CBPF-NF-024/04

Gravitational Collapse, Dissipation and Gravitational Waves*

N. O. Santos
Laboratório Nacional de Computação Científica, 25651-070 Petrópolis RJ, Brazil.
Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro RJ, Brazil.

Abstract

This work is based on the study of matching conditions for a collapsing anisotropic cylindrical perfect fluid, in which I proved that its radial pressure is non null on the surface of the cylinder and proportional to the gravitational radiation produced by the collapsing fluid. This result resembles the one that arises from the radiation - though non-gravitational - in the spherically symmetric collapsing dissipative fluid.

*This paper is dedicated to Professor José Plínio Batista on the occasion of his 70th birthday.

1 Introduction

Spherical gravitational collapse of a dissipative fluid produces outgoing radiation which can be modeled with Vaidya spacetime. The pressure on the surface of the collapsing sphere is non null due to the continuity of the radial flux of momentum [1].

It is generally accepted that gravitational waves carry energy, so a source radiating them should lose mass. If one compares this physical situation to the spherical dissipative collapse - non-gravitationally radiating - which radiates a null fluid, one might expect that gravitational radiation would exert a non null pressure on its collapsing surface as well.

To analyze this problem, I studied the collapse of a cylindrical anisotropic perfect fluid source. In order to do that I firstly presented the scheme for studying the spherical dissipative collapse [1, 2, 3] which is included in the next two sections. In sections 4 and 5, I considered the cylindrical perfect fluid collapse following similar steps [4]. Finally, this article is concluded with a short conclusion.

2 Collapsing dissipative fluid sphere

To start with I assumed a sphere of collapsing perfect fluid with heat flow. Its spherical surface Σ has centre 0 and it is filled with radially moving perfect fluid conducting heat flow, so it has energy momentum tensor

$$T^{-}_{\alpha\beta} = (\mu + P)\, w_\alpha w_\beta + P g_{\alpha\beta} + q_\alpha w_\beta + w_\alpha q_\beta, \tag{1}$$

where μ and P are the proper density and pressure of the fluid, w_α its unit four-velocity, q_α the heat conduction satisfying $q_\alpha w^\alpha = 0$ and $g_{\alpha\beta}$ is the metric tensor of spacetime.

I chose comoving coordinates within Σ and imposed shear-free fluid motion which allows the metric to be written in the form

$$ds^2_- = -A^2 dt^2 + B^2 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{2}$$

where A and B are only functions of t and r. I numbered the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$ and then I had the four-velocity given by

$$w_\alpha = -A\delta^0_\alpha, \tag{3}$$

and the heat flows radially,

$$q^\alpha = q\delta^\alpha_1, \tag{4}$$

where q is a function of t and r.

The collapsing fluid lies within a spherical surface Σ and it must be matched to a suitable exterior. If heat leaves the fluid across Σ, the exterior will not be vacuum, but the outgoing Vaidya spacetime which models the radiation and has metric

$$ds^2_+ = -\left[1 - \frac{2m(v)}{\rho}\right] dv^2 - 2dvd\rho + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where $m(v)$ is the total mass inside Σ and it is a function of the retarded time v. In (5) ρ is a radial coordinate given in a non-comoving frame. Its energy momentum tensor $T^{+}_{\alpha\beta}$ is given by

$$\kappa T^{+}_{\alpha\beta} = -\frac{2}{\rho^2}\frac{dm}{dv}\delta^0_\alpha\delta^0_\beta. \tag{6}$$

3 Junction conditions for the collapsing dissipative fluid sphere

In accordance with Darmois [5] junction conditions, I supposed that the first fundamental form which Σ inherits from the interior metric (2) must be the same as the one it inherits from the exterior metric (5); and similarly, the inherited second fundamental form must be the same. The conditions are necessary and sufficient for a smooth matching without a surface layer.

