The Two-loop Massless $\frac{\lambda}{4!}\varphi^4$ Model in Non-translational Invariant Domain

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We study the $\frac{\lambda}{4!}\varphi^4$ massless scalar field theory in a four-dimensional Euclidean space, where all but one of the coordinates are unbounded. We are considering Dirichlet boundary conditions in two hyperplanes, breaking the translation invariance of the system. We show how to implement the perturbative renormalization up to two-loop level of the theory. First, analyzing the full two and four-point functions at the one-loop level, we shown that the bulk counterterms are sufficient to render the theory finite. Meanwhile, at the two-loop level, we have to introduce also surface counterterms in the bare lagrangian in order to make finite the full two and also four-point Schwinger functions.

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I. INTRODUCTION

In this paper we are interested to show how to implement the renormalization procedure up to two-loop level in the massless $\frac{\lambda}{4!}\varphi^4$ scalar field theory, defined in a four-dimensional Euclidean space with one compactified dimension. Our aim is to shed light on the renormalization procedure in a system defined in a domain where translational symmetry is broken, which must be done for example in the high temperature dimensional reduced quantum chromodynamics (QCD), defined in a finite region.

Quantum chromodynamics is a non-abelian Yang-Mills theory with gauge group $SU(3)$. Since it is assumed that the fermions of the theory transform according to the fundamental representation of the gauge group, each flavour of quark is a triplet of the color group $SU(3)$. Gauge bosons transform according to the adjoint representation. The interaction between the quarks is mediated by the gluons. Due to the non-abelian structure of the theory, the gluons couple not only with the quarks but have also cubic and quartic self-interaction. The self-interaction of the gluons provides the anti-screening of the color charge in QCD. This is responsible for asymptotic freedom and presumably confinement.

The confinement-deconfinement phase transition in QCD may occur in usual matter at sufficiently high temperature or if is strongly compressed [1] [2] [3]. In ultra-relativistic heavy ion collisions, we expect that the plasma of quarks and gluons can be produced. We would like to stress that, although non-equilibrium processes occur in the quark-gluon plasma in the heavy ion collisions, for simplicity in a first approximation we can assume a static situation. Just after the collision hot and compressed nuclear matter is confined in a small region of the space and in such circumstances the volume and surface effects become very important.

In the above described physical situation there are two important points: the first one is that the thermodynamic limit of the infinite volume system can not be used and therefore finite volume effects should be investigated and taken into account. The second point is that the quark-gluon plasma exists in a situation of high temperature, where using the Matsubara formalism to describe high temperature QCD, dimensional reduction must occurs [4] [5] [6] [7]. Dimensional reduction is based in the Appelquist-Carrazone decoupling theorem [8]. From a more fundamental theory, the effective Lagrangian density of this theory can be obtained as some low-energy limit of the fundamental theory where the heavy modes have been removed. There are some interesting physical situations where the decoupling theorem can be used. First, for scalar fields without spontaneous symmetry breaking. Second, in quantum electrodynamics, where a derivative expansion of the photon effective action can be obtained by integrating out the fermionic fields. Also in QCD at least in lowest order in perturbation theory the decoupling theorem works. The decoupling theorem is not valid for example in spontaneous broken gauge theories. It is important to stress that the non-validity of the decoupling theorem means that low-energy experiments can provide information about the high energy physics.

Going back to the heavy-ions collision situation, we can assume that the following scenario appears: in the situation where dimensional reduction occurs, we have an effective theory for the gluons field and also finite size effects for these bosonic fields. To shed light on the renormalization procedure in systems defined in domain where translational symmetry is broken, as for example the high temperature dimensional reduced QCD, in this paper we are interested to investigate scalar models, impose classical boundary condition over the fields. We hope that this study will give us some insight over the most interesting and also more complicated situation as the one mentioned above. Therefore, in this paper we analyze how to implement the perturbative renormalization up to two-loop level of $\frac{\lambda}{4!}\varphi^4$ massless scalar field model defined in a four-dimensional Euclidean space with one compactified dimension.

Finite size effects and the presence of macroscopic structures in different field theory models has been extensively studied in the literature. The critical behavior of the $O(N)$ model in the presence of a surface was a target of intense investigations [9]. The same $O(N)$ model was studied in two different geometries: the periodic cube and the cylinder along one dimension (the time) and finite and periodic in the $(d-1)$ remaining dimensions by Brezin and Zinn-Justin [10]. Finite size effects in QCD [11] and also in different field theory models also has been extensively studied in the literature. Assuming periodic or antiperiodic boundary conditions for bosonic and fermionic models respectively, the translation symmetry is maintained, and surface effects are avoided. Therefore, to avoid surface effects, quantum fields defined in manifolds with periodic or anti-periodic boundary conditions in the spatial section was preferred by many authors [12]. Nevertheless, the case of boundaries conditions that break the translational symmetry deserves our attention.

In the case of hard boundary conditions as for example Dirichlet-Dirichlet(DD) or Neumann-Neumann (NN), the translational invariance is lacking. This fact makes the Feynman diagrams harder to compute that in unbounded space. Moreover the renormalization program is implemented in different way from unbounded or translational invariance systems since some surface divergence appear [13]. For translational invariant systems, one can use the momentum space representation, which is a more convenient framework to analize the ultraviolet divergences of a theory. Translational invariance is preserved for momentum conservation conditions. For non-translational invariant systems a more convenient representation for the n-point Schwinger functions is a mixed momentum coordinate space.

Fosco and Svaiter considered the anisotropic scalar model in a d-dimensional Euclidean space, where all but one of the coordinates are unbounded. Translational invariance along the bounded coordinate which lies in the interval $[0, L]$ is broken because the choice of boundary condition chosen for the hyperplanes at $z = 0$ and $z = L$. Two different possibilities of boundary conditions was considered: (DD) and also (NN), and the renormalization of the two-point function was achieved in the one-loop approximation [14]. Further the renormalization of the four-point function was achieved in the one-loop approximation by Caicedo and Svaiter [15]. Finally Svaiter [16] studied the renormalization of the $\frac{\lambda}{4!}\varphi^4$ massless scalar field model in the one-loop approximation in finite size systems assuming that the system is in thermal equilibrium with a reservoir. Also, still studying surface, edge and corners effects, Rodrigues and Svaiter [17] analysed first the renormalized vacuum fluctuations associated with a massless real scalar field, confined in the interior of a rectangular infinitely long waveguide. A closed form of the analytic continuation of the local zeta function in the interior of the waveguide was obtained and a detailed study of the surface and edge divergences was presented. Next, these authors [18] studied the renormalized stress tensor associated with an electromagnetic field in the interior of a rectangular infinitely long waveguide.

