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A NONLINEAR PROBLEM IN DYNAMIC VISCO-ELASTICITY WITH FRICTION

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ABSTRACT

This paper presents a study of the dynamics of a pile driven into the ground under the action of a pile hammer. Due to the effects of friction one is led to a model consisting of a variational inequality. Results about existence, uniqueness and stability of solutions of an initial value problem for this variational inequality are obtained.

1. INTRODUCTION

The objective of this paper is to present some results obtained in the study of a problem in dynamic visco-elasticity with friction which appears in Foundation Engineering. Such results refer only to the mathematical analysis of the adopted model. In a future paper we shall consider the numerical analysis and present results of some specific experiments.

The problem consists in analyzing the motion of a pile penetrating into the soil under the action of the characteristic force of the pile driver, the resistance force of the soil and the friction on the contact surface. We shall assume the pile to be a one dimensional bar, which is a reasonable model for some types of metallic piles. In consequence, the friction must be simulated through a body force instead of a surface one, and the soil resistance through a force acting on the penetrating tip of the pile.

Our aim is: first, to formulate the problem in terms of a variational inequality derived from Coulomb's dynamic law for friction and the principle of the virtual powers; second, to prove some results about the existence, uniqueness and stability of its solutions.

The justification of the model will be done in section 2 and the mathematical analysis in section 3.

2. *FORMULATION OF THE MODEL*

We can visualize the situation we have at hand through the diagram showed in Figure 1. There, two states of motion are represented, the initial $(t = 0)$ and at some instant $t > 0$.

