Clifford Algebras and the Minimal Representations of the $1D$ N-Extended Supersymmetry Algebra*

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Abstract

The Atiyah-Bott-Shapiro classification of the irreducible Clifford algebra is used to derive general properties of the minimal representations of the $1D$ N-Extended Supersymmetry algebra (the Z_2-graded symmetry algebra of the Supersymmetric Quantum Mechanics) linearly realized on a finite number of fields depending on a real parameter t, the time.

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1 Introduction

The superalgebra of the Supersymmetric Quantum Mechanics ($1D$ N-Extended Supersymmetry Algebra) [1] is a $\mathbf{Z}_2$-graded algebra expressed by N odd generators Q_i ($i = 1, \ldots, N$) and a single even generator H (the hamiltonian). It is defined by the (anti)-commutation relations

$$\begin{array}{rcl} \{Q_i, Q_j\} & = & 2\delta_{ij}H, \\ {[Q_i, H]} & = & 0. \end{array} \qquad (1)$$

The structure of its minimal linear representations realized on a finite number of fields depending on a single real parameter (t, the time) has been substantially elucidated in recent years. Several results have been obtained [2, 3, 4, 5, 6, 7]. They are based on the Atiyah-Bott-Shapiro [8] classification of the irreducible Clifford algebras. In this paper we discuss the recent developments addressing the classification of the (1) representations. In the Conclusions we briefly mention at least one open problem.

The problem we addressing can be stated as follows: to construct and classify, for any given integer N, the linear representations of (1) acting on a finite, minimal, number of fields, even and odd (bosonic and fermionic), depending on t. The generator H has to be represented by a time-derivative, while the Q_i's generators must be realized by finite-dimensional matrices whose entries are either c-numbers or time-derivatives up to a certain power. The representation space we are considering is infinite-dimensional, being given by the set of fundamental fields and their time-derivatives of any order. In the physical literature these representations are called "finite" since they are obtained

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by a finite number of fundamental fields (the situation parallels here the representation theory of chiral algebras [9], given by the generating set of primary fields and their descendants; the time-derivatives of the fundamental fields play, for (1) representations, the role of the descendants in chiral algebra representations). For the same reason, the notion of "minimal representations" is expressed, in the physical literature, as "irreducible representations".

The program of classifying the (1) minimal representations starts with [2], with the recognition that formulating an eigenvalue problem for the hamiltonian H (for an eigenvalue different from zero) reduces the Q_i's anticommutators to, up to normalization, the basic relation for Euclidean Clifford algebra generators. The [2] main result can be stated as follows. The minimal representations of (1), for a given N, are obtained by applying a dressing transformation to a fundamental representation (nowadays called in the literature the "root multiplet"), with equal number of bosonic and fermionic fields. The root multiplet is specified by an associated Euclidean Clifford algebra. As a main corollary, the total number n of bosonic fundamental fields entering a minimal representation equals the total number of fermionic fundamental fields and is expressed, for any given N, by the following relation [2]

$$\begin{aligned} N &= 8l+m, \\ n &= 2^{4l}G(m), \end{aligned} \tag{2}$$

where $l=0,1,2,\ldots$ and $m=1,2,3,4,5,6,7,8$.

$G(m)$ appearing in (2) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(3)

Note the *mod* 8 Bott's periodicity.

An integral $\mathbf{Z}$-grading, compatible with the $\mathbf{Z}_2$-grading of the superalgebra, can be assigned to the fundamental fields and their time-derivatives. In the physical literature, the grading is referred as "mass-dimension". The integral grading will be denoted by z. For convenience, the mass-dimension d will be expressed as $d=\frac{z}{2}$. The hamiltonian H has mass-dimension $d=1$ (its fermionic roots, the Q_i's operators, have mass-dimension $d=\frac{1}{2}$). Bosonic (fermionic) fields have integer (respectively, half-integer) mass-dimension. Each linear representation admitting a finite number of fundamental fields is characterized by its "fields content", i.e. the set of integers $(n_1,n_2,\ldots,n_l)$ specifying the number n_i of fundamental fields of dimension d_i ($d_i=d_1+\frac{i-1}{2}$, with d_1 an arbitrary constant) entering the representation. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the length of the representation (a root representation has length $l=2$). Either $n_1,n_3,\ldots$ correspond to the bosonic fields (therefore $n_2,n_4,\ldots$ specify the fermionic fields) or viceversa. In both cases the equality $n_1+n_3+\ldots=n_2+n_4+\ldots=n$ is guaranteed.