In this paper we will consider an interacting massless scalar model, in a four-dimensional Euclidean space, where the first three coordinates are unbounded and the last one lies in the interval $[0, L]$. We analyze only DD boundary conditions. First we present an algebraic expression in coordinate space for the free propagator which let us to identify the divergences of the n-point Schwinger functions for the interacting theory. This algebraic expression agrees with the result obtained by Lukosz [19]. We would like to stress that instead of assume hard boundary conditions, some authors assumed soft boundary conditions and also treated the boundary as a quantum mechanical object [20]. Here, we prefer to keep hard classical boundary conditions.

The organization of the paper is as follows: In section II we discuss the slab configurations, obtaining some important expressions for the free propagator in order to understand some procedures in the divergences identification. In section III the regularization program is implemented in the one-loop approximation. In section IV the regularization program is implemented in the two-loop approximation. Section V contains our conclusions. In appendix A, expression for the free propagator is introduced. Throughout this paper we use $\hbar = c = 1$.

II. CLASSICAL BOUNDARY CONDITIONS AND SOME PROPERTIES OF THE FREE PROPAGATOR

Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time allowed by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. The $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z(h)$ is formally defined by the following functional integral:

$$Z(h) = \int [d\varphi] \; \exp\left(-S_0 - S_I + \int d^dx \, h(x)\varphi(x)\right), \tag{1}$$

where the action that describes a free scalar field is

$$S_0(\varphi) = \int d^dx \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2(x)\right), \tag{2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I(\varphi) = \int d^dx\,\frac{\lambda}{4!}\,\varphi^4(x). \tag{3}$$

In Eq.(1), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_x d\varphi(x)$. The terms λ and m_0^2 are respectivelly the bare coupling constant and mass squared of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by means of functional derivatives. Note that we are using the same notation for functionals and functions, for example $Z(h)$ instead the usual notation $Z[h]$.

In the weak-coupling perturbative expansion, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant g_0. Let us summarize how to perform the weak-coupling perturbative expansion in the $(\lambda\varphi^4)_d$ theory. The Gaussian functional integral $Z_0(h)$ associated with the Euclidean generating functional $Z(h)$ is

$$Z_0(h) = \mathcal{N}\int [d\varphi]\;\exp\left(-\frac{1}{2}\,\varphi\,K\,\varphi + h\varphi\right). \tag{4}$$

We are using a compact notation and the first term in the right-hand side of Eq.(4) is given by

$$\varphi\,K\,\varphi = \int d^dx \int d^dy\,\varphi(x)K(m_0;\,x,y)\varphi(y). \tag{5}$$

The term that couples linearly the field with the external source is

$$h\varphi = \int d^dx\,\varphi(x)h(x). \tag{6}$$

As usual $\mathcal{N}$ is a normalization factor and the symmetric kernel $K(m_0;x,y)$ is defined by

$$K(m_0;x,y) = (-\Delta + m_0^2\,)\,\delta^d(x-y), \tag{7}$$

where Δ denotes the Laplacian in the Euclidean space R^d. As usual, the normalization factor is defined using the condition $Z_0(h)|_{h=0} = 1$. Therefore $\mathcal{N} = \left(\det(-\Delta + m_0^2)\right)^{\frac{1}{2}}$ but, in the following, we are absorbing this normalization factor in the functional measure. It is convenient to introduce the inverse kernel, i.e. the free two-point Schwinger function $G_0(m_0;x-y)$ which satisfies the identity

$$\int d^dz\,G_0(m_0;x-z)K(m_0;z-y) = \delta^d(x-y). \tag{8}$$

Since Eq.(4) is a Gaussian functional integral, simple manipulations, performing only Gaussian integrals, gives

$$\int [d\varphi]\;e^{-S_0+\int d^dx\,h(x)\varphi(x)} = \exp\left[\frac{1}{2}\int d^dx \int d^dy\;h(x)\,G_0^{(2)}(m_0;x-y)h(y)\right]. \tag{9}$$

Therefore, we have an expression for $Z_0(h)$ in terms of the inverse kernel $G_0^{(2)}(m_0;x-y)$, i.e., in terms of the free two-point Schwinger function. This construction is fundamental to perform the weak-coupling perturbative expansion with the Feynman diagramatic representation of the perturbative series. The non-Gaussian contribution is a perturbation with regard to the remaining terms of the action. It is important to point out that the weak-coupling perturbative expansion can be defined in arbitrary gemetries, and classical boundary conditions must be implemented in the two-point Schwinger function. Another way is to restrict the space of functions that appear in the functional integral.

We are interested to study finite size systems, where the translational invariance is broken. In this situation, we are analyzing the perturbative renormalization for the $\frac{\lambda}{4!}\varphi^4$ massless scalar field model, in the two-loop approximation. Therefore, let us assume boundary conditions over the plates for the massless field $\varphi(x)$. For simplicity we are assuming Dirichlet-Dirichlet boundary conditions i.e.,

$$\varphi(\vec{r},z)|_{z=0} = \varphi(\vec{r},z)|_{z=L} = 0\,, \tag{10}$$

for the free field. Since the translational invariance is not preserved, let us use a Fourier expansion of the fields in the following form

$$\varphi(\vec{r},z)=\frac{1}{(2\pi)^{\frac{d-1}{2}}}\int d^{d-1}p\sum_n \phi_n(\vec{p})\,e^{i\vec{p}.\vec{r}}u_n(z), \tag{11}$$

where the set $u_n(z)$ are the orthonormalized eigenfunctions associated to the operator $-\frac{d^2}{dz^2}$, $\left(-\frac{d}{dz^2}u_n(z)=k_n^2u_n(z)\right)$, and $k_n=\frac{n\pi}{L}$, $n=1,2....$ The orthonormal set correspond to the eigenfunctions of the Hermitian operator $-\frac{d^2}{dz^2}$ defined on a finite interval is given by

$$u_n(z)=\sqrt{\frac{2}{L}}\sin(\frac{n\pi z}{L})\;\; n=1,2... \tag{12}$$

These eigenfunctions satisfy the completeness and orthonormality relations, i.e.,

$$\sum_n u_n(z)u_n^*(z')=\delta(z-z') \tag{13}$$

and

$$\int_0^L dz\,u_n(z)u_{n'}^*(z)=\delta_{n,n'}. \tag{14}$$

Since we are interested to perform the weak coupling expansion, let us first write the free two-point Schwinger function. This free two-point Schwinger function can be expressed in the following form

$$G_0^{(2)}(\vec{r},z,z')=\frac{1}{(2\pi)^{d-1}}\int d^{d-1}p\sum_n e^{i\vec{p}.\vec{r}}u_n(z)u_n^*(z')\,G_{0,n}(\vec{p}), \tag{15}$$

where $G_{0,n}(\vec{p})$ is given by

$$G_{0,n}(\vec{p})=(\vec{p}^2+k_n^2+m^2)^{-1}. \tag{16}$$

Next, we will present some properties of the two-point free Schwinger-function in order to understand the behavior of the interacting field theory in the presence of macroscopic structures. Therefore, in order to understand some procedures used in the identification of the divergences in the Schwinger functions that will appear in the next section, let us analyze some properties of the freet two-point Schwinger function. Substituting Eq.(12) and Eq.(16) in Eq.(15) we get that the free propagator $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ can be written as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z_1}{L}\right)\sin\left(\frac{n\pi z_2}{L}\right)\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)}}{\left(\vec{p}^2+(\frac{n\pi}{L})^2+m^2\right)}. \tag{17}$$

