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PURE AND DILUTE Z(N) SPIN AND GENERALIZED GAUGE LATTICE SYSTEMS: DUALITY AND RELATED CONJECTURES

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ABSTRACT

By generalizing recently introduced quantities (referred to as "thermal transmissivities"), we recover the usual duality results concerning the pure Z(N) spin systems (including the standard Ising and Potts models) as well as generalized gauge systems (plaquettes or more complex simplex) in d- dimensional hypercubic lattices. The essential relationship between duality and simple series- parallel transformation becomes evident. The simplicity of the equations enables conjectures on the approximate critical frontier of the diluted version of the above systems, including some particular asymptotic behaviours which we believe exact. As an illustration the $d = 2$ diluted Z(4) spin system is discussed in some detail: for those regions where exact results are available the agreement is satisfactory.

I - INTRODUCTION

Since the Kramers and Wannier 1941 discussion of the Ising model, duality arguments have been a powerfull tool for discussing the location of critical frontiers (CF) in various statistical systems such as bond percolation (Sykes and Essam 1963) and the N- states Potts model (Potts 1952, Kim and Joseph 1974). Quite general results (concerning in particular the Z(N) models) have been obtained by Wu and Wang 1976, Alcaraz and Köberle 1980, 1981 and Savit 1980.

On different grounds Nelson and Fisher 1975 and Yeomans and Stinchcombe 1979 (among others) have introduced, in the discussion of Ising models, a convenient variable, namely $t \equiv \text{th}\, J/k_BT$ (J is the exchange coupling constant and T the temperature), referred hereafter to as *thermal transmissivity* (Tsallis and Levy 1980 a, Levy at al 1980). This quantity can be extended (Tsallis and Levy 1980 b, Tsallis 1981) to cover the N- states Potts model; its expression is given by

$$t \equiv \frac{1-e^{-NJ/k_BT}}{1+(N-1)\,e^{-NJ/k_BT}} \qquad (1)$$

Remark that in the limit $N \to 1$ t equals $1-e^{-J/k_BT}$ thus reproducing the variable which establishes the isomorphism with the bond percolation problem (Kasteleyn and Fortuin 1969).

The main advantage of the t- variable is to provide a probability - like algorithm to calculate the equivalent transmissivity t_s of a *series* array of two bonds whose transmissivities are t_1 and t_2, namely

$$t_s = t_1\, t_2 \tag{2}$$

If the array is a *parallel* one the equivalent transmissivity t_p satisfies

$$t_p^D = t_1^D\, t_2^D \tag{3}$$

where

$$t_i^D \equiv \frac{1 - t_i}{1 + (N-1)\, t_i} \qquad (i = 1,2,p) \tag{4}$$

The super-script D holds for *dual* (we refer here to the standard duality: see Section II); let us stress that through transformation (4), the series and parallel composition algorithms (respectively Eqs. (2) and (3)) become *one and the same*.

In the present paper (Section II) we extend the transmissivity to cover spin and generalized gauge Z(N) systems which contain several coupling constants and we exhibit that the standard dual transformation can be very simply expressed as a series- parallel transformation.

The simplicity of Eqs. (2) and (3) has enabled quite satisfactory conjectures (Tsallis and Levy 1980 a, Levy et al 1980 and Tsallis 1981) about the CF of the bond-dilute (or even bond-mixed) Ising and Potts models. It seems therefore quite natural to propose (Section III) analogous conjectures for the CF of diluted versions of general Z(N) systems (only d- dimensional hypercubic lattices are considered). The particular case of the d = 2 Z(4) spin system is treated in detail: some already known numerical results are exactly or approximatively recovered and a few predictions are proposed.

II - TRANSMISSIVITY AND DUALITY IN PURE Z(N) MODELS

Let us consider a site (0- simplex) to which we associate a Z(N) random variable $S \equiv e^{i\frac{2\pi n}{N}}$ where $n = 0,1,2,\ldots,N-1$. Then we construct a bond (1-simplex) by joining two such sites (noted 1 and 2) and we associate to it the Z(N) random variable A_1 (the subindex 1 refers to 1-simplex) defined by

$A_1 \equiv S_1^* S_2 = e^{i\,2\pi(n_2-n_1)/N}$. Let $p^{(\alpha)}$ be the probability that this variable takes the value $e^{i\,2\pi\alpha/N}$. We define the N- dimensional vector *transmissivity* $\vec{t}$ through its components given by

$$t^{(\alpha)} = \sum_{\beta=0}^{N-1} p^{(\beta)} e^{i\frac{2\pi}{N}\alpha\beta} \qquad (\alpha = 0,1,2,\ldots,N-1) \tag{5}$$

hence

$$p^{(\beta)} = \frac{1}{N}\sum_{\alpha=0}^{N-1} t^{(\alpha)} e^{-i\frac{2\pi}{N}\alpha\beta} \qquad (\beta = 0,1,2,\ldots,N-1) \tag{5'}$$

Remark that

$$t^{(o)} = 1 \tag{6}$$

and

$$t^{(N-\alpha)} = \left[t^{(\alpha)}\right]^* \tag{7}$$

Remark also that in the case of a Potts bond we have that $p^{(o)} = 1/\left[1+(N-1)\,e^{-NJ/k_BT}\right]$ and, for $\beta \neq 0$, $p^{(\beta)} = e^{-NJ/k_BT}/\left[1+(N-1)\,e^{-NJ/k_BT}\right]$ therefore $t^{(\beta)}$ for $\beta \neq 0$ reduces to expression (1).