The representation theory does not discriminate the overall bosonic or fermionic nature of the representation.

According to [2], if $(n_1, n_2, \ldots, n_l)$ specifies the fields content of an irreducible representation, $(n_l, n_{l-1}, \ldots, n_1)$ specifies the fields content of a dual irreducible representation. Representations such that $n_1 = n_l, n_2 = n_{l-1}, \ldots$ are called "self-dual representations". In [3] it was shown how to extract from the associated Clifford algebras the admissible fields content of the (1) linear finite irreducible representations. We discuss these results in the next Section.

2 Supersymmetric Quantum Mechanics and Clifford algebras

In this Section we give a more detailed account of the connection between representations of the Supersymmetric Quantum Mechanics and Clifford algebras.

According to [2] the length-2 minimal representations of the (1) supersymmetry algebra are uniquely determined by a representation of an associated Clifford algebra. The connection goes as follows. The supersymmetry generators acting on a length-2 irreducible multiplet can be expressed as

$$Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix} \tag{4}$$

where σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of the Clifford algebra relation

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix} \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\eta_{ij} \tag{5}$$

The Q_i's in (4) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks, acting on a multiplet m (thought of as a column vector) which can be either bosonic or fermionic (we conventionally consider a length-2 irreducible multiplet as bosonic if its upper half part of component fields is bosonic and its lower half is fermionic; it is fermionic in the converse case). The connection between Clifford algebra irreps of the Weyl type and minimal representations with minimal length of the N-extended one-dimensional supersymmetry is such that D, the dimensionality of the (Euclidean, in the present case) space-time of the Clifford algebra (5) coincides with the number N of the extended supersymmetries, according to

$\sharp$ of space-time dim. (Weyl-Clifford)	$\Leftrightarrow$	$\sharp$ of extended su.sies (in 1-dim.)
D	$=$	N

(6)

The matrix size of the associated Clifford algebra (equal to $2n$, with n given in (2)) corresponds to the number of (bosonic plus fermionic) fields entering the one-dimensional N-extended supersymmetry irrep.

The classification of Weyl-type Clifford irreps, furnished in [2], can be easily recovered from the well-known classification of Clifford irreps, given in [8] (see also [10] and [11]).

The (4) Q_i's matrices realizing the N-extended supersymmetry algebra (1) on length-2 minimal representations have entries which are either c-numbers or are proportional to the Hamiltonian H. Minimal representations of higher length ($l \geq 3$) are systematically produced [2] through repeated applications of the dressing transformations

$$Q_i \quad \mapsto \quad \widehat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \tag{7}$$

realized by diagonal matrices $S^{(k)}$'s ($k = 1, \ldots, 2n$) with entries ${s^{(k)}}_{ij}$ given by

$${s^{(k)}}_{ij} \quad = \quad \delta_{ij}(1 - \delta_{jk} + \delta_{jk} H) \tag{8}$$

Some remarks are in order [2]

i) the dressed supersymmetry operators ${Q_i}'$ (for a given set of dressing transformations) have entries which are integral powers of H. A subclass of the ${Q_i}'$ s dressed operators is given by the local dressed operators, whose entries are *non-negative* integral powers of H (their entries have no $\frac{1}{H}$ poles). A local representation (minimal representations fall into this class) of an extended supersymmetry is realized by local dressed operators. The number of the extension, given by N' ($N' \leq N$), corresponds to the number of local dressed operators.

ii) The local dressed representation is not necessarily a minimal representation. Since the total number of fields (n bosons and n fermions) is unchanged under dressing, the local dressed representation is a minimal representation iff n and N' satisfy the (2) requirement (with N' in place of N).

