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CANONICAL AND FUNCTIONAL INTEGRAL QUANTIZATION OF YANG-MILLS THEORY

by

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ABSTRACT:

The canonical and functional integral quantization of Yang-Mills theory is discussed. The gauge-fixing 'weak' conditions $A^0_a \approx 0$, $A^3_a \approx 0$ over phase space are found to be very convenient for any gauge group and in the presence of interactions. These conditions fix the gauge for arbitrary strong field and we obtain a description of Yang-Mills field in terms of physical degrees of freedom only.

I - INTRODUCTION:

The Lagrangians of gauge theories in local form are usually written in singular form. Dirac's method [1] for handling constrained dynamical systems may be used to construct the corresponding Hamiltonian formulation which leads to the canonical quantization of the theory. An alternative procedure for quantization is by the functional integral of Feynman [2] with appropriate modifications in the measure as suggested by Faddeev and Popov [3] to take care of the constraints already present in the theory and the additional constraints which must be imposed to fix uniquely a representative in each class of gauge-equivalent fields. In this connection Gribov [4] discovered recently the ambiguity of the Coulomb gauge [5], $\vec{\nabla} \cdot \vec{A} = 0$, in Yang-Mills theory. This gauge fixing condition may become singular for sufficiently strong fields and there is a non-trivial residual gauge freedom. It is thus important to look for other gauge-fixing conditions which avoid such ambiguities to be able to define functional integral. The temporal gauge is a convenient choice to start with. Here the Dirac brackets coincide with canonical brackets. The search for suitable additional constraints or canonical transformations [6] to fix the gauge is greatly simplified. This was done systematically for electromagnetic field interacting with a Dirac field and an external charge[6]. We also showed there that it was not necessary to remove the residual gauge invariance if we understand that the corresponding functional integral acts over the corresponding covariant states. We also pointed out the simple gauge-fixing conditions $A^0 \approx 0$, $A^3 \approx 0$ over the phase-space which was shown to be equivalent to the commonly used Coulomb gauge condition $A^0 \approx 0$ $\vec{\nabla}.\vec{A} \approx 0$.

We discuss in this paper the corresponding ghost-free gauge-fixing conditions $A_a^0 \approx 0$, $A_a^3 \approx 0$ for the Yang-Mills theory for any gauge group [7]. We show that these additional constraints do allow us to fix the gauge even for arbitrary strong gauge fields and we obtain a description of the Yang-Mills fields in terms of physical degrees of freedom only. This is done both in the context of canonical quantization as well as in that of functional integral quantization. The generating functional in the latter case may be integrated over canonical momenta to obtain a convenient representation for the same (Eq.(4.8)). This may then be used to go over to other convenient gauge conditions.

We must mention here the recent attempts to impose gauge-fixing conditions on canonical momenta. Goldstone and Jackiw[8] solved the Gauss' Law constraint equation in temporal gauge ($A_a^0 \approx 0$) for the wave functional for the SU(2) gauge group in momentum represenation and obtained a description of the Yang-Mills theory without any non-physical degrees of freedom. Faddeev, Izergin, Korepin and Semenov-Tiam-Shansky[9] showed that the same results are obtained in the context of functional integral quantization if we impose gauge-fixing conditions on the canonical momenta but not on the vector potential $\vec{A}_a$. These authors could also generalize their method for an arbitrary gauge group as well. It is, however, clear from the equations of motion (Eq.(2.25)) for canonical momenta and the requirement that the additional constraints must hold for all times that such conditions involving canonical momenta would bring in difficult [9] constraint relations to be dealt with.

In Sec. II we briefly review the Dirac's method[1,10] and establish the form of general Hamiltonian in the Yang-Mills theory interacting with a complex spinor field for any gauge group. Temporal gauge is introduced in Sec. III. The additional constraints $A^3_a \approx 0$ are shown to be very convenient ones and an expression of the Hamiltonian is obtained involving physical degrees of freedom only. The functional integral quantization is exposed in Sec. IV and a representation of the generating functional in terms of physical degrees of freedom is obtained in our gauge.

II. HAMILTONIAN DYNAMICS OF NON-ABELIAN GAUGE THEORY. CANONICAL QUANTIZATION.

The action functional for the Yang-Mills theory with self-interacting spinor source fields is written as

$$S = \int L(t)\, dt = \int dt \int d^3x\, \mathcal{L} \qquad (2.1)$$

where

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\Psi}\,(i\gamma.\partial - m)\Psi + g\,\bar{\Psi}\;\gamma.A^a t^a\,\Psi \qquad (2.2)$$

Here the underlying gauge group has n Hermitian generators satisfying

$$\left[T_a, T_b\right] = i\, f_{abc}\, T_c \qquad (2.3)$$

where f_{abc} is real, and completely antisymmetric in a,b,c. The indices assume the values a=1,2,...,n and μ=0,1,2,3;k=1,2,3, in the metric $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$.