Let us now calculate the equivalent transmissivity $\vec{t}_s$ of a *series* array of two Z(N) bonds whose transmissivities are $\vec{t}_1$ and $\vec{t}_2$. If we take into account that the equivalent probabilities are given by

$$p_s^{(\alpha)} = \sum_{\beta=0}^{N-1} p_1^{(\beta)}\, p_2^{(\alpha-\beta)} \tag{8}$$

we immediately obtain that

$$t_s^{(\alpha)} = t_1^{(\alpha)}\, t_2^{(\alpha)} \qquad (\alpha = 0,1,2,\ldots,N-1) \tag{9}$$

If we have instead a *parallel* array, the equivalent probabilities are given by

$$p_p^{(\alpha)} = \frac{p_1^{(\alpha)}\, p_2^{(\alpha)}}{\sum_{\beta=0}^{N-1} p_1^{(\beta)}\, p_2^{(\beta)}} \tag{10}$$

which provides the following relations:

$$t_p^{(\alpha)\,D} = t_1^{(\alpha)\,D}\, t_2^{(\alpha)\,D} \qquad (\alpha = 0,1,2,\ldots,N-1) \tag{11}$$

where

$$\left(t_j^{(\alpha)\,D}\right)^* \equiv \frac{\sum_{\beta=0}^{N-1} t_j^{(\beta)}\, e^{-i\,2\pi\,\alpha\beta/N}}{\sum_{\beta=0}^{N-1} t_j^{(\beta)}} \qquad (j = 1,2,p) \tag{12}$$

Let us stress that through transformation (12) the series and parallel algorithms (Eqs. (9) and (11)) become *one and the same*. The complex conjugation in Eq. (12) ensures that the dual of the dual is the identity.

It is interesting to remark that the real quantity

$$\rho \equiv \frac{1}{\sqrt{N}} \sum_{\alpha=0}^{N-1} t^{(\alpha)} = \sqrt{N}\, p^{(o)}$$

transforms under duality similarly to a resistance (or a conductance), i.e.

$$\rho^D = \frac{1}{\rho}$$

Furthermore the quantity

$$\tau \equiv \frac{\sqrt{N}\,\rho - 1}{N - 1} \tag{13}$$

transforms under duality like the transmissivity of a Potts model (see Eq. (4)), i.e.

$$\tau^D = \frac{1-\tau}{1+(N-1)\tau} \tag{14}$$

Finally we may define another interesting quantity (used in Section III) namely

$$\sigma \equiv \frac{\ell n(\sqrt{N}\,\rho)}{\ell n\, N} = \frac{\ell n\,[1+(N-1)\tau]}{\ell n\, N} \tag{15}$$

We immediately verify that under duality σ transforms like a probability, i.e.

$$\sigma^D = 1 - \sigma \tag{16}$$

This variable generalizes the s- variable introduced in Levy et al 1980 and extended by Tsallis 1981 (see also Tsallis and de Magalhães 1981).

We shall now restate on more general grounds what we have said until now (and by the way clarify the nomenclature introduced in Eq. (12)). Let us consider a square plaquette (hypercubic s- simplex); its border is constituted by 4 bonds (2s (s-1)-simplex). To the i-th bond ((s-1)-simplex) we associate a Z(N) random variable $S_i = e^{i\,2\pi n_i/N}$ and to the plaquette (s- simplex) we associate another Z(N) variable noted $A_2(A_s)$ defined by $A_2 \equiv \prod'_{\{i\}} S_i \equiv S_1^* S_2^* S_3 S_4$ $(A_s \equiv \prod'_{\{i\}} S_i \equiv S_1^* S_2^* \ldots S_s^* S_{s+1} \ldots S_{2s})$ where the prime stands for *oriented* product. To a non elementary plaquette (s- simplex) we shall associate the Z(N) variable $A_2 \equiv \prod'_{\{i\}} S_i$ $(A_s \equiv \prod'_{\{i\}} S_i)$ where the oriented product runs over *all* the bordering bonds ((s-1)- simplex). The plaquette (s- simplex) will be said α- *frustrated* when $A_2(A_s)$ equals $e^{i\,2\pi\alpha/N}$ with $\alpha = 0,1,2,\ldots,$ N-1; it is clear that 0- frustrated corresponds to not frustrated. Let $p^{(\alpha)}$ be the probability that the plaquette (s-simplex) is α- frustrated. Through Eq.(5) we define the transmissivity $\vec{t}$ associated to the plaquette (s- simplex).

Two plaquettes (s- simplex) noted 1 and 2 will be said to be in *series* if they share *one and only one* bordering bond ((s-1)- simplex); the Z(N) random variable associated to this array is obtained by the product $(A_2)_1\ (A_2)_2$ $((A_s)_1 (A_s)_2)$, therefore Eqs. (8) and (9) still hold in the present general picture. Two plaquettes will be said to be in *parallel* if they share the *whole* border; the probability of this array being α- frustrated is still given by Eq.(10)

which implies Eqs. (11) and (12).

It is well known (Yoneya 1978 and Savit 1980) that through duality transformation a s- simplex in a d- dimensional original lattice goes to a (d-s)- simplex in the dual lattice. Consequently the transmissivity $\vec{t}$ of that s- simplex in the original lattice is related to the transmissivity (noted $\vec{t}^{\,D}$) of the (d-s)- simplex in the dual lattice through Eq. (12).