The next step is to show that the two-point free Schwinger function can be written in terms of the variables: r_{12}, z_{12}^- and finally z_{12}^+, where $r_{12}=\frac{|\vec{r}_1-\vec{r}_2|}{L}$, $z_{12}^-=\frac{z_1-z_2}{L}$ and $z_{12}^+=\frac{z_1+z_2}{L}$ respectively. Working in the four-dimensional case and also in the massless situation, a straightforward calculation (see appendix A) give us that $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ can be written as:

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{16\pi^2L^2}\sum_{k=-\infty}^{\infty}\left\{\frac{1}{(k-\frac{|z_{12}^-|}{2})^2+(\frac{r_{12}}{2})^2}-\frac{1}{(k-\frac{z_{12}^+}{2})^2+(\frac{r_{12}}{2})^2}\right\}. \tag{18}$$

The former expression for the two-point Schwinger function was obtained also by Lukosz [19] using the image method. Performing the summations in Eq.(18) (see appendix A), it is possible to find a closed expression for $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$. We get

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{\sinh(\pi r_{12})}{16\pi L^2\,r_{12}}\left\{\frac{\sin(\frac{\pi z_1}{L})\sin(\frac{\pi z_2}{L})}{\left[\sinh^2(\frac{\pi r_{12}}{2})+\sin^2(\frac{\pi z_{12}^-}{2})\right]\left[\sinh^2(\frac{\pi r_{12}}{2})+\sin^2(\frac{\pi z_{12}^+}{2})\right]}\right\}. \tag{19}$$

It is not difficult to show that the two-point Schwinger function $G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2)$ satisfies the following properties:
(i) The free two-point Schwinger function is not negative, i.e., $G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2) \geq 0$, for z_1, $z_2 \in [0, L]$ and $\vec{r}_1$, $\vec{r}_2 \in \mathcal{R}^3$, since we are working in an Euclidean space.
(ii) The free two-point Schwinger function is zero when one of its points are evaluated on the boundaries

$$G_0^{(2)}(\vec{r}_1 - \vec{r}_2, 0, z_2) = G_0^{(2)}(\vec{r}_1 - \vec{r}_2, L, z_2) = G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, 0) = G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, L) = 0\,,$$

since we are assuming Dirichlet boundary conditions.
(iii) The free two-point Schwinger function contain the usual bulk divergences, i.e., when $(\vec{r}_1, z_1) = (\vec{r}_2, z_2)$, it is singular. From the Eq.(18) we can identify three singular terms. Splitting the free two-point Schwinger function in the singular and regular terms we have:

$$\begin{aligned} G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2) \;=\; & \frac{1}{4\pi^2 L^2}\left\{\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right\} \\ & + \frac{1}{4\pi^2 L^2}\left\{\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{(2k - |z_{12}^-|)^2 + r_{12}^2} - \sum_{\substack{k=-\infty \\ k\neq 0,1}}^{\infty} \frac{1}{(2k - z_{12}^+)^2 + r_{12}^2}\right\}. \end{aligned} \tag{20}$$

The first term of the right side of the last equation, is singular only when $\vec{r}_1 = \vec{r}_2$ and $z_1 = z_2$. This is the term that carries the usual bulk divergences. The second term is singular only when $z_1 = z_2 = 0$ and $\vec{r}_1 = \vec{r}_2$. The third term is singular only when $z_1 = z_2 = L$ and $\vec{r}_1 = \vec{r}_2$. These two terms mentioned previously, carries surface divergences. Finally the two last terms do not have singularities.
(iv) When $|\vec{r}_1 - \vec{r}_2|/L \gg 1$ the free propagator behaves like

$$G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2) \sim \frac{1}{2\pi L^2}\frac{e^{-\pi r_{12}}}{r_{12}} \sin\left(\frac{\pi z_1}{L}\right) \sin\left(\frac{\pi z_2}{L}\right)\,, \tag{21}$$

which shows an exponential convergence behavior.
(v) The integral of the variable $\{\vec{r}, z\}$ on a neighborhood around $\{\vec{r}\,', z'\}$ of the free propagator is finite, i.e., $\int_R d^3r\, dz\, G_0^{(2)}(\vec{r} - \vec{r}\,', z, z') < \infty$. See Fig.(1).

Property (v) allow us to shows that the external legs of the Feynman diagrams do not create divergences. Let us suppose we have the integral corresponding to some Feynman diagram,

$$\int_R d^3r dz\; G_0^{(2)}(\vec{r} - \vec{r}\,', z, z')\; F(\vec{r}\,', z')\,, \tag{22}$$

where $G_0^{(2)}(\vec{r} - \vec{r}\,', z, z')$ is some external leg and $F(\vec{r}\,', z')$ describes the remainder part of the diagram. Now in order to proceed we have to use the following statement: for two continuous and positives functions $f(\vec{x})$ and $g(\vec{x})$ defined in a finite region R with the exception of the point $\vec{x}_1$ where $f(\vec{x})$ diverges, then the integral $I = \int_R d^dx\, f(\vec{x})g(\vec{x})$ is finite, if and only if $I' = \int_V d^dx\, f(\vec{x})$ is finite on some neighborhoods V of the point $\vec{x}_1$. With the property (v) and the statement before we can see that external legs from the Feynman diagrams do not generate divergences.

