CONJECTURE ON THE CRITICAL FRONTIER OF THE FULLY ANISOTROPIC HOMOGENEOUS QUENCHED BOND-MIXED POTTS FERROMAGNET IN SQUARE LATTICE

by

Constantino TSALLIS
Centro Brasileiro de Pesquisas Físicas/CNPq
Av. Wenceslau Braz 71 - 22290 - Rio de Janeiro - Brazil

ABSTRACT

We conjecture that the equation

$$\left\langle \frac{\ln\left[1+(q-1)\,e^{-qJ/k_BT_c}\right]}{\ln q} \right\rangle_P + \left\langle \frac{\ln\left[1+(q-1)\,e^{-qJ/k_BT_c}\right]}{\ln q} \right\rangle_{P'} = 1$$

provides a very accurate *approximation* of the yet unknown critical frontier of a fully anisotropic homogeneous quenched bond-mixed q-state Potts *ferromagnet* in square lattice, where the random coupling constant J is distributed according to the laws P(J) and P'(J) for "horizontal" and "vertical" bonds respectively. This equation contains as particular cases a great number of exact (or believed so by us) results as well as a few recent conjectures (which are definitively only approximate).

Many efforts are presently being devoted to the study of the critical properties of random q-state Potts models (characterized by a Hamiltonian $\mathcal{H} = -q \sum_{i,j} J_{ij} \delta_{\sigma_i,\sigma_j}$, where $\sigma_i = 1,2,\ldots,q \quad \forall\, i$), in particular the para-ferromagnetic critical frontier (CF) of the quenched bond-mixed versions in square lattice. We are concerned here with a fully random anisotropic homogeneous version, where we associate, for the coupling constant $J_{ij} \geq 0$ between first-neighbours, the general distribution laws $P(J_{ij})$ for all "horizontal" bonds and $P'(J_{ij})$ for all "vertical" ones. The complete CF of this model is still unknown, but some partial results are already available for the particular case

$$P(J_{ij}) = (1-p)\,\delta(J_{ij}-J_1)+p\delta(J_{ij}-J_2) \qquad (J_1,J_2 \geq 0) \qquad (1.a)$$

$$P'(J_{ij}) = (1-p')\,\delta(J_{ij}-J_1')+p'\delta(J_{ij}-J_2') \qquad (J_1',J_2' \geq 0) \qquad (1.b)$$

To be more precise we have:

a) the *anisotropic pure Potts model* ($[p(1-p)=0;\ p'(1-p')=0]$ or $[J_1/J_2=J_1'/J_2'=1]$) whose critical temperature T_c satisfies (Burkhardt and Southern 1978, Baxter et al 1978), in the let us say $[p=p'=1]$ version,

$$\left[1+(q-1)\,e^{-qJ_2/k_BT_c}\right]\left[1+(q-1)\,e^{-qJ_2'/k_BT_c}\right] = q \qquad (2)$$

for all values of q for which the transition is a second order one ($q \leq 4$);

b) the *anisotropic pure percolation limit* ($[T=0;\ J_1=J_1'=0;\ J_2,J_2'>0]$ or $[T=0;\ J_2=J_2'=0;\ J_1,J_1'>0]$ among other equivalent possibilities) whose CF we believe (see also Southern and Thorpe 1979 and Turban 1980 for the particular case $p/p'=J_2/J_2'=1$ (isotropic model)) to be one and the same for all values of $q\leq 4$ namely (Sykes and Essam 1963), in the let us say $[T=0;\ J_1=J_1'=0;\ J_2=J_2'>0]$ version,

$$p + p' = 1 \tag{3}$$

This CF can be equivalently written in a different form by making use of the $q\to 1$ isomorphism (Kasteleyn and Fortuin 1969), namely, in the let us say $\left[J_1=J_1'=0\ ;\ p \leftrightarrow 1 - e^{-J_2/k_BT};\ p'\leftrightarrow 1 - e^{J_2'/k_BT}\right]$ version,

$$\left(1-e^{-J_2/k_BT_c}\right) + \left(1-e^{-J_2'/k_BT_c}\right) = 1 \tag{3'}$$

or even in a mixed form which generalizes both (3) and (3'), namely

$$p\left(1-e^{-J_2/k_BT_c}\right) + p'\left(1-e^{-J_2'/k_BT_c}\right) = 1 \tag{3''}$$