Let us now perform an application of the present formalism. We shall consider the general ferromagnetic Z(N) bond system in square lattice; its Hamiltonian $\mathcal{H}$ (or action) is given (Alcaraz and Köberle 1980,1981) by

$$\frac{\mathcal{H}}{k_B T} = \sum_{\langle i,j\rangle} h(n_i - n_j) \tag{17}$$

with

$$\begin{aligned} h(n_i-n_j) &= K_1 - \sum_{\beta=1}^{\bar{N}} K_\beta \left[(S_i S_j^*)^\beta + \text{c.c.} \right] \\ &= K_1 - \sum_{\beta=1}^{\bar{N}} 2\, K_\beta \cos\left[\frac{2\pi\beta}{N} (n_i - n_j) \right] \quad ; \end{aligned} \tag{18}$$

the sum of Eq.(17) runs over all the nearest- neighbours and $\bar{N}$ is the integer part of N/2 if $N \geqslant 2$; in the limit $N \to 1$ $\bar{N}$ equals one. The probability that $n_1 - n_2 = \alpha$ (mod. N) is given by

$$p^{(\alpha)} = \frac{e^{-h(\alpha)}}{\sum_{\beta=0}^{N-1} e^{-h(\beta)}} \tag{19}$$

which, through Eq. (5), leads, for $N \geqslant 2$, to

$$t^{(\alpha)} = \frac{\sum_{\beta=0}^{N-1} e^{-h(\beta)} e^{i\,2\pi\alpha\beta/N}}{\sum_{\beta=0}^{N-1} e^{-h(\beta)}} \tag{20}$$

Remark that $t^{(\alpha)} = t^{(N-\alpha)} = \left(t^{(\alpha)}\right)^{*}$. If we consider the particular case of the Potts model (for $N > 2, K_1 = K_2 = \ldots = K_{\overline{N-1}} = \frac{3+(-1)^N}{2} K_{\overline{N}}$ hence $t^{(1)} = t^{(2)} = \ldots = t^{(N-1)}$) we immediately verify that Eq.(20) recovers Eq.(1).

If we substitute now Eq.(20) into Eq.(12) we obtain

$$t^{(\alpha)\,D} = \frac{e^{-h(\alpha)}}{e^{-h(o)}} \tag{21}$$

If we invert finally Eq.(20) and replace into Eq.(21) we obtain

$$t^{(\alpha)\,D} = \frac{\sum_{\beta=0}^{N-1} t^{(\beta)}\, e^{-i\,2\pi\alpha\beta/N}}{\sum_{\beta=0}^{N-1} t^{(\beta)}} \tag{22}$$

which, through notation changes, precisely corresponds to the *exact* dual transformation (Cardy 1980, Alcaraz and Köberle 1980,1981). In the particular case of the Potts model we immediately verify that Eq.(22) recovers Eq.(4). If we take into account the self-duality of the square lattice and the fact we are considering bonds (whose transformed simplex are still bonds) we have that the general self-dual frontier (which contains all the self-dual points and only them) is given by

$$\vec{t} = \vec{t}^{\,D} \tag{23}$$

This equation univoquely determines the location of the critical frontier in the region of the parameter-space where it is unique (Cardy 1980, Alcaraz and Köberle 1980,81).

For the general four-dimensional Z(N) hypercubic lattice gauge model (whose Hamiltonian — invariant through local gauge Z(N) transformation — is analogous to that of Eq.(17)) as well as for the general three-dimensional Z(N) cubic lattice gauge model including Higgs fields it is straightforward to verify that Eq.(12) precisely corresponds to the *exact* dual transformation (Alcaraz and Köberle 1981).

III - DILUTED Z(N) MODELS

We shall now consider a bond-diluted version of the model described by Hamiltonian (17-18); in other words its coupling constants will be now random variables whose probability distribution is

$$P_K(K_1,K_2,\ldots,K_{\bar{N}}) = (1-p) \prod_{\beta=1}^{\bar{N}} \delta(K_\beta) + p \prod_{\beta=1}^{\bar{N}} \delta(K_\beta - K_\beta^o) \tag{24}$$

where $\{K_\beta^o\}$ are known constants. This distribution immediately leads to the distribution P_t for the transmissivities:

$$P_t(t^{(1)},t^{(2)},\ldots,t^{(\bar{N})}) = (1-p) \prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)}) + p \prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)} - t_o^{(\beta)}) \tag{25}$$

where the $\{t_o^{(\beta)}\}$ are related to $\{K_\beta^o\}$ through Eq.(20) with $\{K_\beta^o\}$ playing the role of $\{K_\beta\}$. The probability distribution of the dual variable $\vec{t}^{D}$ is given by

$$P_t^D\left(t^{(1)D},t^{(2)D},\ldots,t^{(\bar{N})D}\right) = (1-p)\prod_{\beta=1}^{\bar{N}}\delta\left(t^{(\beta)D}-1\right)+p\prod_{\beta=1}^{\bar{N}}\delta\left(t^{(\beta)D}-t_0^{(\beta)D}\right) \tag{26}$$

where $\{t_0^{(\beta)D}\}$ is related to $\left\{t_0^{(\beta)}\right\}$ through Eq. (22). The probability distributions $P_\tau(\tau)$ and $P_\tau^D(\tau^D)$ of the variables τ and τ^D respectively defined by Eqs. (13) and (14) are given by

$$P_\tau(\tau) = (1-p)\delta(\tau) + p\delta(\tau-\tau_0) \tag{27}$$

and

$$P_\tau^D(\tau^D) = (1-p)\delta(\tau^D-1) + p\delta\left(\tau^D - \frac{1-\tau_0}{1+(N-1)\tau_0}\right) \tag{28}$$

where τ_0 is related to $\{t_0^{(\beta)}\}$ through Eq. (13) substituting $t^{(\alpha)}$ by $t_0^{(\alpha)}$.