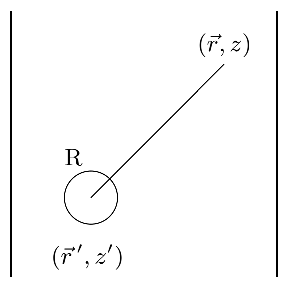

FIG. 1:

III. REGULARIZED TWO AND FOUR-POINT SCHWINGER FUNCTIONS AT ONE-LOOP ORDER

In this section we identify the divergent contribution in the two and four-point Schwinger function at one-loop level. Essentially we use the Eq.(20) in the 1PI diagrams of the Green functions considering their external legs, and the integrations in the coordinate space. We write Eq.(20) as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{4\pi^2L^2}\left[\frac{1}{(z_{12}^-)^2+r_{12}^2}-\frac{1}{(z_{12}^+)^2+r_{12}^2}-\frac{1}{(2-z_{12}^+)^2+r_{12}^2}+h(r_{12},z_1,z_2)\right] \tag{23}$$

where $h(r_{12},z_1,z_2)$ is given by

$$h(r_{12},z_1,z_2)=\frac{1}{4}\sum_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{1}{(k-\frac{|z_{12}^-|}{2})^2+(\frac{r_{12}}{2})^2}-\frac{1}{4}\sum_{\substack{k=-\infty\\k\neq0,1}}^{\infty}\frac{1}{(k-\frac{z_{12}^+}{2})^2+(\frac{r_{12}}{2})^2}. \tag{24}$$

From the property (iii) we see that the three first contributions in the right hand side of Eq.(23) have singularities. Otherwise, the last term is finite in the whole domain where we defined the model. After this briefly introduction, we

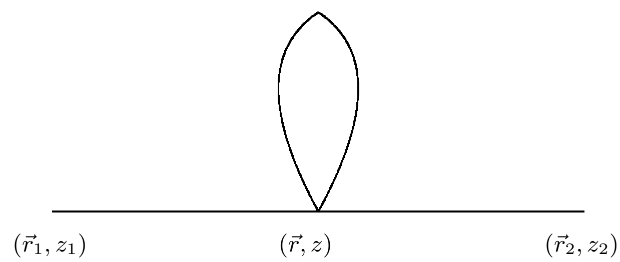

FIG. 2: The two-point function at one-loop level

are able to study the interacting theory. Let us start analyzing the tadpole diagram, displayed in Fig.(2), from which we can write the expression for the one loop two-point Schwinger function $G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$. We have that

$$G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{\lambda}{2}\int d^3r dz\; G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)\;G_0^{(2)}(0,z,z)\;G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)\,. \tag{25}$$

In the following we are generalizing the results obtained by Fosco and Svaiter [14]. Let us begin studying the quantity $G_0^{(2)}(0,z,z)$ that appear in the tadpole defined in Eq.(25). From Eq.(20) we get that $G_0^{(2)}(0,z,z)$ can be written as

$$G_0^{(2)}(0,z,z)=\frac{1}{4\pi^2L^2}\left[A-\frac{1}{(2z/L)^2}-\frac{1}{(2-2z/L)^2}+\sum_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{1}{(2k)^2}-\sum_{\substack{k=-\infty\\k\neq0,1}}^{\infty}\frac{1}{(2k-2z/L)^2}\right]\,, \tag{26}$$

where A is given by

$$\begin{aligned}A &= \lim_{(z_1,\vec{r}_1)\to(z_2,\vec{r}_2)}\frac{L^2}{(z_1-z_2)^2+|\vec{r}_1-\vec{r}_2|^2}\\ &= \lim_{\Lambda\to\infty}\frac{L^2S_4}{8\pi^2}\Lambda^2\,,\end{aligned} \tag{27}$$

$S_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}$ and Λ is an ultraviolet cutoff. In the same way, from Eq.(26) by performing the summations, we get for $G_0^{(2)}(0,z,z)$

$$G_0^{(2)}(0,z,z)=\frac{1}{4\pi^2L^2}\left[A+\frac{\pi^2}{12}-\frac{\pi^2}{4}\frac{1}{\sin^2(\pi z/L)}\right]. \tag{28}$$

Substituting Eq.(27) in Eq.(28) we obtain

$$G_0^{(2)}(0,z,z)=\lim_{\Lambda\to\infty}\frac{S_4}{32\pi^4}\Lambda^2+\frac{1}{48L^2}-\frac{1}{16L^2}\frac{1}{\sin^2(\pi z/L)}. \tag{29}$$

The first term in Eq.(29) is a bulk divergence. Substituting Eq.(29) in Eq.(25) we get

$$\begin{aligned} G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2) &= \lim_{\Lambda\to\infty}\frac{\lambda\,S_4}{64\pi^4}\Lambda^2\int_R d^3r dz\; G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)\;G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z) \\ &+\frac{\lambda}{96L^2}\int_R d^3r dz\; G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)\;G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z) \\ &-\frac{\lambda}{32L^2}\int_R d^3r dz\;\frac{G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)\;G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)}{\sin^2(\pi z/L)}\,. \end{aligned} \tag{30}$$

The first term in the right hand side carries a bulk divergence. The second term is finite. To see this we analyze the integral by sectors. Therefore we have

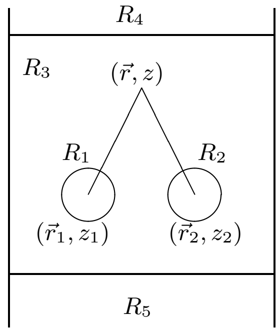

FIG. 3: Regions of integration R_i.

$$\int_R d^3r dz\; G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)\;G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)=\int_{R_1}+\int_{R_2}+\int_{R_3}+\int_{R_4}+\int_{R_5}\,, \tag{31}$$

where each integral are defined in different regions displayed in Fig.3, where the points $(\vec{r}_1,z_1)$ and $(\vec{r}_2,z_2)$ are the centers of the regions R_1 and R_2 respectively. Using the property (v) we have that the integrals on R_1 and R_2 are finite. Since the free propagators $G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)$ and $G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)$ presented in Eq.(31) does not have divergences on R_3 and this region is compact, then the integral on R_3 is finite. The integrals defined in regions R_4 and R_5 also are finite since from the property (iv) the propagator decreases exponentially when one of its points become far from the other. Thus the integral defined by Eq.(31) is finite. Finally we have to study the third integral in the right hand side of Eq.(30). Note that the term $\frac{1}{\sin^2(\pi z/L)}$ diverges when z is evaluated on the boundaries. Nevertheless this integral is convergent, because the products of $G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)$ and $G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)$ take away the divergence. Using Eq.(19), we have that third integral in the right hand side of Eq.(30) is finite. Therefore the one-loop two point Schwinger function only has bulk divergence.