This last expression reduces, for $p/p'=J_2/J_2'=1$, to equations (4) of Southern and Thorpe's paper (1979) and (18) of Turban's paper (1980);

c) the *isotropic bond-dilute almost pure Potts model* ($[p=p'\lesssim 1;\ J_1=J_1'=0;\ J_2=J_2'>0]$ among other equivalent versions) whose critical temperature is characterized by the derivative

$$\frac{1}{T_c(1)} \frac{d\,T_c(p)}{dp}\bigg|_{p=1} = \frac{2\sqrt{q}}{(1+\sqrt{q})\ln(1+\sqrt{q})} \tag{4}$$

which is believed to be exact (Southern and Thorpe 1979; for q=2 see Harris 1974);

d) the *isotropic bond-dilute almost pure percolation limit* ($[T \gtrsim 0;\ p=p' \gtrsim 1/2;\ J_1=J_1'=0;\ J_2=J_2'>0]$ among other equivalent versions) whose critical temperature is characterized by the derivative

$$\frac{d\,e^{-q\,J_2/k_B T_c(p)}}{d\,p}\bigg|_{p=1/2} = \frac{2\ln q}{q-1} \tag{5}$$

which is believed to be exact (Southern and Thorpe 1979; for q=2 see Domany 1978);

e) the *anisotropic equal probability bond-mixed Potts model* ($[p+p'=1;\ J_1/J_1'=J_2/J_2'=1]$ or $[p/p'=J_1/J_2'=J_2/J_1'=1]$) whose critical temperature is, for q=2 (Fisch 1978; see also Oguchi and Ueno 1978, for a slightly less general statement), one and the same for all values of p (remark that p=0 corresponds to the anisotropic pure Potts model), namely, in the let us say $[p+p'=1;\ J_1/J_1'=J_2/J_2'=1]$ version,

$$\left(1+e^{-2J_1/k_B T_c}\right)\left(1+e^{-2J_2/k_B T_c}\right) = 2 \tag{6}$$

Though we have not attempted to prove it, we believe that Fisch's statement (1978) can be extended to *all* values of

$q \leqslant 4$, therefore relation (6) generalizes into

$$\left[1+(q-1)e^{-qJ_1/k_BT_c}\right]\left[1+(q-1)e^{-qJ_2/k_BT_c}\right] = q \qquad (6')$$

Before going on let us introduce some convenient variables and relations. We may associate to a single Potts bond with coupling constant J_i a new variable (referred hereafter as *thermal transmissivity*; see Tsallis and Levy 1980(a), Levy et al 1980 and references therein) defined by

$$t_i \equiv \frac{1-e^{-qJ_i/k_BT}}{1+(q-1)e^{-qJ_i/k_BT}} \in [0,1] \qquad (J_i \geqslant 0) \qquad (7)$$

This variable allows a most simple expression of the equivalent coupling constant J_s associated to a *series* array of two bonds (respectively associated to J_1 and J_2), and this is (Tsallis and Levy 1980(b))

$$t_s = t_1 t_2 \qquad (8)$$

If the array is a *parallel* one, the equivalent transmissivity t_p satisfies

$$t_p^D = t_1^D \, t_2^D \qquad (9)$$

where we have introduced the *dual thermal transmissivity* (of a given one)

$$t_i^D \equiv \frac{1-t_i}{1+(q-1)t_i} \qquad (10)$$

Let us next introduce another variable (which generalizes the one appearing in Levy et al 1980 for q=2) through

$$s \equiv \frac{\ln[1+(q-1)t]}{\ln q} \in [0,1] \tag{11}$$

We remark that, in the limit $q \to 1$, s equals t. For all q we verify the following remarkable property

$$s^D(t) \equiv s(t^D) = 1-s(t) \tag{12}$$

i.e. s *transforms, under duality, like a probability;* this fact is at the core of the central conjecture (presented below) of the present work. Note also that, if we have two bonds, $s_p=s_1+s_2-s_s$ ($s_i \equiv s(t_i)\ \forall i$) which re-states, *for all values of* q, the relation $t_p=[t_1+t_2+(q-2)t_s]/[1+(q-1)t_s]$ between transmissivities (q=1 implies $t_p=t_1+t_2-t_s$); on the other hand s_s will in general differ from $s_1 s_2$, therefore (and only therefore) s_p^D will in general differ from $s_1^D s_2^D$. If we respectively note $P_t(t)$ and $P_s(s)$ the probabilities distributions of t and s, associated to a given one P(J), we have the following relations

$$P(J) = \frac{(1-t)[1+(q-1)t]}{k_B T} P_t(t) = \frac{(q-1)(1-t)}{k_B T \ln q} P_s(s)$$