Following along the lines of Tsallis and Levy 1980 a, Levy et al 1980 and Tsallis 1981 we can now conjecture three slightly different approximations of the CF in the region of the parameter-space (whose dimensionality is $\bar{N}+1$) where the transition is unique. Our three present proposals are

$$\langle t^{(1)}\rangle_{P_t} = \langle t^{(1)}\rangle_{P_t^D} \tag{29}$$

$$\langle \tau \rangle_{P_\tau} = \langle \tau \rangle_{P_\tau^D} \tag{30}$$

and

$$\langle \sigma \rangle_{P_\tau} = \langle \sigma \rangle_{P_\tau^D} \tag{31}$$

which, through use of definitions (13) and (15), respectively lead to

$$p\ t_0^{(1)} = 1-p+p\ t_0^{(1)D} \tag{29'}$$

$$p\ \tau_0 = 1-p+p\ \frac{1-\tau_0}{1+(N-1)\tau_0} \tag{30'}$$

and

$$1+(N-1)\tau_0 = N^{1/2p} \tag{31'}$$

We remark that in the particular case p=1 (pure model) all three Eqs.(29'), (30') and (31') are contained in the *exact* Eq. (23) (as a matter of fact it is known that for the pure case these Eqs. provide the same information if $N < 6$ (Cardy 1980, Alcaraz and Köberle 1980); if $N \geqslant 6$ Eq. (29') or Eq. (30') or Eq.(31') can not univoquely determine the self-dual frontier but only a hypersurface that contains it). In the limit $N \to 1$ all three Eqs. lead to one and the same result namely

$$p\,\tau_0 = 1/2 \tag{32}$$

which is known to be *exact* (Southern and Thorpe 1979, Turban 1980 and Tsallis 1981); we recall that, in the limit $N \to 1$, $t_0^{(1)} = 1 - t_0^{(1)D} = \tau_0$. Furthermore we verify that all three Eqs. provide $\tau_0=1$ for $p = 1/2$ (pure bond percolation limit) and that no solution exists for $p < 1/2$: this result is commonly believed to be *exact* (Southern and Thorpe 1979, Turban 1980 and Tsallis 1981 among others) for the Potts model and we conjecture here that it remains true for the more general model presently discussed. The conjectures (29), (30) and (31) recover, for the Potts model, completely analogous conjectures included in Tsallis and Levy 1980a, Levy et al 1980 and Tsallis 1981 (it is convenient to recall at this point that the present model extends the Potts one only if N $\geqslant$4). In these references it is shown that, for the Potts model, the σ- conjecture (Eq.(31)) is numerically more satisfactory than the others (Eq.(29) and (30)); it is therefore natural to expect that this is still true in the present generalized picture.

From the very beginning we have considered *isotropic* ferromagnetic models but no major difficulty exists if crystalline anisotropy is included. In the particular case of the square lattice we can follow along the lines of Tsallis 1981 and propose for the approximate CF the following equation:

$$\langle \sigma \rangle_P + \langle \sigma \rangle_{P'D} = 1 \tag{33}$$

where P(P') is a general probability distribution for the "horizontal" ("vertical") coupling constants.

We shall now use the Eqs.(29), (30) and (31) to discuss the critical frontier of the Z(4) isotropic bond-dilute model in square lattice. By associating to each site two Ising variables μ_i and ν_i ($\mu_i,\nu_i = \pm 1$) we can write the Z(4) S_i variable as follows:

$$S_i = \frac{1}{\sqrt{2}} \left(\mu_i \, e^{-i\pi/4} + \nu_i \, e^{i\pi/4}\right) \tag{34}$$

Consequently the Hamiltonian (17) can be rewritten

$$\frac{\mathcal{H}}{k_B T} = \sum_{\langle i,j \rangle} \left[K_1 - K_1(\mu_i\mu_j + \nu_i\nu_j) - 2\,K_2\,\mu_i\nu_i\mu_j\nu_j\right] \tag{35}$$

The relevant transmissivities of this random model are given (through Eq.(20)) by

$$t_o^{(1)} = \frac{1 - e^{-4K_1^o}}{1 + 2\, e^{-2(K_1^o + 2K_2^o)} + e^{-4K_1^o}} \tag{36}$$

$$t_o^{(2)} = \frac{1 - 2\, e^{-2(K_1^o + 2K_2^o)} + e^{-4K_1^o}}{1 + 2\, e^{-2(K_1^o + 2K_2^o)} + e^{-4K_1^o}} \tag{36'}$$

and

$$\tau_o = \frac{1}{3} \left(2t_o^{(1)} + t_o^{(2)}\right) \tag{37}$$

The t-, τ- and σ- conjectures respectively provide

$$p = \frac{1 + 2t_o^{(1)} + t_o^{(2)}}{t_o^{(1)} \left(1 + 2t_o^{(1)} + t_o^{(2)}\right) + 2\left(t_o^{(1)} + t_o^{(2)}\right)} \tag{38}$$

$$p = \frac{3\left(1 + 2t_o^{(1)} + t_o^{(2)}\right)}{\left(2t_o^{(1)} + t_o^{(2)}\right)\left(5 + 2t_o^{(1)} + t_o^{(2)}\right)} \tag{39}$$

and

$$p = \frac{\ell n\, 2}{\ell n \left(1 + 2t_o^{(1)} + t_o^{(2)}\right)} \tag{40}$$