The equations of Σ may be written

$$f_- = r - r_\Sigma = 0, \tag{7}$$

$$f_+ = \rho - \rho_\Sigma(v) = 0, \tag{8}$$

where f_- refers to the spacetime interior of Σ and f_+ to the spacetime exterior, and r_Σ is a constant because Σ is a comoving surface forming the boundary of the fluid. I took the coordinates on Σ as $\xi^0 = \tau$, $\xi^2 = \theta$ and $\xi^3 = \phi$.

The conditions on the interior and exterior metrics imposed by the continuity of the first fundamental forms on Σ produced

$$\left(1 - \frac{2m}{\rho} + 2\frac{d\rho}{dv}\right)^{1/2} dv \overset{\Sigma}{=} Adt \overset{\Sigma}{=} d\tau, \tag{9}$$

$$\rho \overset{\Sigma}{=} Br, \tag{10}$$

where $\overset{\Sigma}{=}$ means that both sides of the equation are evaluated on Σ and I assumed

$$1 - \frac{2m}{\rho_\Sigma} + 2\frac{d\rho_\Sigma}{dv} > 0, \tag{11}$$

so that v is a timelike coordinate.

The second fundamental form of Σ is

$$K_{ab}d\xi^a d\xi^b, \quad a, b = 0, 2, 3, \tag{12}$$

where K_{ab} is the extrinsic curvature given on the two sides by

$$K^{\mp}_{ab} = -n^{\mp}_\alpha\left(\frac{\partial^2 x^\alpha}{\partial\xi^a\partial\xi^b} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial\xi^a}\frac{\partial x^\gamma}{\partial\xi^b}\right). \tag{13}$$

The Christoffel symbols are to be calculated from the appropriate exterior or interior metric, (2) or (5), $n^{\mp}_\alpha$ are the outward unit normals to Σ in $f_\mp$, and x^α refers to the equation of Σ in $f_{\mp}$, namely (7) or (8). The non zero $K^{\mp}_{ab}$ are as follows

$$K^{-}_{00} \overset{\Sigma}{=} -\frac{A_{,r}}{AB}, \tag{14}$$

$$K^{-}_{22} \overset{\Sigma}{=} \frac{1}{\sin^2\theta} K^{-}_{33} \overset{\Sigma}{=} r(Br)_{,r}, \tag{15}$$

$$K^{+}_{00} \overset{\Sigma}{=} \dot{\rho}\ddot{v} - \dot{v}\ddot{\rho} - \frac{3m}{\rho^2}\dot{\rho}\dot{v}^2 + \frac{\dot{v}^3}{\rho}\frac{dm}{dv} - \frac{m}{\rho^3}(\rho - 2m)\dot{v}^3, \tag{16}$$

$$K^{+}_{22} \overset{\Sigma}{=} \frac{1}{\sin^2\theta} K^{+}_{33} \overset{\Sigma}{=} (\rho - 2m)\dot{v} + \rho\dot{\rho}, \tag{17}$$

where an overdot means $d/d\tau$.

The complete junction conditions consist of (9) and (10) together with the continuity of $K^{\mp}_{ab}$, namely

$$K^{-}_{00} = K^{+}_{00}, \tag{18}$$

$$K^{-}_{22} = K^{+}_{22}, \tag{19}$$

where the $K^{\mp}_{ab}$ are given by (14)-(17). ¿From (18) and (19) I found

$$m(v) \overset{\Sigma}{=} \frac{Br^3 B^2_{,t}}{2A^2} - \frac{r^3 B^2_{,r}}{2B} - r^2 B_{,r}, \tag{20}$$

which is the total gravitational mass inside Σ, and

$$P \overset{\Sigma}{=} qB. \tag{21}$$

Equation (21) shows that, if there is heat conduction in the spherically symmetric motion of perfect fluid, the pressure on the surface of the sphere does not vanish unless $q \overset{\Sigma}{=} 0$. If $q \overset{\Sigma}{=} 0$, there will be no dissipation of heat from the sphere and the exterior spacetime will be that of Schwarzschild, not Vaidya.