The field strenghts are defined by

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu \qquad (2.4)$$

where g is a coupling constant. The complex spinor field Ψ is an m-component field carrying a (m×m) matrix representation of the gauge group. The corresponding generators are denoted by (m×m) Hermitian matrices t_a which may be reducible. The generators in adjoint representation are (m×m) matrices $(L_a)_{bc}=-i f_{abc}$. The covariant derivative is defined as

$$D_\mu(A) = \partial_\mu - i g T_a A_\mu^a \qquad (2.5a)$$

so that in the adjoint representation, for example,

$$D_\mu^{ab}(A) = \delta_{ab} \partial_\mu - g f_{abc} A_\mu^c \qquad (2.5b)$$

Its space part is

$$\vec{D}^{ab}(\vec{A}) = \delta_{ab} \vec{\nabla} + g f_{abc} \vec{A}_c \tag{2.5c}$$

The theory is invariant under the following local gauge transformation

$$A^\mu(x) \longrightarrow U(x) A^\mu(x) U^{-1}(x) + \frac{i}{g} U(x) \partial^\mu U^{-1}(x) \tag{2.6a}$$

and

$$\Psi(x) \longrightarrow e^{-i \omega_a(x) t_a} \Psi(x) \tag{2.6b}$$

Here $\omega_a(x)$ is any real space-time function, $A^\mu = T_a A^\mu_a$ and $U(x) = \exp\{-i \omega_a(x) T_a\}$ is an element of the gauge group at the space-time point x^μ. It follows that

$$F^{\mu\nu}(x) \longrightarrow U(x) F^{\mu\nu}(x) U^{-1}(x) \tag{2.6c}$$

where $F^{\mu\nu} = T_a F_a^{\mu\nu}$. The infinitesimal transformations read as

$$\delta\Psi(x) = -i\ t_a\ \omega_a(x)\ \Psi(x)$$

$$\delta A_a^\mu(x) = f_{abc}\ \omega_b(x)\ A_c^\mu(x) - \frac{1}{g}\partial^\mu\ \omega_a(x)$$

$$= -\frac{1}{g} D_{ab}^\mu\ \omega_b(x)$$

$$\delta F_a^{\mu\nu}(x) = f_{abc}\ \omega_b(x)\ F_c^{\mu\nu}(x)$$

$$\delta(D^\mu\Psi(x)) = -i\ t_a\ \omega_a(x)\ (D^\mu\ \Psi(x)) \tag{2.7}$$

The Euler-Lagrange equations are

$$D_\mu^{ab}\ F_b^{\mu\nu} = -g\ \overline{\Psi}\ \gamma^\nu\ t^a\ \Psi \tag{2.8a}$$

$$(i\ \not\partial - m)\Psi = -g\ t_a\ \gamma\ .\ A_a\ \Psi \tag{2.8b}$$

We may rewrite Eq.(2.8a) as

$$\partial_\mu F_a^{\mu\nu} = -g Z_a^\nu ,$$

$$Z_a^\nu = \bar{\Psi} \gamma^\nu t^a \Psi - f_{abc} A_\mu^c F_b^{\mu\nu} \tag{2.8c}$$

where Z_a^μ is the conserved Noether's current corresponding to invariance of the theory under global (ω_a=const.) gauge transformations. As a side remark, we may easily show from Eq.(2.8b) that

$$D_\mu^{ab} (\bar{\Psi} \gamma^\mu t_b \Psi) = 0 \tag{2.9}$$

and the identity [11]

$$D_\nu^{ca} D_\mu^{ab} F_b^{\mu\nu} \equiv 0 \tag{2.10}$$

then follows. It is interesting to note that the self-current of the gauge field does not have vanishing covariant derivative.

The Hamiltonian dynamics for the singular Lagrangian of Eq.(2.2) may be constructed by the Dirac's method [1,10]. We define the dynamics on t=const. hyperplanes and all the variations are taken at fixed time. The canonical momenta are

$$\Pi = \frac{\delta L}{\delta \dot{\Psi}} = i\ \Psi^{+}$$

$$\Pi^{a}_{\mu} = \frac{\delta L}{\delta \dot{A}_{a}} = + F^{0\mu}_{a} \qquad (2.11)$$

We rewrite for the gauge fields

$$\vec{\Pi}_{a} = \dot{\vec{A}}_{a} + \vec{D}^{ab}\ A^{0}_{b} \qquad (2.12)$$

$$\Pi_{0}{}^{a} \simeq 0 \qquad (2.13)$$

Here $\vec{\Pi} = (\Pi_1, \Pi_2, \Pi_3)$, $\vec{A} = (A^1, A^2, A^3)$ and $\simeq$ indicates that the primary constraints in Eq.(2.13) are "weak" relations in the sense of Dirac.