Our next step is to analyze the four-point Schwinger function in the one-loop level. Since the free propagator only has singularities when its two points are equal or also when this two points joined are evaluated at the boundaries, we continue our analysis of the integrals only in the domains where the two external points of the free propagators take the same values. The complete four-point function at one-loop level is given by

$$\begin{aligned} G_1^{(4)}(\vec{r}_1,z_1,\vec{r}_2,z_2,\vec{r}_3,z_3,\vec{r}_4,z_4) &= \frac{\lambda^2}{2}\int d^{d-1}r\int d^{d-1}r'\int_0^L dz\int_0^L dz'\;G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)\,\times \\ &\left[G_0^{(2)}(\vec{r}-\vec{r}\,',z,z')\right]^2\;G_0^{(2)}(\vec{r}_3-\vec{r}\,',z_3,z')\;G_0^{(2)}(\vec{r}_4-\vec{r}\,',z_4,z')\,. \end{aligned} \tag{32}$$

For simplicity, in Fig.(5) we define three different regions between the boundaries. The first one, R_1 is concerned when $\{\vec{r}\,',z'\}$ is close to $\{\vec{r},z\}$. In this region the contribution coming from $\left[G_0^{(2)}(\vec{r}-\vec{r}\,',z,z')\right]^2$ is singular. Nevertheless, we still have to analyze if this divergent behavior will appear in the integral defined by Eq.(32). We will show that the singularities will appear only as bulk divergences. In the region R_2 ($z,z'\to 0$ and $\vec{r}\,'\to\vec{r}$) the term $\left[G_0^{(2)}(\vec{r}-\vec{r}\,',z,z')\right]^2$ is also divergent. As we will see, this divergent behavior disappears when we compute the

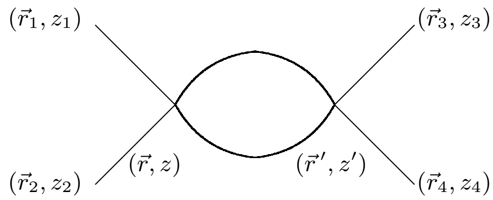

FIG. 4: The four point function at one loop.

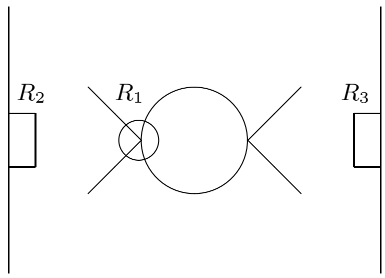

FIG. 5: Regions of integration for the four point function

complete four-point function at one-loop order, defined by Eq.(32). In the region R_3 ($z, z' \to L$ and $\vec{r}\,' \to \vec{r}$) the situation is identical as in the region R_2. Using the same argument that we used before to analyze the convergence of the integral defined by Eq.(22), we can study the convergence of the integral defined by Eq.(32) with the amputated external legs. Therefore we have to study Eq.(32) with the external legs amputated. Therefore we have to study the quantity $\int d^3r dz d^3r' dz' \, \left[G_0^{(2)}(\vec{r}\,' - \vec{r}, z', z)\right]^2$. Substituting Eq.(23) in the former equation we get:

$$\int d^3r dz d^3r' dz' \, \left[G_0^{(2)}(\vec{r}\,' - \vec{r}, z', z)\right]^2 = \frac{1}{(4\pi^2 L^2)^2}\Big(I_1 + I_2 + I_3 + I_4 + I_5\Big) + \text{ finite part} \tag{33}$$

where the integrals I_i, $i = 1, 2, ..$ are given by

$$I_1 = \int d^3r dz d^3r' dz' \, \frac{1}{\left[(z_{12}^-)^2 + r_{12}^2\right]^2}\,, \tag{34}$$

$$I_2 = \int d^3r dz d^3r' dz' \, \left[\frac{1}{\left[(z_{12}^+)^2 + r_{12}^2\right]^2} + \frac{1}{\left[(2 - z_{12}^+)^2 + r_{12}^2\right]^2}\right]\,, \tag{35}$$

$$I_3 = \int d^3r dz d^3r' dz' \, \left[-\frac{2}{\left[(z_{12}^-)^2 + r_{12}^2\right]\left[(z_{12}^+)^2 + r_{12}^2\right]} - \frac{2}{\left[(z_{12}^-)^2 + r_{12}^2\right]\left[(2 - z_{12}^+)^2 + r_{12}^2\right]}\right]\,, \tag{36}$$

$$I_4 = \int d^3r dz d^3r' dz' \, \frac{2}{\left[(z_{12}^+)^2 + r_{12}^2\right]\left[(2 - z_{12}^+)^2 + r_{12}^2\right]}\,, \tag{37}$$

$$I_5 = \int d^3r dz d^3r' dz' \, \left[\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right] h(r_{12}, z_1, z_2)\,. \tag{38}$$

Let us investigate each term of Eq.(33). The integral I_1 must be analyzed only in the region R_1. For this purpose we need an auxiliary result. We can prove that a continuous and positive function $f(x)$ which does not have singularities except for $x = 0$, and $M = \int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^d x\, f(w^2)$ where $w^2 = |\vec{w}|^2$, then there exist ϵ' such that $M = S_d \int_0^{\epsilon'} dw\, w^{d-1} f(w^2)$ where $\epsilon < \epsilon' < \sqrt{d}\,\epsilon$. Then we get

$$I_I = \int_{R_1} d^3r' dz' \, \frac{1}{[(z_{12}^-)^2 + r_{12}^2]^2} = \int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r' \int_{z-\epsilon}^{z+\epsilon} dz' \, \frac{1}{[(z - z')^2 + |\vec{r} - \vec{r}\,'|^2\,]^2}$$

$$= \int_{-\vec{\epsilon}}^{\vec{\epsilon}} \frac{d^4w}{w^4} = S_4 \int_0^{\epsilon\,'} dw\, \frac{w^3}{w^4} = S_4 \ln w\, |_0^{\epsilon'} \;. \qquad (39)$$

Therefore I_1 contributes with a bulk divergence of the type as the one that appears in the theory without boundaries. In the usual renormalization procedure, the contribution coming from I_1 can be eliminated by the usual counterterms. Concerning the contribution coming from I_2 we have that the first term $1/[(z_{12}^{+})^2 + r_{12}^2]^2$ is not singular in the region R_1. In the region R_2, using the same auxiliary result that we used before, we can obtain an upper bound to the contribution coming from this term. We get

$$\int_{R_2} d^3r dz d^3r' dz'\, \frac{1}{[(z+z')^2 + |\vec{r}-\vec{r}\,'|^2\,]^2} < \int d^3r \int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r' \int_0^{\epsilon}\int_0^{\epsilon} dz dz'\, \frac{1}{[z^2 + z'\,^2 + |\vec{r}-\vec{r}\,'|^2\,]^2}$$

$$< \frac{1}{4}\int d^3r \int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^5w\, \frac{1}{w^2} = \frac{1}{12} S_5 \epsilon'\,^3 \int_{R'} d^3r\,. \qquad (40)$$