In particular, distribution (1.a) (analogously for (1.b)) leads to

$$P_t(t) = (1-p)\delta(t-t_1)+p\delta(t-t_2) \tag{1.a'}$$

$$P_s(s) = (1-p)\delta(s-s_1)+p\delta(s-s_2) \tag{1.a''}$$

We are now prepared to re-state, in the new variables, the particular exact results we presented before (models (a) to (e)), and relations (2),(3''),(4),(5) and (6') respectively become

$$t_2 = \frac{1-t_2'}{1+(q-1)t_2'} \tag{2'}$$

$$s_2+s_2'=1 \tag{2''}$$

$$p\ t_2(q=1)+p't_2'(q=1) = 1 \tag{3'''}$$

$$p\ s_2(q=1)+p's_2'(q=1) = 1 \tag{3^{IV}}$$

$$\left.\frac{d\ t_2}{dp}\right|_{p=1} = -\frac{2\sqrt{q}}{(1+\sqrt{q})^2} \tag{4'}$$

$$\left.\frac{ds_2}{dp}\right|_{p=1} = -\frac{2}{\ln q}\ \frac{\sqrt{q}-1}{\sqrt{q}+1} \tag{4''}$$

$$\left.\frac{dt_2}{dp}\right|_{p=1/2} = -\frac{2q\ \ln q}{q-1} \tag{5'}$$

$$\left.\frac{ds_2}{dp}\right|_{p=1/2} = -2 \tag{5''}$$

$$t_1 = \frac{1-t_2}{1+(q-1)t_2} \tag{6''}$$

and $$s_1+s_2 = 1 \tag{6'''}$$

Let us now state our conjecture: the general CF we are looking for is given, within a very good approximation, by

$$\langle s\rangle_{P_s} + \langle s\rangle_{P'_s} = 1 \qquad (1\leqslant q\leqslant 4) \qquad (13)$$

hence

$$\left\langle \ln\left[1+(q-1)e^{-qJ/k_BT_c}\right]\right\rangle_P + \left\langle \ln\left[1+(q-1)e^{-qJ/k_BT_c}\right]\right\rangle_{P'} = \ln q \qquad (13')$$

where <···> means mean value. Relation (13) becomes, for the particular case (1.a)-(1.b),

$$(1-p)s_1+ps_2+(1-p')s'_1+p's'_2 = 1 \qquad (13'')$$

which *exactly* reproduces (2''), (3^{IV}), (5'') and (6'''). In what concerns relation (4'') it partially fails as it leads to $(ds_2/dp)_{p=1} = -1/2$ for *all* values of q, therefore the errors for q=1,2,3 and 4 are respectively 0%, 1%, 2.5% and 4%. Expression (13') reduces, for q=2 and P=P', to Nishimori's conjecture (1979), which he claimed to be *exact* (this is not so, as proved by Aharony and Stephen 1980); furthermore the particular case where P=P' given by (1.a), coincides with the heuristic approximation of Levy et al 1980. In what concerns general values of q, Southern's result (1980) can be reobtained from relation (13'') with $s_1=s'_1=0$ and $p/p'=s_2/s'_2=1$.