We define canonical equal time Poisson brackets for any two functionals f and g by

$$\{f,g\} = \int d^3x \left\{ \frac{\delta f}{\delta A^\mu_a(\vec{x},t)} \frac{\delta g}{\delta \Pi^a_\mu(\vec{x},t)} - \frac{\delta f}{\delta \Pi^a_\mu(\vec{x},t)} \frac{\delta g}{\delta A^\mu_a(\vec{x},t)} + \frac{\delta f}{\delta \Psi_{\underline{a}}(\vec{x},t)} \frac{\delta g}{\delta \Pi_{\underline{a}}(\vec{x},t)} - \frac{\delta f}{\delta \Pi_{\underline{a}}(\vec{x},t)} \frac{\delta g}{\delta \Psi_{\underline{a}}(\vec{x},t)} \right\} \qquad (2.14)$$

The standard non-vanishing brackets are

$$\{ A^\mu_a (\vec{x},t) , \Pi^b_\nu(\vec{y},t) \} = \delta_{ab} \, \delta^\mu_\nu \, \delta^3(\vec{x}-\vec{y})$$

$$\{ \Psi_{\underline{a}}(\vec{x},t) , \Pi_{\underline{b}}(\vec{y},t) \} = \delta_{\underline{ab}} \, \delta^3(\vec{x} - \vec{y}) \qquad (2.15)$$

where a indicates a component of the complex spinor field.

We take as our preliminary Hamiltonian

$$H' = H_c + \int v_a(\vec{x},t)\, \Pi_a^0(\vec{x},t)\, d^3x \tag{2.16}$$

where H_c is the canonical Hamiltonian

$$H_c = \int (\vec{\Pi}_a . \dot{\vec{A}}_a + \Pi \dot{\Psi})\, d^3x - L \tag{2.17}$$

$$= \int d^3x \left\{ \frac{1}{2} \vec{\Pi}_a . \vec{\Pi}_a + \frac{1}{4} F_a^{k\ell} F_{k\ell}^a + \right.$$

$$\left. + \Psi^+ \left[-i\, \vec{\alpha} . (\vec{\nabla} + ig\, t_a \vec{A}_a) + \beta m \right] \Psi + \chi_a A_a^0 \right\} \tag{2.18}$$

Here

$$\chi_a = \vec{D}^{ab} . \vec{\Pi}_b - g\, \Psi^+ t_a \Psi$$

$$= \vec{\nabla} \cdot \vec{\Pi}_a - g\, Z_a^0 \qquad (2.19)$$

and v is an arbitrary functional. The equations of motion are given by

$$\dot{f} = \{f, H'\} + \frac{\partial f}{\partial t}$$

where the second term on the right hand side is evaluated at constant Ψ, Π, Π_μ, A^μ , $\vec{x}$. For the constraints $\Pi_a^0 \tilde{\approx} 0$ to hold for all times, we require

$$\dot{\Pi}_a^0 = \{\Pi_0^a, H'\} = - \frac{\delta H'}{\delta A_a^0} \tilde{\approx} 0$$

This leads to the secondary constraints

$$\chi_a \tilde{\approx} 0 \qquad (2.20)$$

This is the Gauss' Law for the non-abelian gauge theory already contained in Lagrange equation. We also verify that the remaining equations of motion are consistent with Eqs.

(2.8) and

$$\dot{\chi}_a \approx 0 \qquad (2.21)$$

Thus the set of 2n constraints in our theory are $\Pi_a^0 \approx 0$ and $\chi_a \approx 0$. They are in fact first class [1] since they have vanishing Poisson brackets, $\{\Pi_0^a, \Pi_0^b\} \approx 0$, $\{\Pi_0^a, \chi_b\} \approx 0$ and

$$\{\chi_a(\vec{x},t), \chi_b(\vec{y},t)\} = - g\, f_{abc}\, \chi_c(\vec{x},t)\, \delta^3(\vec{x}-\vec{y}) \approx 0 \qquad (2.22)$$

They are, in fact, generators of infinitesimal local gauge transformations [12]. From $\delta A_a^\mu(\vec{x},t) \equiv \{A_a^\mu(\vec{x},t), \varepsilon G(t)\}$ etc.

and Eq.(2.7) we readily obtain

$$\varepsilon G(t) = \int \left[\omega_a \, \psi^+ t_a \, \psi - \frac{1}{g} \, \Pi_\mu^a \, D^\mu_{ab} \, \omega_b\right] d^3x \qquad (2.23)$$

If we may neglect a surface term arising on integration by parts we may rewrite the second term inside the bracket as

$- \frac{1}{g} \left\{ \omega_b \, \chi_b + \Pi_a^0 \, D_0^{ab}(A_0) \, \omega_b \right\}$. Global gauge transformations

of gauge fields are generated by $(Z_a^0 + f_{abc} \, \Pi_0^a \, A_0^c)$ and on

canonical quantization the corresponding integrated "charges" satisfys the commutation relations of the algebra of the group in Eq.(2.3).