Since the region $R\,' \subset R_2$ is finite this integral is convergent. Next, let us analyze the term $1/[(2 - z_{12}^{+})^2 + r_{12}^2]^2$ of I_2 in the region R_3. Since the behavior of the field in each plates (for $z = 0$ and $z = L$) are the same, then the analysis follows the same lines as previous ones and therefore this contribution is also finite. To study I_3, we consider first the term $2/[(z_{12}^{-})^2 + r_{12}^2][(z_{12}^{+})^2 + r_{12}^2]$. This expression must be studied in the regions R_1 and R_2 respectively. In R_1 we can see that the convergence of

$$\int_{R_1} d^3r dz d^3r' dz'\, \frac{1}{[(z-z')^2 + |\vec{r}-\vec{r}\,'|^2]\underbrace{[(z+z')^2 + |\vec{r}-\vec{r}\,'|^2]}_{\text{finite in } R_1}}\,, \qquad (41)$$

depend of the convergence of

$$\int_{R_1} d^3r dz d^3r' dz'\, \frac{1}{(z-z')^2 + |\vec{r}-\vec{r}\,'|^2}. \qquad (42)$$

¿From above arguments we have that Eq.(42) can be written as

$$\int_{-\vec{\epsilon}}^{\vec{\epsilon}} \frac{d^4w}{w^2} = S_4 \int_0^{\epsilon\,'} dw\, w = \frac{S_4 \epsilon'\,^2}{2}\,, \qquad (43)$$

thus Eq.(42) gives a finite contribution. Now we consider the first term of I_3 in the region R_2. For this purpose we will use the following property. Let us take a continuous and positive function $f(x)$ which does not have singularities except for $x = 0$, and $N = \int_0^{\vec{\epsilon}}\int_0^{\vec{\epsilon}} d^l y\, d^m z\, f(y^2 + z^2)$ then there exist ϵ' in such way $N = \frac{S_{l+m+2}}{S_{l+1}S_{m+1}} \int_0^{\epsilon'} dw\, w^{l+m+1} f(w^2)$ where $\epsilon' > 0$. Using this property, we have for the first term of Eq.(35), in the region R_2, that

$$\int d^3r \int_0^{\epsilon} dz \int_z^{z+\epsilon} dz' \int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r' \frac{1}{[(z-z')^2 + |\vec{r}-\vec{r}\,'|^2][(z+z')^2 + |\vec{r}-\vec{r}\,'|^2]}$$

$$< \int d^3r \int_0^{\epsilon} dz \int_0^{\epsilon} du \int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^3v \frac{1}{(u^2+v^2)(z^2+u^2+v^2)} < \frac{1}{4} S_4 \int d^3r \int_0^{\epsilon} dz \int_0^{\epsilon'} dw\, \frac{w}{(z^2+w^2)} = \frac{S_3 S_4}{2 S_1 S_2} \epsilon''. \quad (44)$$

Therefore the first term of I_3 is also finite in R_2. The second term $2/[((z_{12}^{-})^2 + r_{12}^2)((2 - z_{12}^{+})^2 + r_{12}^2)]$ in I_3 must be analyzed also in the regions R_1 and R_3. This analysis follows the same lines as the last case, therefore the contribution coming from this term is also finite.

We have now to study the term I_4. Note that $2/[((z_{12}^{+})^2 + r_{12}^2)((2 - z_{12}^{+})^2 + r_{12}^2)]$ must be analyzed in the regions R_2 and R_3 respectively. Let us start with the region R_2. Using previous arguments we have that the convergence of I_4 depends of the convergence of the following expression

$$\int_0^{\epsilon}\int_0^{\epsilon} dz dz' \int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r' \frac{1}{z^2 + z'^2 + |\vec{r}-\vec{r}\,'|^2} = \frac{S_5}{4}\int_0^{\epsilon'} dw\, \frac{w^4}{w^2} = \frac{S_5}{12}\epsilon'\,^3\,, \qquad (45)$$

which is finite. In the region R_3 our analysis follow the same lines as in the region R_2, thus the integral in the region R_3 is also finite.

Using the same argument that we used before, is not difficult to show that the contribution coming from I_5 is also finite. We conclude that the integral given by Eq.(32) only have bulk divergences. In this way we can conclude that at one-loop level the bulk counterterms are sufficient to render the complete connected Schwinger functions finite. In the next section we will identify the divergent contribution in the connected two-point Schwinger functions at the two-loop order.

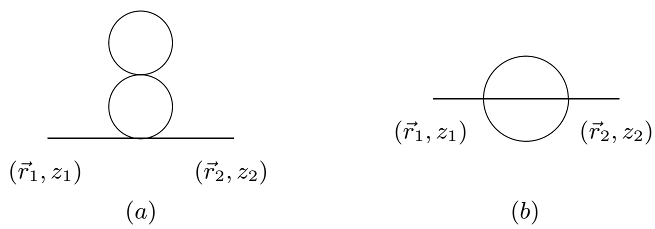

FIG. 6: Two point Schwinger functions at two-loop level

IV. THE DIVERGENCES IN THE TWO-POINT SCHWINGER FUNCTIONS AT TWO-LOOP LEVEL

In this section we will generalize some results obtained by Fosco and Svaiter [14] and also by Caicedo and Svaiter [15]. We will identify the divergent contribution in the connected two-point Schwinger functions at the two-loop order. The diagrams that we are interested to analyze are displayed in Fig.(6). The expression that corresponds to Fig.(6-a) is given by:

$$\frac{\lambda^2}{4}\int d^3r'dz'd^3rdz\, G_0^{(2)}(\vec{r}_1-\vec{r}\,',z_1,z\,')\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2 G_0^{(2)}(0,z,z)\, G_0^{(2)}(\vec{r}_2-\vec{r}\,',z_2,z\,'). \tag{46}$$

Since the external legs in Eq.(46) does not contribute to generate divergences, let us consider only the following integral

$$\int d^3rdz\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2 G_0^{(2)}(0,z,z). \tag{47}$$

Replacing Eq.(29) in Eq.(47) we get

$$\begin{aligned}\int d^3rdz\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2 G_0^{(2)}(0,z,z) \;=\; &\lim_{\Lambda\to\infty}\frac{S_4}{32\pi^4}\Lambda^2\int d^3rdz\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2\\ &+\frac{1}{48L^2}\int d^3rdz\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2\\ &-\frac{1}{16L^2}\int d^3rdz\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2\frac{1}{\sin^2(\pi z/L)}.\end{aligned} \tag{48}$$

The first term and the second one in Eq.(48) can be renormalized introducing only bulk counterterms. The most interesting behavior appears in the last term of this equation. Note that this unrenormalized quantity contains only bulk divercences, since the contribution coming from $\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2$, cancels the surface divergent behavior generated by the $\frac{1}{\sin^2(\pi z/L)}$ term. Nevertheless, after the introduction of a bulk counterterm to render the contribution $\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^2$ finite between the plates, surface divergences appear. Thus this surface divergences must be renormalized. After the introduction of surface and bulk counterterms, the finite contribution coming from the last term of Eq.(48), up to a finite renormalization constant, is given by

$$\frac{1}{16L^2}\int d^3rdz\left[\left(G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right)^2-\frac{1}{(4\pi^2L^2)^2}\frac{1}{[(z_{12}^-)^2+r_{12}^2]^2}\right]\left[\frac{1}{\sin^2(\pi z/L)}-\frac{L^2}{(\pi z)^2}-\frac{L^2}{\pi^2(L-z)^2}\right]. \tag{49}$$

Therefore this term contains an overlapping between bulk and surface couterterms.