All three Eqs. provide qualitatively the same surface (ABCDE in Fig.1) in the $(p,\ t_o^{(1)},\ t_o^{(2)})$ -space. This surface is expected to be a good approximation of the para-ferromagnetic CF in the region where the transition is unique. Let us now consider some limiting cases. In the plane $t_o^{(1)} = 0$ (i.e. $K_1^o = 0$ and $t_o^{(2)} = (1-e^{-2K_2^o})/(1+e^{-2K_2^o})$) we have a bond-dilute Ising model CF (the associated Ising variable being $\mu_i\nu_i$) which corresponds to the line I_1G of Fig.1. We remark that in this case the Hamiltonian (35) is local gauge invariant therefore, in accordance with Elitzur 1975 theorem, $<\mu_i> = <\nu_i> = 0$ (see also Alcaraz and Köberle 1980, 1981) on both sides of the CF. In the plane $t_o^{(2)} = 1$ (i.e. $K_2^o \to \infty$ and $t_o^{(1)} = (1-e^{-4K_1^o})/(1+e^{-4K_1^o})$) we have two CF. The first of them (line I_2D of Fig.1) corresponds to a bond-dilute Ising model (whose coupling constant equals $2K_1^o$) associated to the variable μ_i or ν_i. The second CF (straight line GD in Fig.1) corresponds to the limit of a thermal problem (whose random variable is $\mu_i\nu_i$) which can be considered as a pure bond percolation one. It is then clear that, if the CF is continuous, the surface ABCDE must bifurcate on some line. It is well known that, on the plane $p=1$, this bifurcation occurs on the Potts model ($t_o^{(1)}=t_o^{(2)}=1/3$; point B in Fig.1); it seems plausible that this is still true on the bond-dilute problem (line BD of Figs. 1 and 2). As a direct consequence of the preceeding considerations only the surface ABDE is concerned by Eqs.(38-40). In what concerns the line AED of Fig.1 we have not succeeded in formulating a clear interpretation (one plausible equation for that CF is $pt_o^{(1)} = 1/2$ for the line AE, the line ED being a straight one). To summarize the preceding analysis let us say that in the unitary cube of the $(p, t_o^{(1)}, t_o^{(2)})$ -space three phases exist, namely the paramagnetic (noted P; Z(4) symmetry), the ferromagnetic (noted F; completely broken symmetry) and the "intermediate" (noted I; Z(2) symmetry) ones, characterized by:

$<\mu_i> = <\nu_i> = <\mu_i\nu_i> = 0$ (phase P)

$<\mu_i> \neq 0; <\nu_i> \neq 0; <\mu_i\nu_i> \neq 0$ (phase F)

$<\mu_i> = <\nu_i> = 0; <\mu_i\nu_i> \neq 0$ (phase I)

We can verify directly in the Hamiltonian (35) that $K_2^o = 0$ (hence $t_o^{(1)} = \sqrt{t_o^{(2)}} = (1-e^{-2K_1^o})/(1+e^{-2K_1^o})$) corresponds to the bond-dilute Ising model. In this case Eqs.(38) and (40) recover previous results (Nishimori 1979, Tsallis and Levy 1980(a), Levy et al 1980 and Tsallis 1981).

In Table 1 the most relevant numerical results are presented; we remark that the σ-conjecture is globally rather better than the t- one which in turn is better than the τ-one. In Fig.3 we have presented the critical frontiers associated to different ratios $K_2^{(o)}/K_1^{(o)}$; the errors are expected to be not bigger than the graphical widths.

The σ- conjecture seems to be (see Table 1) asymptotically exact in the limit $p \to 1/2$ (neighbourhood of point D of Fig.1); it provides

$$\left.\frac{d\tau_o}{dp}\right|_{p=1/2} = \frac{1}{3}\left.\frac{d\left[2t_o^{(1)} + t_o^{(2)}\right]}{dp}\right|_{p=1/2} = -\frac{16}{3}\ln 2 \qquad (41)$$

which recovers the exact answers for the Potts ($K_2^o/K_1^o = 1/2$) and Ising ($K_2^o/K_1^o = 0$) models. Eq.(41) leads to an interesting consequence: for *all* ferromagnetic models satisfying $K_2^o/K_1^o < 1/2$, $(dt_1/dp)_{p=1/2}$ equals $-4\ln 2$ for fixed ratio K_2^o/K_1^o, whereas for $K_2^o/K_1^o = 1/2$, $(dt_1/dp)_{p=1/2}$ equals $-\frac{16}{3}\ln 2$.

IV - CONCLUSION

We have herein introduced, for the general Z(N) s- simplex d- dimensional lattice model, convenient variables (transmissivities) which in series composition behave like probabilities. In what concerns parallel arrays it is possible, through a convenient transformation, to put the parallel composition algorithm in the same form as that of the series case. We have exhibited herein that this transformation is *precisely* the well known duality transformation.

The simplicity of the present algorithms enables quite plausible approximate conjectures for the critical frontiers for random Z(N) models. In order to illustrate this type of conjecture, the square lattice Z(4) bond-dilute ferromagnetic model has been discussed in detail. The phase diagram (see Fig.1) exhibits, besides the usual para- and ferromagnetic phases, an intermediate one which is characterized by a partial breakdown of the Z(4) summetry. A numerically interesting result is the p=1/2 limiting slope (Eq.(41)) which, within the present context, is expected to be exact.