A more general Hamiltonian may now be taken to be

$$H'' = H' + \int u_a(\vec{x},t) \; \chi_a(\vec{x},t) \; d^3x \tag{2.24}$$

We find among other equations of motion

$$\dot{A}_a^0 = v_a$$

$$\dot{\vec{A}}_a = \vec{\Pi}_a - \vec{D}^{ab}(A_b^0 + u_b)$$

$$\dot{\Pi}_k^a = D_\ell^{ac} \, F_c^{k\ell} + g \, f_{abc}(A_c^0 + u_c)\, \Pi_k^b - g \, \Psi^+ \, \alpha_k t_a \, \Psi$$

$$\dot{\Pi}_a^0 = \chi_a \simeq 0$$

$$\dot{\chi}_a \simeq 0$$

$$i \, \dot{\Psi} = \left[-i \, \vec{\alpha} . (\vec{\nabla} + ig \, t_a \vec{A}_a) + \beta m \right] \Psi$$

$$- g \, t_a (A_a^0 + u_a) \Psi \tag{2.25}$$

III - TEMPORAL GAUGE, $A^0_a \approx 0$. GAUGE-FIXING CONDITIONS [7] $A^a_3 \approx 0$, $A^a_0 \approx 0$.

The 2n first class constraint equations $\Pi^a_0 \approx 0$, $\chi_a \approx 0$ show that the dynamics over phase space is constrained. We may exclude some or all of the redundant variables using Dirac's suggestion [1, 10] We introduce suitable supplementary constraints so that a first class constraint becomes a second class , that is, it does not have vanishing Poisson bracket with all the constraints. We may then define modified Poisson brackets - Dirac brackets - with respect to the set of now second class constraint. The initial first class constraint relations together with the other second class weak relations may now be set as strong (equality) relations inside the Dirac brackets. The Hamiltonian dynamics built over Dirac brackets automatically takes care of the constraints without requiring to solve the constraint relations explicitly.

The choice of supplementary constraints $\{\chi\}$ is rather arbitrary. We may set down some necessary conditions. It is clear that we also require $\dot{\chi} = 0$. The set $\{\chi\}$ should be such that we are able to solve for the arbitrary functionals u_a and v_a appearing in Eq.(2.24). This in turn implies that det.$||\{ \chi , \phi\}||$, where $\{\phi\}$ is the set of surviving first class constraints, should be nonvanishing and well defined. The same determinant appears also in the prescription [3] of quantization by functional integral over phase space of constrained systems. We will look for additional constraints such that the determinant be a constant so that it effectivaly drops out of functional integral and Gribov type ambiguities are avoided.

Temporal gauge, $A_a^0 \approx 0$, is a simple choice **to start**. The set $\Pi_a^0 \approx 0$ is now second class and $\det \| \{A_a^0 , \Pi_b^0\} \| = \text{const.}$

For these constraints to hold for all times we require $\dot{A}_0^a \approx 0$ which fixes $v_a = 0$ in view of Eq.(2.25). The constraints χ_a continue to be first class corresponding to the residual gauge invariance, in the temporal gauge, with respect to time independent gauge transofrmations. Defining Dirac brackets

$$\{f,g\}^* = \{f,g\} + \int d^3z \left[\{f,A_0^a(\vec{z},t)\} \{\Pi_0^a(\vec{z},t),g\} - (A_0^a \leftrightarrow \Pi_0^a)\right] \tag{3.1}$$

we find that $\{f, \Pi_0^a\}^* = \{f,A_a^0\}^* = 0$ and the equations of motion are now given by

$$\frac{df}{dt} = \{f,\tilde{H}\}^* + \frac{\partial f}{\partial t} \tag{3.2}$$

so that $\Pi_0^a = 0$, $A_a^0 = 0$ hold effectivelly as strong relations.

Here $\tilde{H}$ is obtained from H" by putting $A_0^a = 0$ and $v_a = 0$.

We remark that if we had set $A_0^a = 0$ naively at the very begining we would have missed the Gauss' Law constraints $\chi_a \approx 0$. We do, however, obtain $\dot{\chi}_a \approx 0$ instead. From Eq.(3.1) we conclude that $\{f,g\}^*$ may be written in the form of Eq.(2.14) by just dropping the terms involving functional derivatives with respect to A_a^0. The non-vanishing standard brackets are now

$$\{A_a^k(\vec{x},t), \quad \Pi_\ell^b(\vec{y},t)\}^* = \delta_{ab} \quad \delta_\ell^k \; \delta^3(\vec{x}-\vec{y})$$

$$\{\Psi_{\underline{a}}(\vec{x},t) \; , \; \Pi_{\underline{b}}(\vec{y},t)\}^* = \delta_{\underline{ab}} \; \delta^3(\vec{x}-\vec{y}) \qquad (3.3)$$