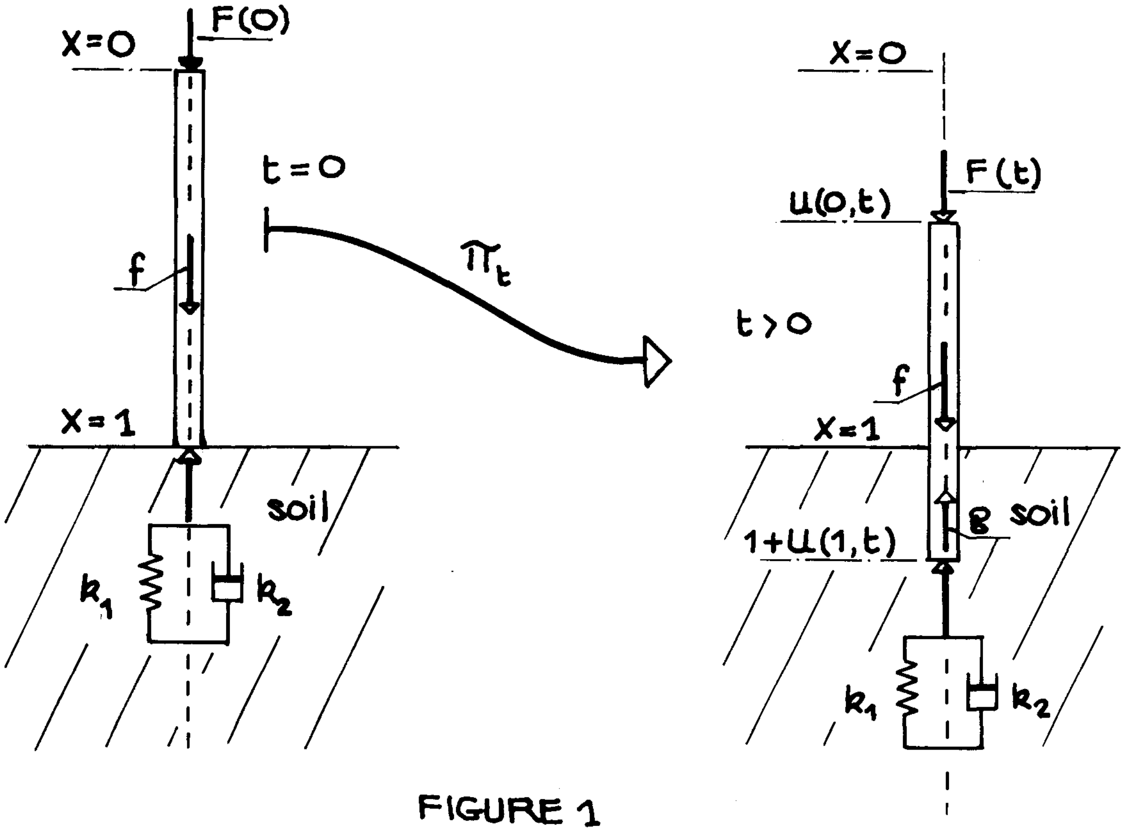

FIGURE 1

The two states are related by the motion mapping

$$\pi_t : [0,L] \longrightarrow [u(0,t),L+u(L,t)] ,$$

that carries a "particle" x from the initial configuration of the pile to x+u(x,t), which is its position at time t, u(x,t) being the displacement field.

The parameters involved are L = length of the pile, k_1 = soil elastic resistance coefficient and k_2 = soil viscous resistance coefficient.

The forces acting on the pile are:

$F(t)$ = force transmitted to the system through the hammer of the pile driver; its action is exerted only on the "particle" $x = 0$;

$f(x,t)$ = body force; it is due to an external field;

$-k_1 u(L,t)$ = reaction force due to the elastic properties of the soil; its action is only exerted on the penetrating tip of the pile, that is, on the "particle" $x = L$;

$-k_2 \dot{u}(L,t)$ = reaction force due to the viscosity of the soil; its action is only exerted on the "particle" $x=L$;

$g(x,t)$ = friction force which starts acting on a "particle" x just when it penetrates into the ground, that is, when $x+u(x,t) > L$; this implies from the very beginning a representation in the form $g(x,t) = H(x+u-L) \cdot \tilde{g}(x,t)$, where H is the Heaviside function and $\tilde{g}$ would be the actual value of the friction.

The force due to friction is in general a surface contact force, but in this one-dimensional model of the pile it will be a body force with a proper dimensional coefficient. This is a natural idea for such a specific situation. Its behavior will be given by Coulomb's law, which says (see Duvaut-Lions [1]): "at a time t and at any point of the contact region,

(2.1) (i) if $|g| < F|F_N|$ then $\dot{u} = 0$,

(ii) if $|g| = F|F_N|$ then there exists a $\lambda \geq 0$ such that $\dot{u} = -\lambda g$,

where F is the friction coefficient, F_N is the force normal to the structure, and $\dot{u} = \frac{\partial u}{\partial t}$ is the velocity field". In our case F_N is the pressure exerted by the soil onto the pile, so that

$$|F_N| = K\gamma\ell\cdot(x+u-L) \quad (2.2)$$

where K is the Rankine coefficient $(0.4 \leq K \leq 3)$, γ is the specific weight of the soil, and ℓ is the perimeter of the cross section of the pile at x.

Coulomb's law (2.1)(i)— (2.1)(ii) implies the following relation, satisfied at any time t and for all contact points:

$$g(v-\dot{u}) + F|F_N|(|v|-|\dot{u}|) \geq 0 \quad (2.3)$$

where $\dot{u}$ is the actual velocity of the system and v would be a "virtual velocity field".

Indeed, if $|g| < F\,|F_N|$, relation (2.3) would be $g\,v + F\,|F_N|\ |v| \geq 0$, which is obviously true. If $|g| = F\,|F_N|$, relation (2.3) reduces to $g\,v + F\,|F_N|\ |v| \geq 0$, which also holds.