We still have to analyze the sunset diagram. The expression corresponding to the Fig.(6-b) is given by

$$\frac{\lambda^2}{6}\int d^3r'dz'd^3rdz\, G_0^{(2)}(\vec{r}_1-\vec{r}\,',z_1,z\,')\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z\,',z)\right]^3 G_0^{(2)}(\vec{r}_2-\vec{r}\,',z_2,z\,'). \tag{50}$$

Again, the external legs does not contribute to generate divergences, and therefore let us study the amputated diagram, i.e., without external legs. We have

$$\int d^3rdzd^3r'dz'\left[G_0^{(2)}(\vec{r}\,'-\vec{r},z',z)\right]^3=\frac{1}{(4\pi^2L^2)^3}\Big(I_1+I_2+...+I_{12}\Big)+\text{ finite part}, \tag{51}$$

where

$$I_1 = \int d^3r dz d^3r' dz' \; \frac{1}{[(z_{12}^-)^2 + r_{12}^2]^3}, \tag{52}$$

$$I_2 = \int d^3r dz d^3r' dz' \; \frac{1}{[(z_{12}^+)^2 + r_{12}^2]^3}, \tag{53}$$

$$I_3 = \int d^3r dz d^3r' dz' \; \frac{1}{[(2 - z_{12}^+)^2 + r_{12}^2]^3}, \tag{54}$$

$$I_4 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^-)^2 + r_{12}^2]^2[(z_{12}^+)^2 + r_{12}^2]}, \tag{55}$$

$$I_5 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^-)^2 + r_{12}^2]^2[(2 - z_{12}^+)^2 + r_{12}^2]}, \tag{56}$$

$$I_6 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2]^2}, \tag{57}$$

$$I_7 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^-)^2 + r_{12}^2][(2 - z_{12}^+)^2 + r_{12}^2]^2}, \tag{58}$$

$$I_8 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^+)^2 + r_{12}^2]^2[(2 - z_{12}^+)^2 + r_{12}^2]}, \tag{59}$$

$$I_9 = \int d^3r dz d^3r' dz' \; \frac{2}{[(z_{12}^+)^2 + r_{12}^2][(2 - z_{12}^+)^2 + r_{12}^2]^2}, \tag{60}$$

$$I_{10} = \int d^3r dz d^3r' dz' \; \frac{6}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2][(2 - z_{12}^+)^2 + r_{12}^2]}, \tag{61}$$

$$I_{11} = \int d^3r dz d^3r' dz' \; \left[\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right]^2 h(r_{12}, z_1, z_2), \tag{62}$$

$$I_{12} = \int d^3r dz d^3r' dz' \; \left[\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right] h^2(r_{12}, z_1, z_2). \tag{63}$$

Let us analyze each contributions coming from each terms of Eq.(51). The first integral I_1 given by Eq.(52) is divergent in R_1. In general we can show that

$$\int_{R_1} d^3r' dz' \frac{1}{\left[(z_{12}^-)^2 + r_{12}^2\right]^n} = \begin{cases} \text{finite} & n < 2 \\ \infty & n \geq 2 \end{cases} \tag{64}$$

Using the above result we can see that the integrals I_3, I_4 and the first integral of I_{11} are divergent. These integrals contain bulk divergences which must be removed introducing bulk counterterms. Next let us analyze the contribution coming from the integral I_2 in the region R_2. Using previous arguments and considering the external legs we get

$$\int_{R_2} d^3r dz d^3r' dz' \; \frac{zz'}{[(z_{12}^+)^2 + r_{12}^2]^3} < \int d^3r \int_{-\vec{\epsilon}}^{\vec{\epsilon}} \int_0^{\epsilon} \int_0^{\epsilon} d^3w dz dz' \; \frac{zz'}{(z^2 + z'^{\,2} + w^2)^3} < \frac{S_7}{S_2^2}\epsilon' \int d^3r. \tag{65}$$

Therefore this term gives a finite contribution to the Eq.(50). The contribution from the integral I_6 to the integral must be studied in region R_2. In this case we have to consider the external legs, and the property: let us take a function $f(x,y)$ positive which does not have singularities except for $(x,y) = (0,0)$, $I = \int_0^\epsilon \int_0^\epsilon dx\, dy \; f(x,y)$ then, $I < \int_0^\epsilon dx \int_x^{x+\epsilon} dy \; f(x,y) + \int_0^\epsilon dy \int_y^{y+\epsilon} dy \; f(x,y)$, we get

$$\int_{R_2} d^3r dz d^3r' dz' \; \frac{zz'}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2]^2} < 2 \int d^3r \int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^3w \int_0^{\epsilon} dz \int_0^{\epsilon} du \; \frac{z(z+u)}{(u^2 + w^2)(u^2 + z^2 + w^2)^2}. \tag{66}$$

¿From above arguments we have that the contribution from the integral I_6 is smaller than

$$S_4 \int d^3r \int_0^{\epsilon} dz \int_0^{\epsilon'} ds \; \frac{z^2 s}{(s^2 + z^2)^2} + 2S_3 \int d^3r \int_0^{\epsilon} dz \int_0^{\epsilon} du \int_0^{\epsilon'} ds \; \frac{zus^2}{(u^2 + s^2)(u^2 + s^2 + z^2)^2}$$
$$< \frac{S_4 S_5}{S_2 S_3}\epsilon'' \int d^3r + 2\frac{(S_5)^2}{(S_2)^2 S_3} \int d^3r \int_0^{\epsilon'''} dw \; \frac{w^4}{w^4} < \left(\frac{S_4 S_5}{S_2 S_3}\epsilon'' + 2\frac{(S_5)^2}{(S_2)^2 S_3}\epsilon'''\right) \int d^3r. \tag{67}$$

We conclude that the integral I_6 is finite. Also integrating the contribution coming from the term I_8 on R_2 we get

$$\int_{R_2} d^3r dz d^3r' dz' \, \frac{1}{[(z_{12}^+)^2 + r_{12}^2]^2 \underbrace{[(2 - z_{12}^+)^2 + r_{12}^2]}_{\text{finite}}}. \tag{68}$$