We have seen in Section II that the functional form of the relevant transformations does not depend on s (order of the s- simplex). Consequently the conjectural picture presented in Section III should hold for general Z(N) $\frac{d}{2}$- simplex- random ferromagnetic models in d- dimensional hypercubic lattices, thus reinforcing the common belief that the gauge four- dimensional systems are very similar to the bond two- dimensional ones. In particular for the $\frac{d}{2}$ - simplex- *dilute* model the ferromagnetic phase disappears at a probability $p=1/2$. This remark suggests the possibility for defining a generalized s- simplex percolation whose critical probability is expected to be 1/2 for $s=d/2$ and 1 for $s=d$, and which possibly corresponds to a generalization of the Kasteleyn and Fortuin 1969 $N\to 1$ limit.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 Phase diagram of the bond-dilute Z(4) model in square lattice (the point E is here located according to the results obtained through the present approximations; it is however possible that the exact p_0 equals 1/2). B (I_1, I_2 and I_3) is (are) the pure Potts (Ising) critical point(s); the line BD (I_1G and I_2D) corresponds to bond-dilute Potts (Ising) model(s). P, F and I denote the para-, ferromagnetic and intermediate phases.

Fig. 2 Fixed p sections of the phase diagram of Fig. 1. (a) p=1; (b) p=0.8; (c) p=0.7; (d) p=0.6; (e) p =0.53. The line BD corresponds to the bond-dilute Potts model.

Fig. 3 Fixed K_2^o/K_1^o ratio sections of the phase diagram of the bond-dilute Z(4) model in square lattice. (a) K_2^o/K_1^o=0.5 (Potts); (b) K_2^o/K_1^o=0.3; (c) K_2^o/K_1^o=0.25; (d) K_2^o/K_1^o=0 (Ising); (e) K_2^o/K_1^o=-0.3. P and F denote the para- and ferromagnetic phases.

Table 1 Relevant quantities (calculated through the t-, τ - and σ- conjectures) associated to the phase diagram represented in Fig. 1 (where the point E is located at $p = p_0$). See the text for the values followed by (?). (a) Wu and Lin 1974; (b) Sykes and Essam 1963; (c) Baxter 1973; (d) Southern and Thorpe 1979; (e) Kramers and Wannier 1941; (f) Domany 1978; (g) Harris 1974.