At this point we invoke the "principle of virtual powers", since for the study of friction problems it is necessary to postulate "global" equilibrium conditions. Following P. Germain [2], "in a Galilean frame, and for an absolute chronology, the virtual power associated to the inertial forces in a system $\mathcal{S}$ is equal to the power generated by all the forces applied to the system, internal as well as external, and for any considered virtual motion of system $\mathcal{S}$". In our case,

$$(2.4) \qquad \int_0^L A\rho\, \ddot{u}\,(v-\dot{u})dx = -\int_0^L A\sigma(v_x-\dot{u}_x)dx$$

$$+ \left[-k_1 u(L,t) - k_2\dot{u}(L,t)\right]\left[v(L) - \dot{u}(L,t)\right]$$

$$+ F(t)\left[v(0) - \dot{u}(0,t)\right]$$

$$+ \int_0^L Af(v-\dot{u})dx$$

$$+ \int_0^L H(x+u-L)\tilde{g}(v-\dot{u})dx,$$

where σ = stress tensor, ρ = density of the pile, A = area of the cross-section of the pile, $v-\dot{u}$ = virtual velocity field, and $u_x = \frac{\partial u}{\partial x}$ is the linearized strain tensor.

Assuming a linear visco-elastic behavior law for the pile in the form

$$(2.5) \qquad \sigma = au_x + b\dot{u}_x ,$$

we imply its classification in the domain of the "visco-elastic solids" (with "short memory"!). This assumption, which will be taken as part of our model, together with the consideration of (2.3), transforms (2.4) into

$$(2.6)\qquad (\rho A\ddot{u}\ ,\ v-\dot{u}) + a(Au_x\ ,\ v_x - \dot{u}_x) + b(A\dot{u}_x\ ,\ v_x - \dot{u}_x) + [k_1 u(L,t) + k_2\dot{u}(L,t)]\ [v(L) - \dot{u}(L,t)] + (\mathcal{F}H|F_N|\ ,\ |v| - |\dot{u}|) \geq (Af\ ,\ v-\dot{u}) + F(t)\ [v(0) - \dot{u}(0,t)]\ ,\quad \forall\ v,$$

where $(u_1\ ,\ u_2) = \int_0^L u_1(x)\ u_2(x)dx.$

Such a transformation of the equation is necessary because Coulomb's law does not give information about g when $|g|<\mathcal{F}|F_N|$.

If we define, for each u, the functional

$$J(u;.)\ :\ H^1(0,L) \longrightarrow \mathbb{R}$$

$$v \longrightarrow \gamma K\mathcal{F}\int_0^L \ell(x)H(x+u-L)(x+u-L)|v(x)|dx,$$

with $H^1(0,L)$ being the usual Sobolev space, we can rewrite the equilibrium condition (2.6), and the problem of the motion of the pile will be described mathematically by:

(2.7) $$u(x,0) = 0,$$

(2.8) $$\dot{u}(x,0) = 0,$$

(2.9) $$(\rho A\ddot{u} , v-\dot{u}) + a(Au_x , v_x - \dot{u}_x) + b(A\dot{u}_x , v_x - \dot{u}_x) + [k_1 u(L,t) + k_2\dot{u}(L,t)] \cdot [v(L) - \dot{u}(L,t)] + J(u;v) - J(u,\dot{u}) \geq (Af , v-\dot{u}) + F(t)[v(0) - \dot{u}(0,t)] , \quad \forall\, v \in H^1(0,L).$$

Written in this form the "pile driver problem" reduces to a meaningful mathematical question: an initial value problem for a variational inequality of evolution type. To establish it, the physical information contained in Coulomb's dynamic friction law was fully used, as well as the conservation law implicit in the principle of virtual powers. For the rest of this paper we shall pursue answers to the question: what conditions on the data f and F would guarantee existence, uniqueness and stability of solutions of (2.7)—(2.9).

3. AN EXISTENCE THEORY

For the sake of simplicity we shall adopt in this section a system of units in which $a = b = k_1 = k_2 = L = 1$. Furthermore, we assume $\rho(x) \equiv A(x) \equiv \ell(x) \equiv 1$. We can easily see that we are not losing generality, since the general situation can be handled under natural hypothesis for those three given functions.

In respect to the notation we adopt the convention $\Omega = (0,1)$ and

$$H^1(\Omega) = \{v \in L^2(\Omega) \mid v_x \in L^2(\Omega)\} .$$

With the scalar product

$$(u,v)_1 = (u,v) + (u_x , v_x)$$

$$= \int_0^1 u\, v \; dx + \int_0^1 u_x \, v_x \; dx ,$$

$H^1(\Omega)$ is a Hilbert space. Functions in $H^1(\Omega)$ are continuous and it makes sense to take $u(y)$, $y \in \overline{\Omega}$.