iii) The dressing changes the dimension of the fields of the original multiplet m. Under the $S^{(k)}$ dressing transformation (7), $m \mapsto S^{(k)} m$, all fields entering m are unchanged apart from the k-th one (denoted, e.g., as φ_k and mapped to $\dot{\varphi}_k$). Its dimension is changed from $[k] \mapsto [k] + 1$. This is why the dressing changes the length of a minimal representation. As an example, if the original length-2 multiplet m is a bosonic multiplet with d 0 mass-dimension bosonic fields and d $\frac{1}{2}$ mass-dimension fermionic fields (in the following such a multiplet will be denoted as $(x_i; \psi_j) \equiv (d, d)_{s=0}$, for $i, j = 1, \ldots, d$), then $S^{(k)} m$, for $k \leq d$, corresponds to a length-3 multiplet with $d - 1$ bosonic fields of 0 mass-dimension, d fermionic fields of $\frac{1}{2}$ mass-dimension and a single bosonic field of mass-dimension 1 (in the following we employ the notation $(d - 1, d, 1)_{s=0}$ for such a multiplet of fields).

When looking purely at the representation properties of a given multiplet the assignment of an overall mass-dimension s is arbitrary, since the supersymmetry transformations of the fields are not affected by s. Introducing an overall mass-dimension is useful for tensoring multiplets and becomes essential for physical applications, e.g. in the construction of supersymmetric invariant terms entering an action.

In the above multiplet l denotes its length, d_l the number of auxiliary fields of highest mass-dimension transforming as time-derivatives. The total number of odd-indexed equal the total number of even-indexed fields, i.e. $d_1 + d_3 + \ldots = d_2 + d_4 + \ldots = d$. The multiplet is bosonic if the odd-indexed fields are bosonic and the even-indexed fields are fermionic (the multiplet is fermionic in the converse case). For a bosonic multiplet the auxiliary fields are bosonic (fermionic) if the length l is an odd (even) number.

Just like the overall mass-dimension assignment, the assignment of a bosonic (fermionic) character to a multiplet is arbitrary since the mutual transformation properties of the fields inside a multiplet are not affected by its statistics. Therefore, multiplets always appear in dually related pairs so that to any bosonic multiplet there exists its fermionic counterpart with the same transformation properties.

Throughout this paper we assign integer valued mass-dimension to bosonic multiplets and half-integer valued mass-dimension to fermionic multiplets.

As pointed out before, the most general $(d_1, d_2, \ldots, d_l)$ multiplet is recovered as a dressing of its corresponding N-extended length-2 (d, d) multiplet. In [2] it was shown that all dressed supersymmetry operators producing any length-3 multiplet (of the form $(d-p, d, p)$ for $p = 1, \ldots, d-1$) are of local type. Therefore, for length-3 multiplets, we have $N' = N$. This implies, in particular, that the $(d-p, d, p)$ multiplets are non-equivalent irreps of the N-extended one-dimensional supersymmetry. As concerns length $l \geq 4$ multiplets, the general problem of finding minimal representations was not addressed in [2]. It was shown, as a specific example, that the dressing of the length-2 $(4, 4)$ irrep of $N = 4$, realized through the series of mappings $(4, 4) \mapsto (1, 4, 3) \mapsto (1, 3, 3, 1)$, produces at the end a length-4 multiplet $(1, 3, 3, 1)$ carrying only three local supersymmetries ($N' = 3$). Since the relation (2) is satisfied when setting the number of extended supersymmetries acting on a multiplet equal to 3 and the total number of bosonic (fermionic) fields entering a minimal representation equal to 4, as a consequence, the $(1, 3, 3, 1)$ multiplet corresponds to a minimal representation of the $N = 3$ extended supersymmetry.

Based on an algorithmic construction of representatives of Clifford irreps, an iterative method to compute the admissible field contents of the minimal representations for arbitrary N values of the extended supersymmetry was presented in [3].

3 Admissible field contents

It is now possible to plug the information contained in Clifford algebras and apply the construction outlined in the previous Section to compute the admissible field content for the length-4 representations for arbitrary values of N. This construction was done in [3]. We present here the list of length-4 field content up to $N \leq 11$.

Up to $N = 8$ we have

$N = 1$	NO
$N = 2$	NO
$N = 3$	$(1, 3, 3, 1)$
$N = 4$	NO
$N = 5$	$(1, 5, 7, 3)$, $(3, 7, 5, 1)$, $(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$
$N = 6$	$(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$
$N = 7$	$(1, 7, 7, 1)$
$N = 8$	NO

(9)

Since there are no length-l irreps with $l \geq 5$ for $N \leq 9$, the above list, together with the already known length-2 and length-3 irreps, provides the complete classification of the admissible field content of the minimal representations for $N \leq 8$.