Let us stress at this point that although relation (13) (and consequently (13")) is in general *not exact*, we claim it to be a very good approximation everywhere ($\forall T_c$) and for $1 \leq q \leq 4$ (the error in the s- variable is expected to be less than one percent in the worse case (q=4) and the worse region (middle way between the equal concentration and pure cases, in the bond-dilute particular case)). More specifically, and besides the well known exact results, we believe that:

A) relation (13) is exact for q=1 (therefore generalizing relation (3"), as for this case the problem becomes isomorphic to bond percolation and the s- variable (which is now identical to the t- variable) *strictly* behaves as a probability (in particular $s_s = s_1 s_2$ and $s_p^D = s_1^D s_2^D$);

B) relation (13") is asymptotically exact for all $q \leq 4$ for the *anisotropic slightly bond-mixed model* ($[(J_2-J_1)/k_BT \to 0$ and $(J_2'-J_1')/k_BT \to 0]$ therefore $[(s_2-s_1)\to 0$ and $(s_2'-s_1')\to 0]$)

i.e.

$$s_2+s_2' \sim 1 + (1-p)(s_2-s_1) + (1-p')(s_2'-s_1') \qquad (14)$$

hence

$$\left[1+(q-1)e^{-qJ_2/k_BT_c}\right]\left[1+(q-1)e^{-qJ_2'/k_BT_c}\right]$$

$$\sim q\left[1-\frac{q(J_2-J_1)}{k_BT_c}\;\frac{(1-p)(q-1)}{e^{qJ_2/k_BT_c}+(q-1)}\right]\left[1-\frac{q(J_2'-J_1')}{k_BT_c}\;\frac{(1-p')(q-1)}{e^{qJ_2'/k_BT_c}+(q-1)}\right]$$

(14')

which respectively generalize (2") and (2); see Fig. 1.a for the isotropic case $[p/p'=s_1/s_1'=s_2/s_2'=1]$;

C) relation (13") is asymptotically exact for all $q\leq 4$ for the *anisotropic almost equal probability bond-mixed model* (distributions (1.a)-(1.b) with $p\simeq 1/2$ and $p'\simeq 1/2$), i.e.

$$s_1+s_2+s_1'+s_2' \sim 2\left[1+(s_2-s_1)\left(\frac{1}{2}-p\right)+(s_2'-s_1')\left(\frac{1}{2}-p'\right)\right] \tag{15}$$

hence

$$\left[1+(q-1)e^{-qJ_1/k_BT_c}\right]\left[1+(q-1)e^{-qJ_2/k_BT_c}\right]\left[1+(q-1)e^{-qJ_1'/k_BT_c}\right]\left[1+(q-1)e^{-qJ_2'/k_BT_c}\right] \sim$$

$$q^2\left[1+(1-2p)\ln\frac{1+(q-1)e^{-qJ_2/k_BT_c}}{1+(q-1)e^{-qJ_1/k_BT_c}}+(1-2p')\ln\frac{1+(q-1)e^{-qJ_2'/k_BT_c}}{1+(q-1)e^{-qJ_1'/k_BT_c}}\right] \tag{15'}$$

which generalize (5") and (6''') (hence (5) and (6')); see Fig. 1.a for the isotropic case $[p/p'=s_1/s_1'=s_2/s_2'=1]$;

D) relation (13) is asymptotically exact for all $q\leq 4$ in the limit $T\to 0$ of the following *generalized bond-dilute anisotropic model:*

$$P(J) = (1-p)\,\delta(J) + pR(J) \tag{16.a}$$

$$P'(J) = (1-p')\,\delta(J) + p'R'(J) \tag{16.b}$$

where the distribution laws $R(J)$ and $R'(J)$ satisfy

$$\int_0^\infty R(J)\,dJ = \int_0^\infty R'(J)\,dJ = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_0^\varepsilon R(J)\,dJ =$$

$\lim_{\varepsilon\to 0}\int_0^\varepsilon R'(J)\,dJ = 0$ (i.e. both $R(J)$ and $R'(J)$ do not grow, in the limit $J \to 0$, as $1/J$ or faster). The standard particular case $R(J)=\delta(J-J_2)$ and $R'(J)=\delta(J-J'_2)$ leads to

$$p+p' \sim 1+p(1-s_2)+p'(1-s'_2) \tag{17}$$

hence

$$p\left[1-\frac{q-1}{\ln q}\,e^{-qJ_2/k_BT_c}\right]+p'\left[1-\frac{q-1}{\ln q}\,e^{-qJ'_2/k_BT_c}\right]\sim 1 \tag{17'}$$

which generalize (3^{IV}) and (5") (hence (3") and (5)); see Fig. 1.b for the case $R(J)=R'(J)=\delta(J-J_2)$;

E) the CF (13") has a definite location with respect to the unknown exact one, in the sense that they have in common the following (and, for $q\neq 1$, probably *only* the following) regions (and their trivially equivalent ones):