The redundant variables A_0^a (and Π_0^a) drop out as well as no time dependent gauge transformations are permitted any more. The independent canonical variables in the temporal gauge have canonical brackets equal to their Dirac brackets and are very simple to work with. Unlike in the case of Coulomb gauge no fields appear on the right hand side of the above Dirac brackets. The ghost loops in the Feynmann rules are thus avoided in quantized theory. This gauge is also a very convenient starting point to look for additional constraints to fix the gauge completely. We may, for example, take the additional n constraints to be $\chi'_a \equiv (\Lambda_a - c_a(t)) \approx 0$ where $\vec{\nabla}\Lambda_a$ is the longitudinal component of $\vec{A}_a$ and $c_a = c_a(t)$ are space-independent functions. Such a choice was shown to be very convenient for the abelian case[6]. However, for the non-abelian case we run into trouble. We have now

$$\{\chi'_b(\vec{x},t), \chi_a(\vec{y},t)\}^* = -i\,\delta_{ab}\,\delta^3(\vec{x}-\vec{y}) +$$

$$+ g\,\{\Lambda_b(\vec{x},t), f_{acd}\,\vec{A}_d \cdot \vec{\Pi}_c\}^* \qquad (3.4a)$$

$$\dot{\chi}'_b = \frac{\partial c_b}{\partial t} + \{\Lambda_b, H_c\}^* + \int d^3x\, u_a\, \{\chi'_b, \chi_a\}^* \approx 0 \qquad (3.4b)$$

Since det.$||\{\chi'_b, \chi_a\}^*||$ now depends on the gauge potentials it may for sufficiently strong fields become singular and we are unable to solve Eq.(3.4b) for functionals u_a. The same ambiguity makes this gauge rather inconvenient for functional integral quantization.

The Eq.(3.2) results in Eqs.(2.25) with A^0_a set to zero.

We notice, for example, that

$$\dot{A}_a^3 = \Pi_3^a - (\partial_3 u_a + g f_{abc} A_c^3 u_b) \qquad (3.5)$$

It is thus suggested in view of the discussion above that a convenient choice for an additional set of constraints in the temporal gauge would be simply $A_3^a \approx 0$. Eq.(3.5) then implies $\Pi_3^a \approx \partial_3 u_a$ since we must require $\dot{A}_3^a \approx 0$ for $A_3^a \approx 0$ to hold for all times. The iterative property of Dirac brackets is assured in view of $\{\Pi_a^0, A_b^3\}^* = \{A_a^0, A_b^3\}^* \approx 0$. Moreover, we find

$$\{A_a^3, A_b^3\}^* \approx 0 \quad ,$$

$$C_{ab}(\vec{x},\vec{y}) \equiv \{\chi_a(\vec{x},t), A_b^3(\vec{y},t)\}^* = -\Big[\delta_{ab}\,\partial_3^x\,\delta^3(\vec{x}-\vec{y}) + g f_{abc} A_c^3\,\delta^3(\vec{x}-\vec{y})\Big]$$

$$\approx -\delta_{ab}\,\partial_3^x\,\delta^3(\vec{x}-\vec{y}) \quad ,$$

$$\{\chi_a(\vec{x},t),\ \chi_b(\vec{y},t)\}^* = -g\, f_{abc}\, \chi_c(\vec{x},t)\ \delta^3(\vec{x}-\vec{y}) \simeq 0$$

(3.6)

The constraints $\{A^3_a\}$, $\{\chi_a\}$ are now second class and det. $||\{\chi_a, A^3_b\}^*||$ = det. $||-\delta_{ab}\ \partial_3||$ does not depend on the gauge field. It will be absorbed as a constant in the normalization factor on quantization by functional integral(Sec.IV). Since no more first class constraints are left the gauge gets essentially fixed. We may define the new Dirac brackets by

$$\{f,g\}^{**} = \{f,g\}^* - \int d^3z \int d^3z' \Big(\{f,\chi_a(\vec{z},t)\}^*\ C^{-1}_{ab}(\vec{z},\vec{z}') \ .$$

$$. \ \{A^3_b(\vec{z}',t),\ g\}^*\Big)$$

$$= \quad \{f,g\}^* + \int d^3z \int d^3z' \ F(\vec{z},\vec{z}')\Big(\{f,\chi_a(\vec{z},t)\}^* \ .$$

$$. \ \{A^3_a(\vec{z}',t),\ g\}^* + \{f,A^3_a(\vec{z},t)\}^* .$$

$$. \ \{\chi_a(\vec{z}',t),\ g\}^*\Big) \qquad (3.7)$$

Here, defining the notation $\bar{x} = (x^1, x^2)$, $\bar{\nabla} = (\partial_1, \partial_2)$ etc.,

$$F(\vec{z}, \vec{z}') = g(z^3, z'^3)\ \delta^2(\bar{z} - \bar{z}')$$

$$K(\vec{z}, \vec{z}') = G(z^3, z'^3)\ \delta^2(\bar{z} - \bar{z}') \tag{3.8}$$

and $g(\tau, \tau')$ is the Green's function satisfying

$$\partial_\tau g(\tau, \tau') = -\ \partial_{\tau'}\ g(\tau, \tau') = \delta(\tau - \tau') \tag{3.9}$$