If X is a Banach space with norm denoted by $|\cdot|_X$, we denote by $L^p(0,T;X)$ the space of (classes of) measurable functions $t \to f(t)$ from $[0,T] \to X$ (for the measure dt) such that

$$\left[\int_0^T |f(t)|_X^p \; dt \right]^{1/p} = |f|_{L^p(0,T;X)} < \infty ,$$

, if $p \neq \infty$,

$$\operatorname*{ess\,sup}_{t\varepsilon(0,T)} |f(t)|_X = |f|_{L^\infty(0,T;X)} < \infty \quad ,$$

otherwise.

This is a Banach space.

At this point we mention a result that describes a situation we will constantly encounter in the following. Let V and H be two Hilbert spaces, with $V \subset H$, V dense in H. Identifying H with its dual, H is then identified with a subspace of the dual V' of V, whence $V \subset H \subset V'$. Let then v be given by

(3.1) $$v \,\varepsilon\, L^2(0,T;V) \ , \ \frac{dv}{dt} \,\varepsilon\, L^2(0,T;V').$$

It is shown (see Lions-Magenes [4], Chap. 1) that

(3.2) "after possible modification on a set of measure zero, the function $t \to v(t)$ is continuous from $[0,T] \to H$."

The following theorem gives an existence theory for (2.7)—(2.9).

<u>Theorem 3.1</u>: Let

(3.3) $$f \ , \ \dot{f} \,\varepsilon\, L^2(0,\infty\,;L^2(\Omega)) \ ,$$

(3.4) $$F \ , \ \dot{F} \,\varepsilon\, L^2(0,\infty\,;\mathbb{R}) \ , \ \text{and} \ F(0) = 0 .$$

Then, for any given $T > 0$, there exists a unique $u \,\varepsilon\, L^\infty(0,T;H^1(\Omega))$

such that

(3.5) $\quad \dot{u} \in L^{\infty}(0,T;H^{1}(\Omega))$,

(3.6) $\quad \ddot{u} \in L^{\infty}(0,T;L^{2}(\Omega))$,

(3.7) $\quad u(0) = 0,$

(3.8) $\quad \dot{u}(0) = 0,$

(3.9)
$$(\ddot{u} , v-\dot{u}) + (u_x , v_x - \dot{u}_x) + (\dot{u}_x , v_x-\dot{u}_x) + [u(1,t) + \dot{u}(1,t)] \; [v(1)-\dot{u}(1,t)] + J(u ; v) - J(u ; \dot{u}) \geq$$
$$\geq (f , v-\dot{u}) + F(t) \; [v(0) - \dot{u}(0,t)] \;,$$
$$\forall \; v \in H^{1}(\Omega) \;, \quad \text{a.e. } t \in (0,T).$$

Proof. An outline of our demonstration is the following:

Step 1 - Uniqueness;

Step 2 - Convex regularization of J and reduction of (3.9) to a variational equation;

Step 3 - Solution of this new equation through Galerkin's method;

Step 4 - Proof of the existence statements of the theorem.

Step 1. We assume that u_1 and u_2 are two solutions and take $u = u_1 - u_2$. Choosing $v = \dot{u}_2$ in (3.9) written for u_1, $v = \dot{u}_1$ in (3.9) written for u_2 , and adding the results, we obtain

$$(\ddot{u}, \dot{u}) + (u_x, \dot{u}_x) + (\dot{u}_x, \dot{u}_x)$$

$$+ [u(1,t) + \dot{u}(1,t)]\, \dot{u}(1,t) \leq J(u_1;\dot{u}_2) -$$

$$- J(u_1;\dot{u}_1) + J(u_2;\dot{u}_1) - J(u_2;\dot{u}_2).$$

Hence,

$$\frac{1}{2}\frac{d}{dt}|\dot{u}|^2(t) + \frac{1}{2}\frac{d}{dt}|u_x|^2(t) + |\dot{u}_x|^2(t) + \frac{1}{2}\frac{d}{dt}u^2(1,t)$$