α) ($s_1/s_2=s'_1/s'_2=1$; $s_2+s'_2=1$; $\forall p$; $\forall p'$) which generalizes the line ($s_1=s_2=s'_1=s'_2=1/2$; $\forall p=p'$) of the Fig. 1.a;

β) ($p=p'=1$; $s_2+s'_2=1$; $\forall s_1$; $\forall s'_1$) which generalizes the line ($p=p'=1$; $s_2=s'_2=1/2$; $\forall s_1=s'_1$) of the Fig. 1.a;

γ) ($p=p'=1/2$; $s_1+s_2+s'_1+s'_2=2$) which generalizes the line ($p=p'=1/2$; $s_1/s'_1=s_2/s'_2=1$; $s_1+s_2=1$) of the Fig. 1.a;

δ) ($s_1=s'_1=0$; $s_2=s'_2=1$; $p+p'=1$) which is represented in Fig. 1.b.

Furthermore, for the isotropic case $[p/p'=s_1/s'_1=s_2/s'_2=1]$ considered in Fig. 1.a, the unknown exact surface ($q\neq 1$) lays (see also Tsallis and Levy 1980 (a) and Levy et al 1980) on the low- (high-) s_2 side of the CF $(1-p)s_1+ps_2=1/2$ for $(1>p>1/2;\ s_1<1/2)$ and $(0<p<1/2;\ s_1>1/2)$ ($(1>p>1/2;\ s_1>1/2)$ and $(0<p<1/2;\ s_1<1/2)$). Analogously, for the bond-dilute case $[s_1=s_1'=0;\ s_2=s_2']$ considered in Fig. 1.b, the unknown exact surface ($q\neq 1$) lays on the low $-s_2$ side of the CF $(p+p')s_2=1$ for all (p,p') such that $1<p+p'<2$.

Let us conclude by saying that, for numerical purposes, the conjectural equation (13') for the critical frontier of the ferromagnetic fully anisotropic homogeneous quenched bond-mixed Potts model in square lattice we have been considering here is certainly quite satisfactory, and, for analytical purposes, it leads to a great number of particular asymptotic behaviours (eqs. (14'), (15') and (17')) which we believe to be exact. Furthermore, we can speculate that Fisch's statement (1978) for the *standard* quenched bond-mixed *Ising* ferromagnet can be extended as follows: the same critical temperature T_c might be shared by a whole class of *generalized* quenched bond-mixed *Potts* ferromagnets, where half (*any half*) of the bonds have coupling constants distributed according to $P(J)$, and the other half according to $P'(J)$. Clearly the fully anisotropic model we have been considering in this paper is but an element of this class.

Stimulating discussions with SVF Levy, ACN Magalhães and EMF Curado are gratefully acknowledged.

REFERENCES

Aharony A and Stephen MJ 1980, J.Phys. C 13, L407-14.

Baxter RJ, Temperley HNV and Ashley SE 1978, Proc. R. Roc. A 358, 535-59.

Burkhardt TW and Southern BW 1978, J.Phys. A 11, L247-51.

Domany E 1978, J. Phys. C 11, L337-42.

Fisch R 1978, J. Statist. Phys. 18, 111-4.

Harris AB 1974, J. Phys. C 7, 1671-92.

Kasteleyn PW and Fortuin CM 1969, J. Phys. Soc. Japan (suppl.) 26, 11-4.

Levy SVF, Tsallis C and Curado EMF 1980, Phys. Rev. B 21, 2991-8.

Nishimori H 1979, J. Phys. C 12, L905-8.

Oguchi T and Ueno Y.1978, J. Phys. Soc. Japan 44, 1449-54.

Southern BW 1980, J. Phys. C 13, L285-7.

Southern BW and Thorpe MF 1979, J.Phys. C 12, 5351-60.

Sykes MF and Essam JW 1963, Phys. Rev. Lett. 10, 3-4.

Tsallis C and Levy SVF 1980; (a) J. Phys. C 13, 465-70; (b) to be published.

Turban L 1980, J. Phys. C 13, L13-9.