For discussion below we also need the Green's function $G(\tau, \tau')$ satisfying

$$\frac{\partial^2 G}{\partial \tau^2} = \frac{\partial^2 G}{\partial \tau'^2} = \delta(\tau - \tau') \tag{3.10}$$

We may make the explicit choice $g = \frac{1}{2}\ \varepsilon(\tau - \tau')$ and

$G = \frac{1}{2} |\tau - \tau'|$. We verify $\{f, A_a^3\}^{**} = \{f, \chi_a\}^{**} = 0$ and that the equations of motion are now given by

$$\frac{df}{dt} = \{f, H\}^{**} + \frac{\partial f}{\partial t} \qquad (3.11)$$

where H is the canonical Hamiltonian of Eq.(2.18) with A_a^0 and A_3^a put to zero. Inside the new Dirac brackets

$$A_a^3 = 0 \quad ,$$

and

$$\chi_a \equiv \partial_3 \, \Pi_3^a + \bar{D}^{ab} \cdot \bar{\Pi}_b - g \, \Psi^+ \, t_a \, \Psi = 0 \qquad (3.12)$$

where $\bar{D}^{ab} = \delta_{ab} \, \bar{\nabla} + g \, f_{abc} \, \bar{A}_c$, are now strong relations and we have incorporated in the final Hamiltonian formulation all the constraints. From Eqs.(3.12) and (3.9) we may express Π_3^a variables as dependent solely on $\bar{\Pi}_a$ and $\bar{A}_a$. Thus the final Hamiltonian involves only the two physical degrees of freedom, for each 'a' corresponding to a massless gauge field.

It is now given by

$$H = \int d^3x \left\{ \frac{1}{2} \overline{\Pi}_a \cdot \overline{\Pi}_a + \frac{1}{4} F^a_{k\ell} F^{k\ell}_a + \psi^+ \left[- i\vec{\alpha} \cdot (\vec{\nabla} + igt_a\overline{A}_a) + \beta m \right] \psi \right\} -$$

$$- \frac{1}{2} \int d^3x \int d^3y \left[\overline{D}^{ab} \cdot \overline{\Pi}_b(\vec{y},t) - g\, \psi^+ t_a \psi \right] K(\vec{x},\vec{y}) \left[\overline{D}^{ad} \cdot \overline{\Pi}_d(\vec{x},t) - g\psi^+ t_d\, \psi(\vec{x},t) \right] . \qquad (3.13)$$

where we have used $\Pi^a_3 = \partial_3 u_a$ and $K(\vec{x},\vec{y})$ is defined in Eq. (3.8). The last term contains a sort of Coulomb self-energy term. In abelian case $\overline{D}^{ab}$ gets replaced by $\overline{\nabla}$ and t_a is absent. We will elaborate on it in Sec. IV. The standard brackets are also simple

$$\{A^i_a(\vec{x},t),\ \Pi^b_j(\vec{y},t)\}^{**} = \delta_{ab}\ \delta^i_j\ \ \delta^3(\vec{x},\vec{y}) \quad ,$$

$$\{A^i_a \; , \; A^j_b\}^{**} = \{\Pi^i_a \; , \; \Pi^j_b\}^{**} = 0 \qquad ,$$

$$\{\Psi_{\underline{a}}(\vec{x},t) \, , \; \Pi_{\underline{b}}(\vec{y},t)\}^{**} = \delta_{\underline{ab}} \, \delta^3(\vec{x}-\vec{y}) \text{ etc.}$$

(3.14)

where $i,j=1,2$. We also list some other ones

$$\{\Pi^a_3(\vec{x},t) \, , \; \Pi^b_3(\vec{y},t)\}^{**} = -g \, f_{abc}\left[\Pi^c_3(\vec{x},t) - \Pi^c_3(\vec{y},t)\right] F(\vec{x},\vec{y}$$

$$\{\Pi^a_3(\vec{x},t) \, , \; \Pi^b_i(\vec{y},t)\}^{**} = g \, f_{abc} \, \Pi^c_i(\vec{y},t) \; F(\vec{x},\vec{y}) \qquad ,$$

$$\{\Pi^a_3(\vec{x},t) \, , \; A^i_b(\vec{y},t)\}^{**} = \left[\delta_{ab} \, \partial^x_i + g \, f_{abc} \, A^i_c(\vec{y},t)\right] F(\vec{x},\vec{y}$$