Using the fact that the integral $\int_{R_2} d^3r dz d^3r' dz' \, \frac{1}{[(z_{12}^+)^2 + r_{12}^2]^2}$ is finite, we have that this integral also is convergent in R_2. The contribution from the term I_{10} on R_2 is given by

$$\int_{R_2} d^3r dz d^3r' dz' \, \frac{1}{[(z_{12}^-)^2 + r_{12}^2]\,[(z_{12}^+)^2 + r_{12}^2]\,\underbrace{[(2 - z_{12}^+)^2 + r_{12}^2]}_{\text{finite}}}. \tag{69}$$

Since the integral $\int_{R_2} d^3r dz d^3r' dz' \, \frac{1}{[(z_{12}^-)^2 + r_{12}^2]\,[(z_{12}^+)^2 + r_{12}^2]}$ is finite, then the integral defined by Eq.(69) is convergent in R_2. The contribution coming from the terms I_{11} contain only a bulk divergence. Otherwise, the contributions coming from the terms I_{12} is finite. We conclude that we need only bulk counterterms to render the integral defined by Eq.(50) finite. The same analysis can be done for the four-point Schwinger function in the two-loop approximation. We obtained that only bulk divergences appear in the full four-point function.

V. CONCLUSIONS

In this paper we are interested to show how to implement the renormalization procedure in systems where the translational invariance is broken by the presence of macroscopic structures. For the sake of simplicity we are studying the self-interacting massless scalar field theory in a four-dimensional Euclidean space. We impose that one coordinate is defined in a compact domain, introducing two parallel mirrors where we are assuming Dirichlet-Dirichlet boundaries conditions. Note that although there are some similarities with the finite temperature field theory using the Matsubara formalism, in thermal systems appears only bulk divergences, as for example in the case of system where we assume periodic boundary conditions. In non-translational invariant systems, in general to render the theory finite it is necessary to introduce surface countertems.

In this work we generalize some results obtained by Fosco and Svaiter [14] and also by Caicedo and Svaiter [15]. We identify the divergences of the Schwinger functions in the massless self-interacting scalar field theory up to the two-loop approximation. First, analyzing the full two and four-point Schwinger functions at the one-loop level, we shown that the bulk counterterms are sufficient to render the theory finite. Second, at the two-loop level, we have to introduce surface counterterms in the bare lagrangian in order to make finite the full two and also four-point Schwinger functions. The most interesting behavior appears in the "double scoop" diagram given by Eq.(46). The amputated diagram is given by Eq.(47) and we are interested in the last term of Eq.(48). This unrenormalized quantity contains only bulk divercences. Nevertheless, after the introduction of a bulk counterterm to render the contribution finite between the plates, surface divergences appear. Thus this surface divergences must be renormalized. Therefore this term contains an overlapping between bulk and surface couterterms. This procedure can be generalized to the n-loop level. The inclusion of the counterterm in the lagrangian up two-loop level with the full renormalized action and the general algorithm to identify the surface and bulk conterterms in the n-loop level will be left to a future work.

VI. ACKNOWLEDGMENT

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APPENDIX A

In this appendix we will derive an useful representation for the free two-point Schwinger function. Starting from Eq.(17), we have that $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ is given by

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{2}{(2\pi)^{d-1}L}\sum_{n=1}^{\infty}\int d^{d-1}p\,\sin\Big(\frac{n\pi z_1}{L}\Big)\sin\Big(\frac{n\pi z_2}{L}\Big)\frac{e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)}}{[\vec{p}^2+(\frac{n\pi}{L})^2+m^2]}\,. \tag{A.1}$$

Using the variables $u=\frac{z_1-z_2}{L}$ and $v=\frac{z_1+z_2}{L}$ defined respectively in the region $u\in[-1,1]$ and $v\in[0,2]$, and also making use of a trigonometric identity and performing the sum that appear in Eq.(A.1) [21] we obtain

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{2}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)}}{(\vec{p}^2+m^2)^{\frac{1}{2}}}\left[\frac{\cosh\Big(L(1-|u|)(\vec{p}^2+m^2)^{\frac{1}{2}}\Big)}{\sinh\Big(L(\vec{p}^2+m^2)^{\frac{1}{2}}\Big)}-\frac{\cosh\Big(L(1-v)(\vec{p}^2+m^2)^{\frac{1}{2}}\Big)}{\sinh\Big(L(\vec{p}^2+m^2)^{\frac{1}{2}}\Big)}\right]. \tag{A.2}$$

Taking $m=0$, $d=4$, and integrating the angular part, it is possible to show that $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ can be written as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{-i}{2(2\pi)^2r'L^2}\int_0^{\infty}dx\left(e^{ixr'}-e^{-ixr'}\right)\left[\frac{\cosh\left((1-|u|)\,x\right)}{\sinh x}-\frac{\cosh\left((1-v)\,x\right)}{\sinh x}\right], \tag{A.3}$$

where the variable r' is defined by: $r'\equiv\frac{|\vec{r}_1-\vec{r}_2|}{L}$. Making use of the following integral representation of the product between the Gamma function and the Riemann zeta function [21]

$$\int_0^{\infty}dx\,\frac{x^{z-1}e^{-\beta x}}{e^{px}-1}=\frac{\Gamma(z)}{p^z}\zeta(z,\frac{\beta}{p}+1), \tag{A.4}$$

where $Re(z)>1$, $Re(\frac{\beta}{p})>-1$ and the Riemann zeta function $\zeta(z,q)$ is defined by

$$\zeta(z,q)=\sum_{k=0}^{\infty}\frac{1}{(k+q)^z},\quad q\neq 0,-1,-2.. \tag{A.5}$$

then, it is possible to write $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{16\pi^2L^2}\left[\sum_{k=-\infty}^{\infty}\frac{1}{(k-\frac{|u|}{2})^2+(\frac{r'}{2})^2}-\sum_{k=-\infty}^{\infty}\frac{1}{(k-\frac{v}{2})^2+(\frac{r'}{2})^2}\right]. \tag{A.6}$$

Finally, using the following identity:

$$\sum_{k=-\infty}^{\infty}\frac{1}{(k-z)^2+r^2}=\frac{\pi}{2r}\frac{\sinh(2\pi r)}{\sinh^2(\pi r)+\sin^2(\pi z)}, \tag{A.7}$$

we obtain the representation for the two-point Schwinger function that we need to proceed in our analysis. Using the above equation in Eq. (A.6) we get

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{\sinh(\pi r')}{16\pi L^2r'}\left[\frac{\sin(\frac{\pi z_1}{L})\sin(\frac{\pi z_2}{L})}{\left[\sinh^2(\frac{\pi r'}{2})+\sin^2(\frac{\pi u}{2})\right]\left[\sinh^2(\frac{\pi r'}{2})+\sin^2(\frac{\pi v}{2})\right]}\right]. \tag{A.8}$$

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