CAPTION FOR FIGURE

Fig. 1 The conjectural equation (13") represents the q- state Potts model critical surface in a 6-dimensional parameter space: two typical particular cases are represented here, where the shadowed regions are believed to be *asymptotically exact* for all $q \leqslant 4$. (a) $p/p'=s_1/s_1'=s_2/s_2'=1$, hence $(1-p)s_1+ps_2=1/2$ (the lines ($p=1$; $s_2=1/2$; $\forall\, s_1$) and ($p=0$; $s_1=1/2$; $\forall\, s_2$) are known to be exact as well); (b) $s_1=s_1'=0$ and $s_2=s_2'$, hence $(p+p')s_2=1$ (the point ($p=p'=1$; $s_2=1/2$) is known to be exact as well).

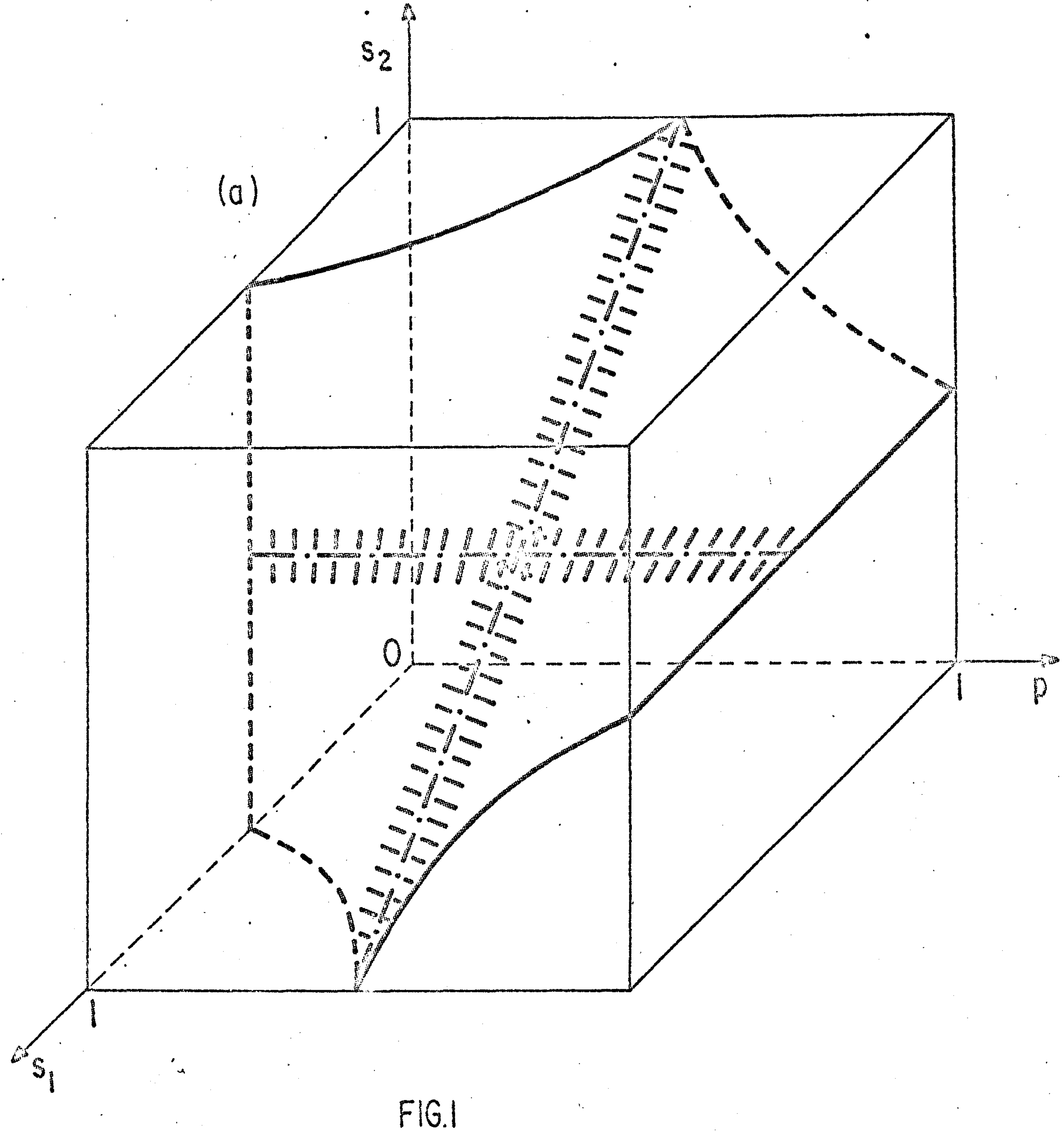

FIG.1

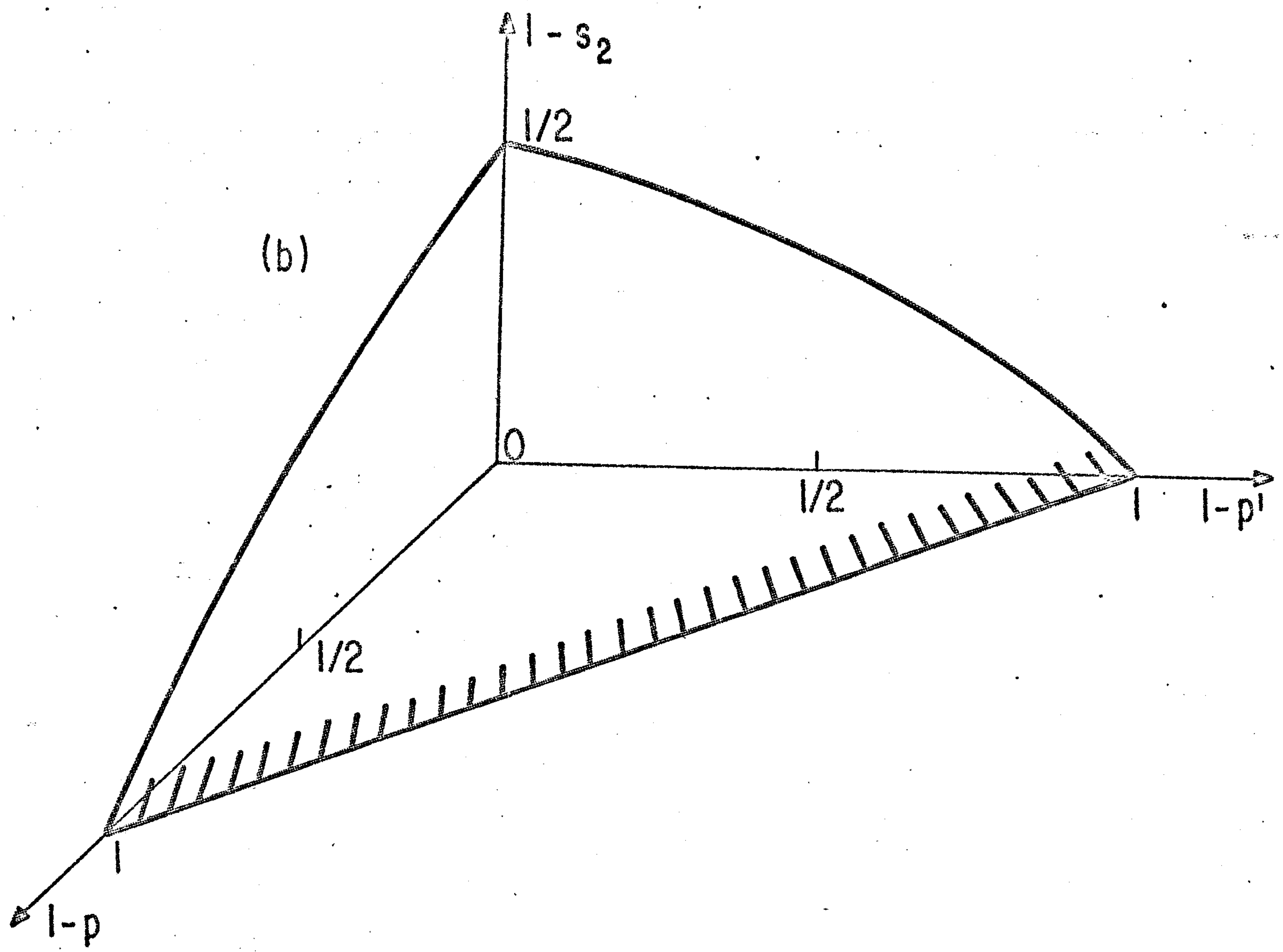

FIG.1