$$= -{}^y\bar{D}^{ba}_i \; F(\vec{x},\vec{y}) \qquad ,$$

$$\{\partial_3 \;\; \Pi^a_3(\vec{x},t) \, , \; A^i_b(\vec{y},t)\}^{**} = -{}^y\bar{D}^{ba}_i \; \delta^3(\vec{x}-\vec{y}) \qquad ,$$

$$\{\partial_3 \;\; \Pi^a_3(\vec{x},t) \, , \; \Psi(\vec{y},t)\}^{**} = g\{\Psi^+ t_a \; \Psi, \; \Psi(\vec{y},t)\}^{**} =$$

$$= ig \; t_a \; \Psi \; \delta^3(\vec{x}-\vec{y}) \qquad . \qquad (3.15)$$

We confirm that the correct equations of motion given in Eq.(2.25) are generated by Eq.(3.11) for the independent variables. We remark also that in our gauge $F_a^{23} = + \partial_3 A_a^2 =$

$= (\partial^2 A_a^{T3} - \partial^3 A_a^{T2})$ and $F_a^{31} = - \partial_3 A_a^1 = (\partial^3 A_a^{T1} - \partial^1 A_a^{T3})$

where $\vec{A}_a^T$ are tranverse components of $\vec{A}_a$ satisfying $\vec{\nabla} \cdot \vec{A}_a^T = 0$.

The canonical quantization is performed as usual by replacing the new Dirac brackets by commutator or anticommutator between corresponding operators in a self consistent manner. Appealing to the quantization of free fields, say, we are led to

$$\left[\hat{A}_a^i(\vec{x},t),\ \hat{\Pi}_j^b(\vec{y},t)\right] = i\hbar\, \delta_{ab}\, \delta^3(\vec{x}-\vec{y})$$

$$\left\{\hat{\Psi}_{\underline{a}}(\vec{x},t),\ \hat{\Psi}_{\underline{b}}^+(\vec{y},t)\right\} = \hbar\, \delta_{\underline{ab}}\, \delta^3(\vec{x}-\vec{y}) \qquad (3.16)$$

We have no ghost loops in the Feynman rules in our gauge. We will forego the discussion of Poincaré covariance and study the quantization by Feynman functional integral.

We remark before leaving this section that any supplementary condition involving canonical momenta is bound to bring in difficult constraint equations especially when we are dealing with arbitrary gauge group and interactions are present.

This is evident from Eq. (2.25)and the imposition that the additional constraints must hold for all times.

IV - FUNCTIONAL INTEGRAL QUANTIZATION IN $A^0_a \approx 0$, $A^3_a \approx 0$ GAUGE:

The quantization of constrained Hamiltonian systems may also be done by expressing formally the generating functional for the evolution operator (S-matrix) by means of the phase space functional integral[3] . In our $A^0_a \approx 0$, $A^3_a \approx 0$ gauge it is given by [7]

$$Z = N \int [d\,\Pi^a_\mu]\;[d\,A^\mu_a]\;\prod_{a,x}\;\delta(A^0_a(x))\;\delta(\Pi^0_a(x)\;\delta(A^3_a(x))\,\delta(\chi_a(x))\,.$$

$$.\,\det\,||\{A^0_a,\,\Pi^0_b\}||\,.\,\det\,||\{A^3_a,\,\chi_b\}||\;e^{iS} \tag{4.1}$$

where N is a normalization factor and

$$S = \int \Big[\Pi^a_0 \dot{A}^a_0 - \dot{\vec{\Pi}}_a \cdot \vec{A}_a - \frac{1}{2}\,\vec{\Pi}_a \cdot \vec{\Pi}_a - \frac{1}{4}\,F^{k\ell}_a\,F^a_{k\ell} - \chi_a(A^0_a + u_a) +$$

$$- \Pi^0_a\,v_a\Big]\,d^4x \tag{4.2}$$

and $\delta(A_a^3)\ \delta(\chi_a) = \delta(A_a^3)\ \delta(\partial_3\ \Pi_3^a + \overline{\nabla}\cdot\ \overline{\Pi}_a + g\ f_{abc}\ \overline{\Pi}_b\cdot\ \overline{A}_c)$

For simplicity in writing are ignore the fermion field. The determinants above are constant and may be absorbed in the normalization factor and we integrate over A_a^0, Π_a^0 and A_a^3 using delta functionals. It amounts to putting these variables to zero in Eq.(4.2). We make use of the exponential representation

$$\delta(\chi) = \int e^{-i \int \omega\ \chi\ d^4x} \prod_x \frac{d\omega(x)}{2\Pi} \qquad (4.3)$$

so that the action in the functional integral takes the form

$$S = \int \left[+ \overline{\Pi}_a\cdot\dot{\overline{A}}_a - \frac{1}{2}\Pi_3^a\ \Pi_3^a - \frac{1}{2}\overline{\Pi}_a\cdot\overline{\Pi}_a - \frac{1}{4}F_{k\ell}^a F_a^{k\ell} - \right.$$

$$\left. - \omega_a\ (\partial_3\ \Pi_3^a + \overline{D}^{ab}\cdot\ \overline{\Pi}_b) \right] d^4x$$