Please note that the length-4 irrep of $N = 3$, $(1,3,3,1)$, is self-dual under the [3] mass-dimension duality exchange discussed in the Introduction, while two of the non-equivalent length-4 $N = 5$ irreps are self-dual, $(2,6,6,2)$ and $(1,7,7,1)$. The remaining ones are pair-wise dually related ($(1,5,7,3) \Leftrightarrow (3,7,5,1)$ and $(1,6,7,2) \Leftrightarrow (2,7,6,1)$).

The $N = 9$ length-4 minimal representation (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 16 - h$, $d_2 = 16 - k$. The complete list of $N = 9$ length-4 fields content is expressed by h, k satisfying the constraint

$$h + k \leq 8. \tag{10}$$

$N = 10$ is the lowest supersymmetry admitting length-5 minimal representations. The field content of its length-4 minimal representations is given by (d_1, d_2, d_3, d_4), expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 32 - h$, $d_2 = 32 - k$. If we set

$$r = min(h, k) \tag{11}$$

the non-equivalent length-4 field content is given by the ordered pair of positive integers h, k satisfying the constraint

$$h + k + r \leq 24. \tag{12}$$

For $N = 11$ the length-4 fields content (d_1, d_2, d_3, d_4) is expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$. Setting as before $r = min(h, k)$ and introducing the $s(r)$ function defined through

$$s(r) = \left\{ \begin{array}{cc} 8 - r & \text{for } r = 1, \ldots, 7 \\ 0 & \text{otherwise} \end{array} \right\} \tag{13}$$

we can express the constraints on h, k as

$$h + k + r - s(r) \leq 48. \tag{14}$$

4 Supersymmetry graphs and their connectivity

In this Section we describe, largely based on [12], the graphical interpretation of the minimal supersymmetry representations and discuss, based on [6] and [7], their connectivity properties.

An association can be made between N-colored oriented graphs and the linear supersymmetry transformations. The identification goes as follows. The fundamental fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X - Y$ plane. The Y coordinate can be chosen to correspond to the mass-dimension d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the X axis. The higher dimensional fields have positive, integer or half-integer values of Y. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the N Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented N-colored graph with $2n$ vertices (see (2)). The graph is such that precisely N edges, one for each color, are linked to any given vertex which represents either a 0-mass dimension or a $\frac{1}{2}$-mass dimension field.

Despite the fact that the presentation of the graph is not unique, certain of its features only depend on the class of the supersymmetry transformations. We introduce now, following [6], the invariant characterization. An unoriented "color-blind" graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black). For simplicity, we discuss here the invariant characterization of the graphs associated to the length $l = 3$ irreducible representation that will be discussed in the following (the generalization of the invariant characterization to graphs of arbitrary length is straightforward, see [6]). They admit fields content $(k, n, n-k)$. The corresponding fields are denotes as x_p (for 0-mass dimension), ψ_q (for $\frac{1}{2}$ mass-dimension) and g_r (the 1 mass-dimension auxiliary fields), where $p =, 1, \ldots, k$, $q = 1, \ldots, n$ and $r = 1, \ldots, n-k$.

The connectivity of the associated length $l = 3$ color-blind graph can be expressed through the connectivity symbol ψ_g, expressed as

$$\psi_g \quad = \quad (m_1)_{s_1} + (m_2)_{s_2} + \ldots + (m_Z)_{s_Z}. \tag{15}$$

The ψ_g symbol encodes the information on the partition of the n $\frac{1}{2}$-mass dimension fields (vertices) into the sets of m_z vertices ($z = 1, \ldots, Z$) with s_z edges connecting them to the $n-k$ 1-mass dimension auxiliary fields. We have

$$m_1 + m_2 + \ldots + m_Z \quad = \quad n, \tag{16}$$

while $s_z \neq s_{z'}$ for $z \neq z'$.

The connectivity symbol is an invariant characterization of the class of the irreducible supersymmetry transformations.