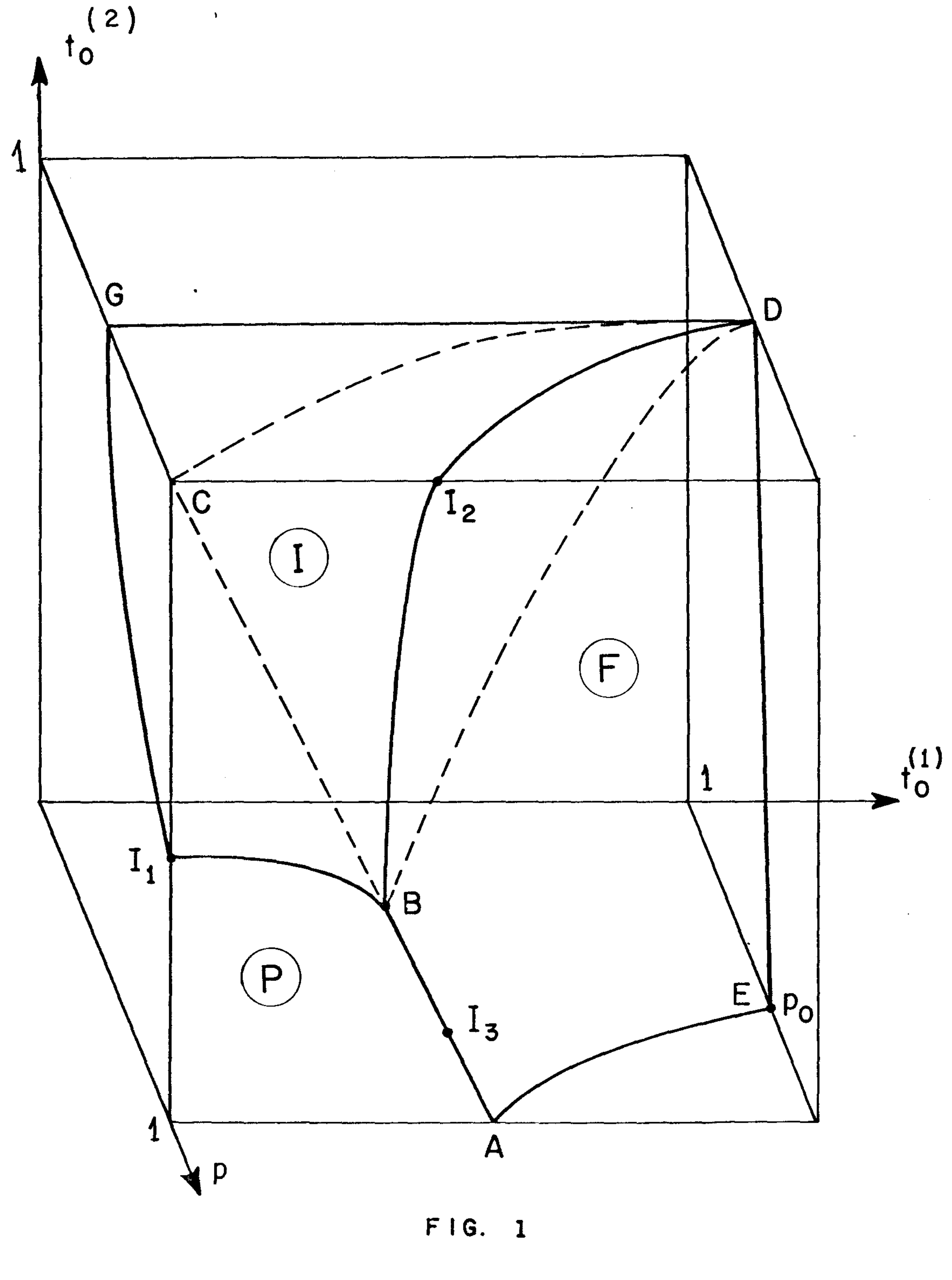

FIG. 1

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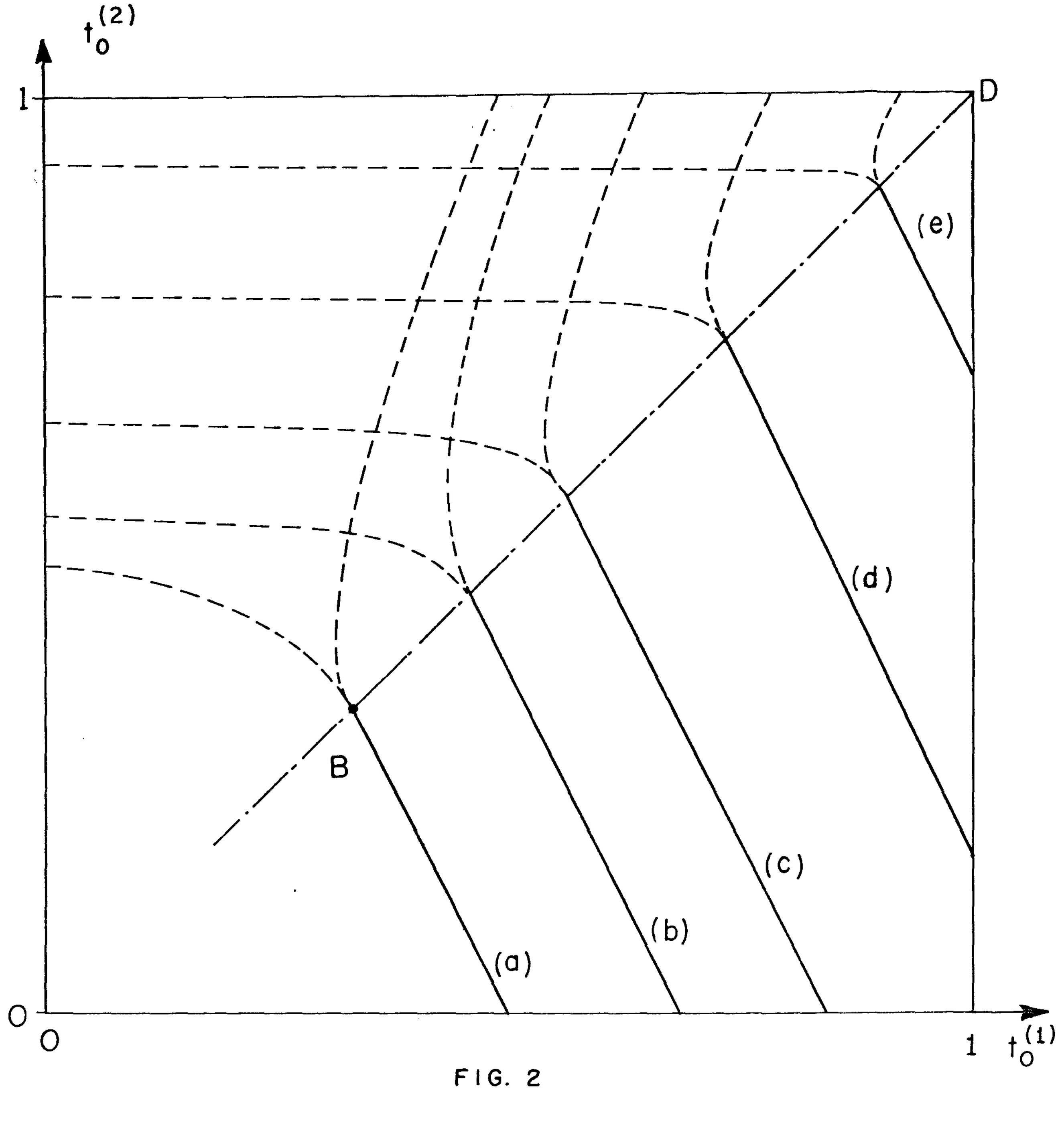