$$+ \dot{u}^2(1,t) \leq \gamma K F \int_0^1 \left[|\dot{u}_2(x,t)| - |\dot{u}_1(x,t)|\right] \cdot$$

$$\cdot \left[H(x+u_1-1)(x+u_1-1) - H(x+u_2-1)(x+u_2-1)\right]dx$$

$$\leq \frac{1}{2}\gamma K F\{|u|^2(t) + |\dot{u}|^2(t)\},$$

where $|f| = \sqrt{(f,f)}$. If we integrate from 0 to t:

$$|\dot{u}|^2(t) + u^2(1,t) + |u_x|^2(t) + 2\int_0^t \dot{u}^2(1,\tau)d\tau + \tag{3.10}$$

$$+ 2\int_0^t |\dot{u}_x|^2(\tau)d\tau \leq \gamma K F \int_0^t \{|u|^2(\tau) + |\dot{u}|^2(\tau)\}d\tau.$$

But, in view of

$$u(x,t) = u(1,t) - \int_x^1 u_x(\zeta,t)d\zeta ,$$

(3.11) $$\int_0^1 u^2(x,t)dx \le 2 \left[u^2(1,t) + \int_0^1 u_x^2(\zeta,t)d\zeta\right].$$

By theorem 3.5 of [3],

(3.12) $$\sup_{x\varepsilon[0,1]} u(x,t) \le C\left[\int_0^1 u^2(x,t)dx + \int_0^1 u_x^2(x,t)dx\right]^{\frac{1}{2}},$$

$$\forall\ t ,$$

with C independent of u. Hence (3.11) and (3.12) imply that

$$|u|_1 = \sqrt{(u,u)_1} = \left[\int_0^1 u^2 dx + \int_0^1 u_x^2\ dx\right]^{\frac{1}{2}} ,$$

and

$$\left[u^2(1) + \int_0^1 u_x^2\ dx\right]^{\frac{1}{2}}$$

are equivalent norms for $H^1(\Omega)$.

Carrying this information into (3.10) we get

$$|\dot{u}|^2(t) + |u|_1^2(t) + \int_0^t |\dot{u}|_1^2(\tau)d\tau$$

$$\le C \int_0^t \{|\dot{u}|^2(\tau) + |u|_1^2(\tau)\}d\tau,$$

so that, by Gronwall's lemma,

$$|\dot{u}|^2(t) + |u|_1^2(t) = 0,$$

that is, $u_1 = u_2$. It should be noticed that, at some point of this argument, we made use of the initial conditions (3.7) and (3.8).

<u>Step 2</u>. For a given $\varepsilon > 0$ we take functions $\tilde{\psi}_\varepsilon(\lambda)$, $\phi_\varepsilon(\lambda)$, convex and sufficiently smooth, such that $\phi_\varepsilon(\lambda) = |\lambda|$ if $|\lambda| \geq \varepsilon$ and $\tilde{\psi}_\varepsilon(\lambda) = \gamma FK\, H(\lambda)\lambda$ if $|\lambda| \geq \varepsilon$. Denoting the function $\tilde{\psi}_\varepsilon[u(x)-(1-x)]$ by $\psi_\varepsilon(u)$, we define

$$J_\varepsilon(u;v) = \int_0^1 \psi_\varepsilon(u)\ \phi_\varepsilon(v)\ dx,$$

for $u, v \in H^1(\Omega)$, and consider an "approximate" regular problem substituting J by J_ε in (3.9). Such a problem is:

(3.13) $\quad u_\varepsilon \in L^\infty(0,T;H^1(\Omega))$,

(3.14) $\quad \dot{u}_\varepsilon \in L^\infty(0,T;H^1(\Omega))$,

(3.15) $\quad \ddot{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega))$,

(3.16) $\quad u_\varepsilon(0) = 0$,

(3.17) $\quad \dot{u}_\varepsilon(0) = 0$,

$$(3.18)\qquad (\ddot{u}_\varepsilon\, , v-\dot{u}_\varepsilon) + (u_{\varepsilon x}\, , v_x - \dot{u}_{\varepsilon x}) + (\dot{u}_{\varepsilon x}\, , v_x - \dot{u}_{\varepsilon x}) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\ [v(1)-\dot{u}_\varepsilon(1,t)] + J_\varepsilon(u_\varepsilon\, ; v) - J_\varepsilon(u_\varepsilon\, ; \dot{u}_\varepsilon) \geq$$

$$\geq (f, v-\dot{u}_\varepsilon) + F(t)\ [v(0) - \dot{u}_\varepsilon(0,t)]\ ,$$

$$\forall\ v \in H^1(\Omega)\ ,\quad \text{a.e.}\quad t \in (0,T).$$

Our claim is that in this new situation (3.18) is equivalent to a variational equation. Taking in (3.18) $v = \dot{u}_\varepsilon + \lambda w$, $\lambda > 0$, $w \in H^1(\Omega)$, dividing by λ , and carrying out the limit as $\lambda \to 0$, we get

$$(\ddot{u}_\varepsilon\, , w) + (u_{\varepsilon x}\, , w_x) + (\dot{u}_{\varepsilon x}\, , w_x) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\ w(1) + (\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon\ (\dot{u}_\varepsilon), w) \geq (f,w) + F(t)\ w(0).$$

Since we could have taken $-w$ in place of w , u_ε satisfies

$$(3.19)\qquad (\ddot{u}_\varepsilon\, , w) + (u_{\varepsilon x}\, , w_x) + (\dot{u}_{\varepsilon x}\, , w_x) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\ w(1) + (\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon(\dot{u}_\varepsilon),\ w) = (f,w) + F(t)\ w(0)\ ,$$

$$(3.18)\qquad (\ddot{u}_\varepsilon\,,\,v-\dot{u}_\varepsilon) + (u_{\varepsilon x}\,,\,v_x - \dot{u}_{\varepsilon x})$$

$$+ (\dot{u}_{\varepsilon x}\,,\,v_x - \dot{u}_{\varepsilon x}) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\ [v(1)-\dot{u}_\varepsilon(1,t)]$$

$$+ J_\varepsilon(u_\varepsilon\,;\,v) - J_\varepsilon(u_\varepsilon\,;\,\dot{u}_\varepsilon) \geq$$

$$\geq (f,v-\dot{u}_\varepsilon) + F(t)\ [v(0) - \dot{u}_\varepsilon(0,t)] \quad ,$$

$$\forall\ v \in H^1(\Omega)\ ,\quad \text{a.e.}\quad t \in (0,T).$$

Our claim is that in this new situation (3.18) is equivalent to a variational equation. Taking in (3.18) $v = \dot{u}_\varepsilon + \lambda w$, $\lambda > 0$, $w \in H^1(\Omega)$, dividing by λ , and carrying out the limit as $\lambda \to 0$, we get

$$(\ddot{u}_\varepsilon\,,\,w) + (u_{\varepsilon x}\,,\,w_x) + (\dot{u}_{\varepsilon x}\,,\,w_x) + [u_\varepsilon(1,t) +$$

$$+ \dot{u}_\varepsilon(1,t)]\ w(1) + (\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon\ (\dot{u}_\varepsilon),w)$$

$$\geq (f,w) + F(t)\ w(0).$$

Since we could have taken $-w$ in place of w , u_ε satisfies

$$(3.19)\qquad (\ddot{u}_\varepsilon\,,\,w) + (u_{\varepsilon x}\,,\,w_x) + (\dot{u}_{\varepsilon x}\,,\,w_x) + [u_\varepsilon(1,t)$$

$$+ \dot{u}_\varepsilon(1,t)]\ w(1) + (\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon(\dot{u}_\varepsilon),\ w)$$

$$= (f,w) + F(t)\ w(0)\ ,$$

for any $w \in H^1(\Omega)$, a.e. $t \in (0,T)$.