The connectivity symbol ψ_g can be used to induce a map $\widetilde{\psi_g}$ from the set of graphs Gr into the set of integers $\mathbf{Z}$ ($\widetilde{\psi_g} : Gr \rightarrow \mathbf{Z}$) s.t. $W \in \mathbf{Z}$ is given by

$$W \quad = \quad \prod_{z=1}^{Z} (p_{2z-1}^{m_z})(p_{2z}^{s_z}), \tag{17}$$

where the p_w's, $w = 1, 2, 3, \ldots$, denote the ordered set of prime integers $(2, 3, 5, \ldots)$. With the above definition two inequivalent connectivities induce two distinct integers W, W' ($W' \neq W$).

5 $N=4$ decompositions and connectivities of the $N=5$ minimal representations

The $N=5$ minimal representations contain a total number of 8 bosonic and 8 fermionic fields. The $N=5$ minimal representations can be decomposed into two sets of $N=4$ minimal representations whose vertices (component fields) are linked together by the 5^{th} supersymmetry. The $N=4$ representations contain 4 bosonic and 4 fermionic fields associated to different mass-dimensions.

The length-2 and length-4 $N=5$ minimal representations admit a unique decomposition into $N=4$ representations. The situation is different for the length-3 $N=5$ minimal representations whose fields content is given by $(n,8,8-n)$, for $n=1,2,\ldots,7$. They admit the following decompositions in terms of $(k,4,4-k)$ and $(n-k,4,4-n+k)$ $N=4$ representations:

$$(n,8,8-n) \quad = \quad (k,4,4-k)+(n-k,4,4-n+k). \tag{18}$$

It is convenient to express n as

$$n \quad = \quad 4+\epsilon m, \tag{19}$$

where $\epsilon=\pm 1$, while $m=0,1,2,3$.

The inequivalent values of k are given by the integers

$$k \quad = \quad \frac{1}{2}(1+\epsilon)m, \frac{1}{2}(1+\epsilon)m+1,\ldots,\frac{1}{2}(1+\epsilon)m+[\frac{4-m}{2}], \tag{20}$$

where the square brackets refers to the integral part.

The ψ_g connectivity symbol can be easily computed for each such decomposition. We obtain, in terms of n and k,

$$\psi_g \quad = \quad (4-k)_{5+k-n}+(k)_{4+k-n}+(4+k-n)_{5-k}+(n-k)_{4-k}. \tag{21}$$

For any given n, the ψ_g connectivity symbol differs for inequivalent values of k. This implies, as a corollary, that the decomposition into $N=4$ representations specified by different, inequivalent values of k produces inequivalent $N=5$ minimal representations (no matter which supersymmetry generator is picked up as the "fifth").

We define as "Δ" the number of degeneracies, i.e. the number of inequivalent minimal representations with the same fields content. Δ is computed to be

$$\Delta \quad = \quad [\frac{4-m}{2}]+1, \tag{22}$$

The results for the inequivalent $N=5$ length-3 minimal representations can be summarized in the following table

fields cont.	$N=4$ decomp.	ψ_g connectivities	labels
$(1,8,7)$	$(0,4,4)+(1,4,3)$	3_5+5_4	
$(2,8,6)$	$(0,4,4)+(2,4,2)$	$2_5+2_4+4_3$	A
	$(1,4,3)+(1,4,3)$	6_4+2_3	B
$(3,8,5)$	$(0,4,4)+(3,8,5)$	$1_5+3_4+4_2$	A
	$(1,4,3)+(2,4,2)$	$2_4+5_3+1_2$	B
$(4,8,4)$	$(0,4,4)+(4,4,0)$	4_4+4_1	A
	$(1,4,3)+(3,4,1)$	$1_4+3_3+3_2+1_1$	B
	$(2,4,2)+(2,4,2)$	4_3+4_2	C
$(5,8,3)$	$(1,4,3)+(4,4,0)$	$4_3+3_1+1_0$	A
	$(2,4,2)+(3,4,1)$	$1_3+5_2+2_1$	B
$(6,8,2)$	$(2,4,2)+(4,4,0)$	$4_2+2_1+2_0$	A
	$(3,4,1)+(3,4,1)$	2_2+6_1	B
$(7,8,1)$	$(3,4,1)+(4,4,0)$	5_1+3_0	

(23)

The last column specifies the labels assigned, in terms of increasing values of k, to each inequivalent $N=5$ minimal representations.