(4.4)

Here $\bar{D}^{ab} \equiv (\delta_{ab} \bar{\nabla} + g\, f_{abc} \bar{A}_c)$ and an over bar , defined in Sec.III, indicates the component (1,2) of a 3-vector. The functional integration over Π_3^a variables is Gaussian and the coefficient of quadratic term is constant. It is readily calculated and Π_3^a are simply replaced by $(\partial_3 \omega_a)$. The integration over ω_a is then calculated by the shift transformation $\omega_a \rightarrow \omega_a + \omega_a^0$ as usual . The ω_a^0 are chosen so as to remove the linear term $\omega_a (\bar{D} \cdot \bar{\Pi})_a$. We drop out a factor corresponding to a path integral over ω_a with free action $\int d^4x (\partial_3 \omega_a)^2 d^4x$. This may be absorbed in the normalization. Thus we finally get

$$Z = N \int e^{iS} \left[d\Pi^a_1\right] \left[d\Pi^a_2\right] \left[dA^1_a\right] \left[dA^2_a\right] \qquad (4.5)$$

where

$$S = \int d^4x \left\{ \left(\bar{\Pi} \cdot \dot{\bar{A}} - \frac{1}{2} \bar{\Pi}^2 - \frac{1}{4} F^a_{k\ell} F^{k\ell}_a \right) + \frac{1}{2} \int d^3y \;\cdot \right.$$

$$\left. \cdot\, \bar{D}^{ab} \cdot \bar{\Pi}_b (\vec{y},t) K(\vec{x},\vec{y}) \bar{D}^{ad} \cdot \bar{\Pi}_d (\vec{x},t) \right\} \qquad (4.6)$$

The corresponding Hamiltonian, as expected, is the same as in Eq.(3.13)for canonical quantization case. In the presence of spinor field source $(\overline{D}.\overline{\Pi})_a$ is replaced by $(\overline{D}.\overline{\Pi})_a - g\Psi^+ t_a \Psi$ in Eq.(4.6). For the abelian case it reduces to $(\overline{\nabla}.\overline{\Pi} - g\,\Psi^+\Psi)$. In this case the Hamiltonian can be rewritten in terms only of transverse components $\vec{A}_T$, $\vec{\Pi}_T$ of the gauge field and $\vec{\Pi}_T = \dot{\vec{A}}_T$. The kinetic terms like $\hat{\Psi}^+ [- i\vec{\alpha}.(\vec{\nabla}+i\,g\,\hat{\vec{A}})]\hat{\Psi}$ are handled by a unitary transformation[6] which, for example, takes the present term to $\hat{\Psi}^+ [- i\vec{\alpha}\,.(\vec{\nabla} + i\,g\,\vec{A}_T)]\hat{\Psi}$ while unaffecting the gauge field part already rewritten in terms of transverse components. The Coulomb self-energy interaction term is separted out.

The gauge-fixing conditions $A_a^0 \approx 0$, $A_3^0 \approx 0$ over phase space thus fix the the gauge even for arbitrary strong gauge fields and we obtain a description of the Yang-Mills field for any gauge group in terms of physical degrees of freedom only. The last term in Eq.(4.6) contains cubic and quartic terms. We may,however, integrate Eq.(4.1) over A_a^0, Π_a^0, use the Eq.(4.3) and integrate over $\vec{\Pi}_a$ using the shift transformations,

$$\Pi_3^a \rightarrow \Pi_3^a + \partial_3\,\omega_a$$

$$\overline{\Pi}_a \rightarrow \overline{\Pi}_a + (\dot{\overline{A}}_a + \overline{D}^{ab}\,\omega_b) \tag{4.7}$$

A Gaussian functional integral involving solely the variables $\vec{\Pi}_a$ may be absorbed in the normalization factor to obtain the following expression for the generating functional

$$Z = N \int \left[d\, A_a^k\right]\left[d\, A_a^0\right] \prod_{x,a} \delta(A_a^3(x))\; e^{iS} \tag{4.8}$$

where

$$S = -\frac{1}{4} \int \left[\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\, f_{abc}\, A_b^\mu\, A_c^\nu\right]^2 d^4x \tag{4.9}$$

and we have formally rewritten $\omega_a \equiv A_a^0$. This representation may then be used as a starting point to go over to other convenient gauge conditions[13] where Feynman rules are given in manisfestly covariant form. It is also clear that for an arbitrary constant unit vector $\vec{n}$, $A_a^3 \simeq 0$ is replaced by $(\vec{n}.\vec{A}_a) \simeq 0$ and $\bar{A}_a$ represents a vector tranverse to the direction $\vec{n}$. The delta functional in Eq.(4.8) gets substituted by $\delta(\vec{n}.\vec{A}_a)$.

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