Conversely, taking $w = v - \dot{u}_\varepsilon$ in (3.19), and summing $J_\varepsilon(u_\varepsilon ; v) - J_\varepsilon(u_\varepsilon ; \dot{u}_\varepsilon)$ to both sides we reach (3.18) thanks to the convexity property

$$[\phi_\varepsilon(v) - \phi_\varepsilon(\dot{u}_\varepsilon)] - \phi'_\varepsilon(\dot{u}_\varepsilon)\,(v-\dot{u}_\varepsilon) \geq 0.$$

Step 3. Now by Galerkin's technique we construct a solution for problem (3.13)—(3.17), (3.19).

Let (V_h, p_h, r_h) be a convergent approximation of $H^1(\Omega)$, that is,

(3.20) $$V_h = \mathbb{R}^{N_h}, \; N_h \text{ integer}, \; N_h \to \infty \text{ as } h \downarrow 0 ;$$

(3.21) $p_h : V_h \to H^1(\Omega)$, isomorphism from V_h onto its closed range P_h in $H^1(\Omega)$;

(3.22) $r_h : H^1(\Omega) \to V_h$, linear continuous operator from $H^1(\Omega)$ onto V_h ;

(3.23) $$\lim_{h\to 0} |v - p_h r_h v|_1 = 0 \quad , \quad \forall\, v \in H^1(\Omega).$$

The Galerkin approximations to u_ε associated to the scheme (V_h, p_h, r_h) are mappings $u_\varepsilon^h : [0,T] \to V_h$, such that

$$(3.24)\qquad (p_h \ddot{u}^h_\varepsilon , p_h v_h) + (Dp_h u^h_\varepsilon , Dp_h v_h) + (Dp_h \dot{u}^h_\varepsilon, Dp_h v_h) + [p_h u^h_\varepsilon(1,t) + p_h \dot{u}^h_\varepsilon(1,t)]\, p_h v_h(1) + (\psi_\varepsilon(p_h u^h_\varepsilon)\phi'_\varepsilon(p_h \dot{u}^h_\varepsilon) , p_h v_h) = (f,p_h v_h) + F(t)p_h v_h(0) , \quad \forall\, v_h \in V_h ,$$

$$(3.25)\qquad u^h_\varepsilon(0) = 0 ,$$

$$(3.26)\qquad \dot{u}^h_\varepsilon(0) = 0 ,$$

where $D = \frac{\partial}{\partial x}$.

Conditions (3.24)—(3.26) for u^h_ε represent a second order ordinary differential system with dimension N_h and the required initial data. In view of the smoothness of the associated function, such a system has a unique local solution which can be extended to $[0,T]$ in case $(u^h_\varepsilon , \dot{u}^h_\varepsilon)$ is bounded on the interval. In the sequel we shall prove two a priori estimates (I and II) for (3.24) which will fulfill that boundedness condition, so generating a smooth $u^h_\varepsilon(t)$, $t \varepsilon [0,T]$.

For estimate I we take $v_h = \dot{u}^h_\varepsilon$ in (3.24), getting

$$(3.27)\qquad \frac{1}{2}\frac{d}{dt}|p_h\dot{u}^h_\varepsilon|^2 + \frac{1}{2}\frac{d}{dt}|Dp_h u^h_\varepsilon|^2 + |Dp_h\dot{u}^h_\varepsilon|^2$$

$$+ \frac{1}{2}\frac{d}{dt}[p_h u^h_\varepsilon(1,t)]^2 + [p_h\dot{u}^h_\varepsilon(1,t)]^2$$

$$+ \int_0^1 \psi_\varepsilon(p_h u^h_\varepsilon)\phi'_\varepsilon(p_h\dot{u}^h_\varepsilon)\, p_h\dot{u}^h_\varepsilon\, dx$$

$$= (f, p_h\dot{u}^h_\varepsilon) + F(t)\, p_h\dot{u}^h_\varepsilon(0,t).$$

Now we observe that

$$|p_h u^h_\varepsilon|_1^2(t) \leq 3\{[p_h u^h_\varepsilon(1,t)]^2 + |Dp_h u^h_\varepsilon|^2(t)\},$$

and also that we can always choose the regularizations so that

$$\psi_\varepsilon(p_h u^h_\varepsilon) \geq 0 \quad , \quad \phi'_\varepsilon(p_h\dot{u}^h_\varepsilon)\, p_h\dot{u}^h_\varepsilon \geq 0.$$