FIG. 2

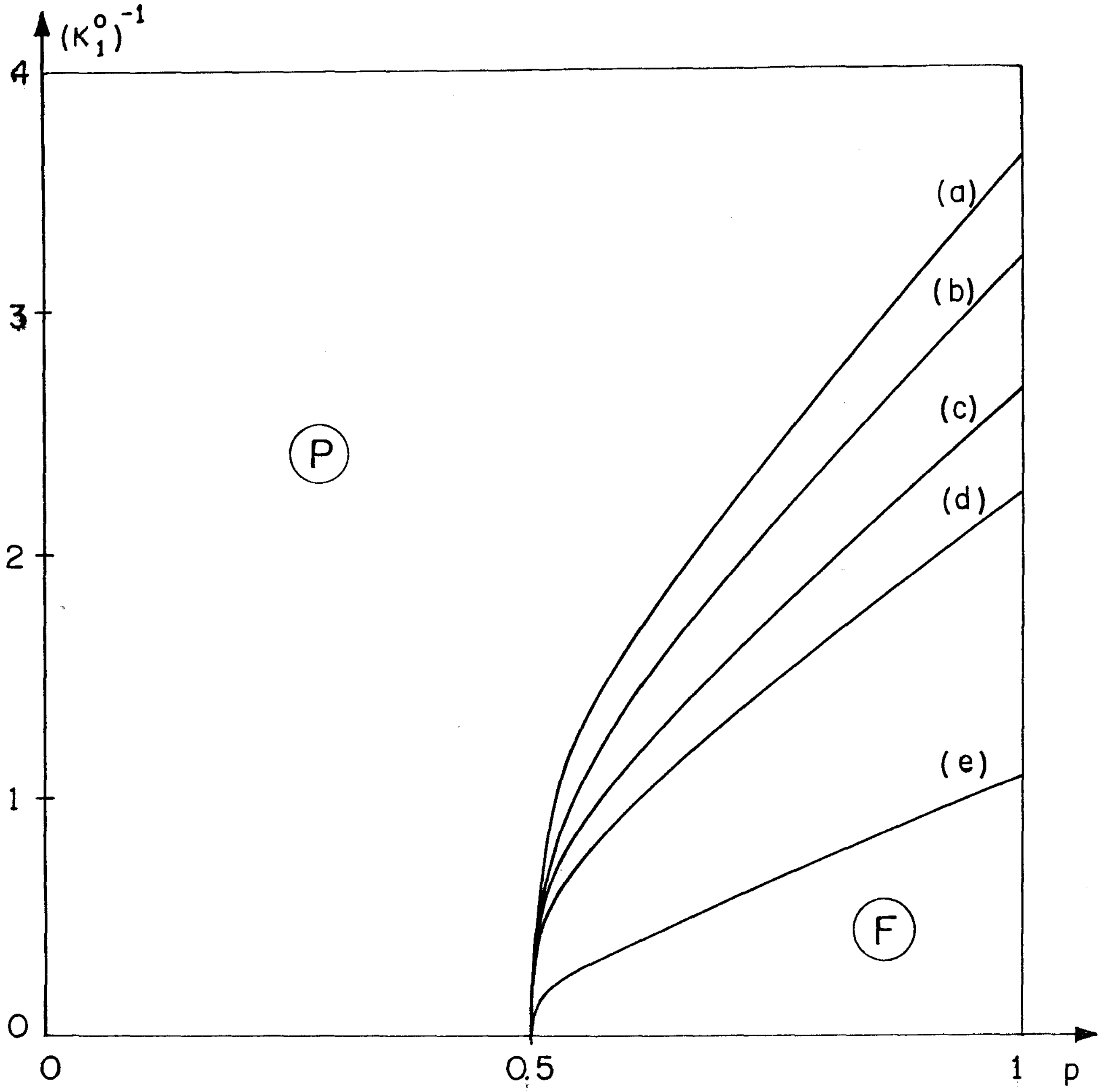

FIG. 3

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TABLE 1

CONJECTURES		t (Eq. (38))	τ (Eq. (39))	σ (Eq. (40))	EXACT
p_o		$\frac{3}{5} = .6$	$\frac{9}{14} \simeq .64$	$\frac{\ell n2}{\ell n3} \simeq .63$	$\frac{1}{2}$ (?)
Plane $t_o^{(1)}=1$	$-\frac{dt_o^{(2)}}{dp}\Big\vert_{p=1/2,\ t_o^{(2)}=1}$	16	$\frac{48}{5} = 9.6$	$16\ell n2 \simeq 11.1$	∞ (?)
	$-\frac{dt_o^{(2)}}{dp}\Big\vert_{p=p_o,\ t_o^{(2)}=0}$	$\frac{25}{4} = 6.25$	$\frac{196}{39} \simeq 5.03$	$\frac{3(\ell n3)^2}{\ell n2} \simeq 5.22$	∞ (?)
Plane $t_o^{(2)}=0$	$-\frac{dt_o^{(1)}}{dp}\Big\vert_{p=1,\ t_o^{(1)}=\frac{1}{2}}$	$\frac{2}{3} \simeq .67$	$\frac{3}{4} = .75$	$\ell n2 \simeq .69$	$\frac{1}{2}$ (?)
	$-\frac{dt_o^{(1)}}{dp}\Big\vert_{p=p_o,\ t_o^{(1)}=1}$	$\frac{25}{11} \simeq 2.27$	$\frac{98}{39} \simeq 2.51$	$\frac{3(\ell n3)^2}{2\ \ell n2} \simeq 2.61$	2 (?)
Plane $p=1$	$-\frac{dt_1}{dt_2}\Big\vert_{t_o^{(1)} \in \left[\frac{1}{3}, \frac{1}{2}\right]}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (a)
Plane $t_o^{(1)}=t_o^{(2)}=t_o$	p_c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (b)
	t_c	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$ (c)
	$-\frac{dt_o}{dp}\Big\vert_{p=\frac{1}{2},\ t_o=1}$	$\frac{16}{5} = 3.2$	$\frac{16}{5} = 3.2$	$\frac{16}{3}\ell n2 \simeq 3.70$	$\frac{16}{3}\ell n2$ (d)
	$-\frac{dt_o}{dp}\Big\vert_{p=1,\ t_o=\frac{1}{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}\ell n2 \simeq .46$	$\frac{1}{3\ell n2} \simeq .48$ (d)
Surface $t^{(2)}=\left(t^{(1)}\right)^2 \equiv t_o^2$	p_c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (b)
	t_c	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$ (e)
	$-\frac{dt_o}{dp}\Big\vert_{p=\frac{1}{2},\ t_o=1}$	$\frac{8}{3} \simeq 2.67$	$\frac{12}{5} = 2.4$	$4\ell n2 \simeq 2.77$	$4\ell n2$ (f)
	$-\frac{dt_o}{dp}\Big\vert_{p=1,\ t_o=\sqrt{2}-1}$	$\frac{1}{2}$	$\frac{3}{4\sqrt{2}} \simeq .530$	$\frac{\ell n2}{\sqrt{2}} \simeq .490$	$(6\sqrt{2}-8) \simeq .485$ (g)