A physical interpretation of (21) can be given as follows:

Consider the radial flux of momentum on both sides of Σ, given by

$$F^{\mp} = e^{\mp\alpha} n^{\mp\beta} T^{\mp}_{\alpha\beta}, \tag{22}$$

where $e^{\mp\alpha}$ is a unit tangent vector in the τ direction of Σ, which implies in

$$F^{-} \overset{\Sigma}{=} -qB, \tag{23}$$

$$\kappa F^{+} \overset{\Sigma}{=} \frac{2}{\rho^2}\frac{dm}{dv}\dot{v}^2. \tag{24}$$

¿From the matching conditions and the field equations I could obtain

$$\kappa P \overset{\Sigma}{=} -\frac{2}{\rho^2}\frac{dm}{dv}\dot{v}^2, \tag{25}$$

so it is clear from (23)-(25) that (21) is equivalent to $F^- = F^+$, that is, to the continuity of the radial flux of momentum across Σ.

It is believed that gravitational waves carry energy, so a source radiating them would lose mass. So I suggested the following problem:

If we considered a collapsing fluid that might produce gravitational waves, would it produce a similar result to (21)? In other words, would the gravitational radiation exert a non null pressure on the collapsing surface? To try to answer this question I studied, in the next sections, the cylindrical gravitational collapse of a perfect fluid, using a scheme similar to the one presented above.

4 Collapsing perfect fluid cylinder

Some symbols used in the previous sections are repeated here and their meaning should not be mixed up.

I considered a collapsing cylinder filled with anisotropic non-dissipative fluid bounded by a cylindrical surface Σ and with energy momentum tensor given by

$$T^-_{\alpha\beta} = (\mu + P_r)V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_\phi - P_r)K_\alpha K_\beta + (P_z - P_r)S_\alpha S_\beta, \tag{26}$$

where μ is the energy density, P_r, P_z and P_ϕ are the principal stresses and V_α, K_α and S_α are vectors satisfying

$$V^\alpha V_\alpha = -1, \;\; K^\alpha K_\alpha = S^\alpha S_\alpha = 1, \;\; V^\alpha K_\alpha = V^\alpha S_\alpha = K^\alpha S_\alpha = 0. \tag{27}$$

I then assumed the general time dependent cylindrically symmetric metric

$$ds^2_- = -A^2(dt^2 - dr^2) + B^2 dz^2 + C^2 d\phi^2, \tag{28}$$

where A, B and C are functions of t and r. To represent cylindrical symmetry, I imposed the following ranges on the coordinates

$$-\infty \leq t \leq \infty, \;\; 0 \leq r, \;\; -\infty < z < \infty, \;\; 0 \leq \phi \leq 2\pi. \tag{29}$$

I numbered the coordinates $x^0 = t$, $x^1 = r$, $x^2 = z$ and $x^3 = \phi$ and chose the fluid to be comoving in this coordinate system, hence from (27) and (28)

$$V_\alpha = -A\delta^0_\alpha, \;\; K_\alpha = C\delta^3_\alpha, \;\; S_\alpha = B\delta^2_\alpha. \tag{30}$$

For the exterior vacuum spacetime of the cylindrical surface Σ, I took the metric in Einstein-Rosen coordinates,

$$ds^2_+ = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dz^2 + R^2 e^{-2\psi} d\phi^2, \tag{31}$$

where γ and ψ are functions of T and R and for the fields equations $R_{\alpha\beta} = 0$ I had

$$\psi_{,TT} - \psi_{,RR} - \frac{\psi_{,R}}{R} = 0, \tag{32}$$

$$\gamma_{,T} = 2R\psi_{,T}\psi_{,R}, \tag{33}$$

$$\gamma_{,R} = R(\psi^2_{,T} + \psi^2_{,R}). \tag{34}$$

5 Junction conditions for the collapsing perfect fluid cylinder

I took again the Darmois junction conditions [5] as in section 3 and considered the interior metric to Σ given by (28) and the exterior (31). The equations of Σ may be written

$$g_- = r - r_\Sigma = 0 \ , \tag{35}$$

$$g_+ = R - R_\Sigma(T) = 0 \ , \tag{36}$$

where g_- refers to the spacetime interior of Σ and g_+ to the spacetime exterior, and r_Σ is a constant since Σ is a comoving surface forming the boundary of the fluid. I took the coordinates on Σ as $\xi^0 = \tau$, $\xi^2 = z$ and $\xi^3 = \phi$.