For the sake of clarity we show two unoriented graphs (the figures 1 and 2) associated to the two inequivalent $N=5$ minimal representations (A and respectively B) with same fields content $(2,8,6)$. Similar graphical presentations can be straightforwardly drawn for all other inequivalent $N=5$ minimal representations.

6 Fusion algebra

The tensor product of linear minimal representations can be decomposed into their irreducible constituents. This decomposition contains useful information in the construction of bilinear (in general, multilinear) terms entering a supersymmetric invariant action. We recall that the auxiliary fields in a given representation transform as a total derivative (a time derivative in one dimension). Useful information concerning the decomposition of the tensor products of the minimal representations can be encoded in the so-called fusion algebra of the irreps and their supersymmetric vacua. The notion of a fusion algebra of the supersymmetric vacua of the N-extended one dimensional supersymmetry, introduced in [3], is constructed by analogy with the fusion algebra for rational conformal field theories. Fusion algebras can also be nicely presented in terms of their associated graphs. We explicitly present here the $N=1$ and $N=2$ fusion graphs (with two subcases for each N, according to whether or not the irreps are distinguished w.r.t. their bosonic/fermionic statistics). Let us discuss here how to present the [3] results in graphical form. The minimal representations correspond to points. N_{ij}^k oriented lines (with arrows) connect the $[j]$ and the $[k]$ minimal representations if the decomposition $[i]\times[j]={N_{ij}}^k[k]$ holds. The arrows are dropped from the lines if the $[j]$ and $[k]$ minimal representations can be interchanged. The $[i]$ minimal representation should correspond to a generator of the fusion algebra. This means that the whole set of $N_l={N_{lj}}^k$ fusion matrices is produced as the sum of powers of the $N_i={N_{ij}}^k$ fusion matrix.

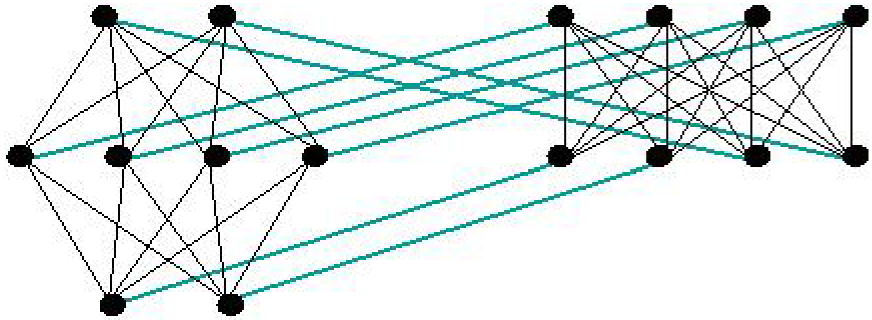

Figure 1: Graph of the $N = 5$ $(4, 8, 4)$ multiplet of $4_4 + 4_1$ connectivity (type A).

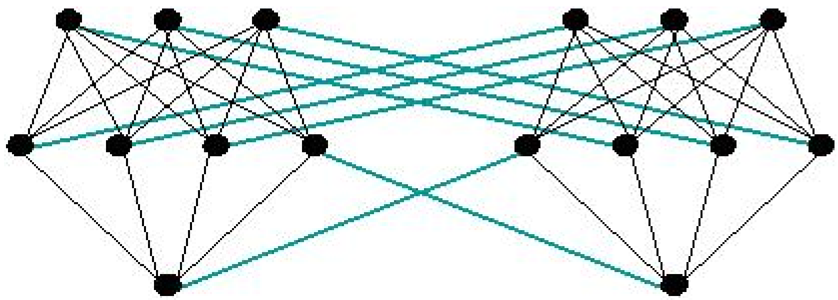

Figure 2: Graph of the $N = 5$ $(4, 8, 4)$ multiplet of $4_3 + 4_2$ connectivity (type B).

