $\beta \stackrel{q\to1}{\longrightarrow} \frac{1}{16}$; taking into account that the $(P_\sigma, P_0), (P_0, P_\sigma)$ terms disappear in the classical limit (cf. eq. (39), (40)) when applied to $\omega \wedge \omega$, we find the expected limit $Z^{ij}{}_{kl} \stackrel{q\to1}{\longrightarrow} \delta^i_l \delta^j_k$.

The Cartan-Maurer equations are deduced as before, and are given by (35) where now Z is the ω permutator of eq. (43) (Note: for explicit calculations the expression (42) is more convenient). Again the $\omega_b{}^b$ terms drop out since Z admits the expansion $Z_{c_1}{}^{c_2}{}_{d_1}{}^{d_2}|^{a_1}{}_{a_2}{}^{b_1}{}_{b_2}\omega_{a_1}{}^{a_2} \wedge \omega_{b_1}{}^{b_2} = -\omega_{d_1}{}^{d_2} \wedge \omega_{c_1}{}^{c_2} + O(q - q^{-1})$.

In conclusion: we have found an explicit (and computable) expression for the Cartan-Maurer equations of the B_n, C_n, D_n q-groups. This opens the possibility of constructing gauge theories of these q-groups, following the procedure used in [6] for the A_{n-1} q-groups.

Finally, let us comment on the differential calculus presented by the authors of ref. [10]. Their definition of exterior product in the B, C, D case differs from ours (and from the one adopted in [11,12,13]), and essentially amounts to require that $(P_\sigma, P_0)\omega \wedge \omega = 0$, $(P_0, P_\sigma)\omega \wedge \omega = 0$, besides (37). This has one advantage: the term $(P_\sigma, P_0) + (P_0, P_\sigma)$ disappears in the expression (43). The disadvantage is that the defining formula $\omega \wedge \omega = (I - \Lambda)\omega \otimes \omega$ does not hold any more for the B, C, D series, so that the general treatment of ref. [11] and the constructive procedure of ref.s [12] do not apply.

References

[1] V. Drinfeld, Sov. Math. Dokl. **32** (1985) 254.

[2] M. Jimbo, Lett. Math. Phys. **10** (1985) 63; **11** (1986) 247.

[3] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Algebra and Analysis, **1** (1987) 178.

[4] S. Majid, Int. J. Mod. Phys. **A5** (1990) 1.

[5] A.P. Isaev and Z. Popowicz, *q-Trace for the Quantum Groups and q-deformed Yang-Mills Theory*, Wroclaw preprint ITP UWr 786-91; I.Y. Aref'eva and I.V. Volovich, Mod. Phys. Lett **A6** (1991) 893; Phys. Lett. **B264** (1991) 62; L. Castellani, Phys. Lett. **B279** (1992) 291 and Phys. Lett. **B292** (1992) 93; S. Meljanac and S. Pallua, *Classical field lagrangian and deformed algebras*, DFPF/92/TH/15; A. Dimakis and F. Müller-Hoissen, *Noncommutative differential calculus, gauge theory and gravitation*, GOET-TP-33/92; K. Wu and R.-J. Zhang, Commun. Theor. Phys. **17** (1992) 175; T. Brzeziński and S. Majid, *Quantum group gauge theory on quantum spaces*, DAMTP/92-27 and *Quantum group gauge theory on classical spaces*, DAMTP/92-51.

[6] L. Castellani, $U_q(N)$ *gauge theories*, Torino preprint DFTT 74/92.

[7] A. Connes, Publ. Math. IHES Vol. **62** (1986) 41; *Géométrie non commutative* (Inter Editions, Paris, 1990).

[8] S. Majid, Classs. Quantum Grav. **5** (1988) 1587; Int. J. Mod. Phys. **A5** (1990) 4689.

[9] P. Aschieri and L. Castellani, *An introduction to non-commutative differential geometry on quantum groups*, preprint CERN-TH.6565/92, DFTT-22/92 (1992), to be publ. in Int. J. Mod. Phys. A.

[10] U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, Commun. Math. Phys. **142** (1991) 605.

[11] S.L. Woronowicz, Publ. RIMS, Kyoto Univ., Vol. **23**, 1 (1987) 117; Commun. Math. Phys. **111** (1987) 613 and Commun. Math. Phys. **122**, (1989) 125.

[12] B. Jurčo, Lett. Math. Phys. **22** (1991) 177.

[13] B. Zumino, *Introduction to the Differential Geometry of Quantum Groups*, LBL-31432 and UCB-PTH-62/91, notes of a plenary talk given at the 10-th IAMP Conf., Leipzig (1991); P. Schupp, P. Watts and B. Zumino, *Bicovariant quantum algebras and quantum Lie algebras*, LBL-32315, UCB-PTH-92/14.