The conditions on the interior and exterior metrics imposed by the continuity of the first fundamental forms on Σ produced

$$e^{\gamma-\psi}\left[1-\left(\frac{dR}{dT}\right)^2\right]^{1/2} dT \overset{\Sigma}{=} Adt \overset{\Sigma}{=} d\tau, \tag{37}$$

$$e^{\psi} \overset{\Sigma}{=} B, \tag{38}$$

$$e^{-\psi}R \overset{\Sigma}{=} C, \tag{39}$$

where I assumed

$$1-\left(\frac{dR_\Sigma}{dT}\right)^2 > 0, \tag{40}$$

so that T is a timelike coordinate.

For the continuity of the second fundamental form (12) I calculated the extrinsic curvature (13) using the same procedure as in section 3 but with the interior (28) and exterior (31) metrics. After a long calculation I obtained the following non zero components of $K^{\mp}_{ab}$

$$K^-_{00} \overset{\Sigma}{=} -\frac{A_{,r}}{A^2}, \tag{41}$$

$$K^-_{22} \overset{\Sigma}{=} \frac{BB_{,r}}{A}, \tag{42}$$

$$K^-_{33} \overset{\Sigma}{=} \frac{CC_{,r}}{A}, \tag{43}$$

$$K^+_{00} \overset{\Sigma}{=} e^{2(\gamma-\psi)}\Big\{\ddot{T}\dot{R} - \ddot{R}\dot{T} \\ -(\dot{T}^2-\dot{R}^2)\left[\dot{R}(\gamma_{,T}-\psi_{,T}) + \dot{T}(\gamma_{,R}-\psi_{,R})\right]\Big\}, \tag{44}$$

$$K^+_{22} \overset{\Sigma}{=} e^{2\psi}(\dot{R}\psi_{,T} + \dot{T}\psi_{,R}), \tag{45}$$

$$K^+_{33} \overset{\Sigma}{=} -e^{-2\psi}R^2\left(\dot{R}\psi_{,T} + \dot{T}\psi_{,R} - \frac{\dot{T}}{R}\right). \tag{46}$$

The complete junction conditions consist of (37-39) together with the continuity of K_{ab}, namely

$$K^-_{00} \overset{\Sigma}{=} K^+_{00}, \tag{47}$$

$$K_{22}^{-} \overset{\Sigma}{=} K_{22}^{+}, \tag{48}$$
$$K_{33}^{-} \overset{\Sigma}{=} K_{33}^{+}. \tag{49}$$

¿From (47)-(49) with the field equations and the matching conditions (37)-(39) I found

$$\kappa P_r \overset{\Sigma}{=} (\dot{R}\psi_{,R})^2 - \frac{\dot{R}}{\dot{T}}(\dot{T}^2 - \dot{R}^2)\frac{\gamma_{,T}}{R}. \tag{50}$$

The result (50) shows that the radial pressure P_r on the surface Σ of the collapsing perfect fluid is non null due to momentum flux of the gravitational wave emerging from the cylinder. If the cylindrical fluid source is static, then $\dot{R} = 0$ and $P_r = 0$ on the surface Σ as expected.

6 Conclusion

I first presented a review of the study of matching conditions for dissipative gravitational spherical collapse. The main result of this study is that the pressure on the surface of the collapsing surface is non null. Its physical interpretation is justified through the continuity of the radial flux of momentum across this surface. Following this result, I inquired if gravitational radiation might have a similar behaviour. In order to answer this question, I studied the matching conditions of a collapsing anisotropic perfect fluid cylinder. I then showed that, in fact, the radial pressure is not null on the matching cylindrical surface, but proportional to the gravitational radiation. If the system is static I will reobtain the usual result that the radial pressure is zero on the boundary surface.

References

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