Let us discuss explicitly the $N = 2$ case. We obtain the following list of four minimal representations (if we discriminate their statistics):

$$[1] \equiv (2,2)_{Bos};\ [2] \equiv (1,2,1)_{Bos};\ [3] \equiv (2,2)_{Fer};\ [4] \equiv (1,2,1)_{Fer} \tag{24}$$

The corresponding $N = 2$ fusion algebra is realized in terms of four 4×4, mutually commuting, matrices given by

$$N_1 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv X; \tag{25}$$

$$N_2 = N_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Y; \tag{26}$$

$$N_3 = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Z. \tag{27}$$

The fusion algebra admits three distinct elements, X, Y, Z and one generator (we can choose either X or Z), due to the relations

$$Y = \frac{1}{8}(X^3 - 2X) \quad , \quad Z = -\frac{1}{4}(X^3 - 6X^2 + 4X). \tag{28}$$

The vector space spanned by X, Y, Z is closed under multiplication

$$\begin{aligned} X^2 = Z^2 = ZX &= X + 2Y + Z, \\ XY = Y^2 = YZ &= 4Y. \end{aligned} \tag{29}$$

This fusion algebra corresponds to the "smiling face" graph below. We obtain the following four tables for the fusion graphs of the $N = 1$ and $N = 2$ supersymmetric quantum mechanics minimal representations. The "A" cases below correspond to ignore the statistics (bosonic/fermionic) of the given minimal representations. In the "B" cases, the number of fundamental minimal representations is doubled w.r.t. the previous ones, in order to take the statistics of the minimal representations into account. We have

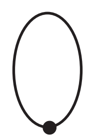

Figure 3: Fusion graph of the N=1 superalgebra (A case, no bosons/fermions distinction).

Figure 4: Fusion graph of the N=1 superalgebra (B case, bosons/fermions distinction).

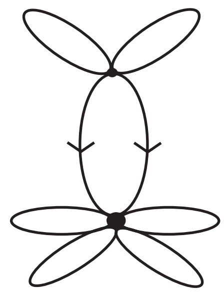

Figure 5: Fusion graph of the N=2 superalgebra (A case, no bosons/fermions distinction).

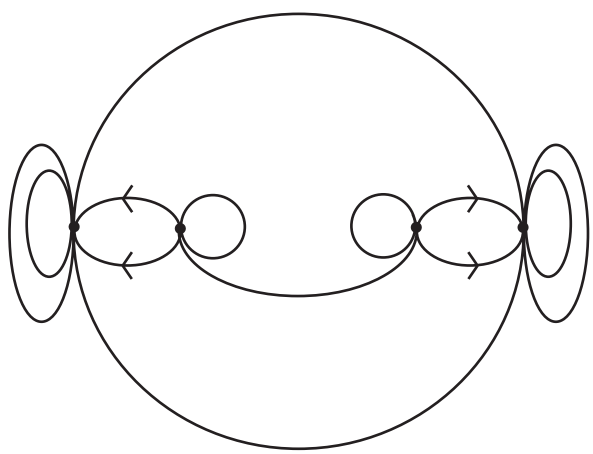

Figure 6: Fusion graph of the N=2 superalgebra (B case, bosons/fermions distinction), "the smiling face". From left to right the four points correspond to the $[2] - [1] - [3] - [4]$ irreps, respectively. The lines are generated by the $N_1 \equiv X$ fusion matrix, see ([13]).

7 Conclusions

In this paper we have reviewed the present state of the art concerning the classification of the minimal representations of the (1) $1D$ N-Extended Superalgebra of the Supersymmetric Quantum Mechanics. Since we have already discussed it elsewhere [14] we did not mention here the vast range of physical implications of the Supersymmetric Quantum Mechanics (like the possibility which offers to investigate in a simplified context the properties of higher-dimensional supersymmetric theory of grandunification, supergravity and M-theory). We focused instead on the mathematical aspects of the representation theory and its relation with the Clifford algebras. We detailed the classification of [3] of the admissible fields content of the minimal representations and the classifications given in [6] and [7] of the admissible connectivities of the graphs associated to the minimal representations. It is quite appropriate to emphasize at a Conference devoted to Clifford algebras that these results were obtained by applying Clifford algebras to the representation problem of the Supersymmetric Quantum Mechanics. Other results, like the construction of the fusion algebra associated to the tensoring of the representations, were also presented. From a mathematical point of view some questions still have to be answered. They need to be attacked by using more powerful methods than the ones here discussed, relying on combinatorics. An issue which deserves being mentioned concerns the possibility, for a sufficiently large value of N, that a dressing of the root multiplet of minimal length produced by an admissible non-diagonal dressing matrix, could end up in a minimal representation which cannot be expressed in graphical form with the type of graphs introduced in Section **4**.

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