Hence, integrating (3.27) from 0 to t and using the initial conditions:

$$\frac{1}{2}|p_h\dot{u}^h_\varepsilon|^2(t) + \frac{1}{6}|p_h u^h_\varepsilon|_1^2(t) + \frac{1}{3}\int_0^t |p_h\dot{u}^h_\varepsilon|_1^2(\tau)d\tau$$

$$\leq \int_0^t \{|f|(\tau)|p_h\dot{u}^h_\varepsilon|(\tau) + |F(\tau)|\ |p_h\dot{u}^h_\varepsilon(0,\tau)|\}d\tau$$

$$\leq \alpha\int_0^t |p_h\dot{u}^h_\varepsilon|_1^2(\tau)d\tau + C^1_\alpha\int_0^\infty \{|f|^2(\tau) + F^2(\tau)\}d\tau,$$

for any $\alpha > 0$. Observe in the last step the use of the trace theorem in $H^1(\Omega)$ and the independence of the constant C^1_α with respect to ε, h and T.

Now we choose $\alpha = \frac{1}{6}$ to obtain estimate I:

$$(3.28)\qquad |p_h\dot{u}_\varepsilon^h|^2_{L^\infty(0,T);L^2(\Omega))} + |p_h u_\varepsilon^h|^2_{L^\infty(0,T;H^1(\Omega))}$$

$$+ |p_h\dot{u}_\varepsilon^h|^2_{L^2(0,T;H^1(\Omega))}$$

$$\leq 6C^1_{\frac{1}{6}} \{|f|^2_{L^2(0,\infty;L^2(\Omega))} + |F|^2_{L^2(0,\infty;\mathbb{R})}\}.$$

For estimate II we first differentiate equation (3.24) with respect to t:

$$(p_h\dddot{u}_\varepsilon^h , p_h v_h) + (Dp_h\dot{u}_\varepsilon^h , Dp_h v_h) + (Dp_h\ddot{u}_\varepsilon^h , Dp_h v_h)$$

$$+ [p_h\dot{u}_\varepsilon^h(1,t) + p_h\ddot{u}_\varepsilon^h(1,t)]\ p_h v_h(1) +$$

$$+ (\frac{d}{dt}\ \psi_\varepsilon(p_h u_\varepsilon^h)\phi_\varepsilon'(p_h\dot{u}_\varepsilon^h),\ p_h v_h) +$$

$$+ (\psi_\varepsilon(p_h u_\varepsilon^h)\ \frac{d}{dt}\phi_\varepsilon'(p_h\dot{u}_\varepsilon^h)\ ,\ p_h v_h)$$

$$= (\dot{f} , p_h v_h) + \dot{F}(t)p_h v_h(0)\ ,$$

$$v_h \in V_h\ ,$$

and then take $v_h = \ddot{u}_\varepsilon^h$, to get

$$(3.29)\qquad \frac{1}{2}\frac{d}{dt}|p_h\ddot{u}^h_\varepsilon|^2(t) + \frac{1}{2}\frac{d}{dt}|Dp_h\dot{u}^h_\varepsilon|^2(t) +$$

$$+ |Dp_h\ddot{u}^h_\varepsilon|^2(t) + \frac{1}{2}\frac{d}{dt}[p_h\dot{u}^h_\varepsilon(1,t)]^2 +$$

$$+ [p_h\ddot{u}^h_\varepsilon(1,t)]^2 +$$

$$+ (\psi'_\varepsilon(p_h u^h_\varepsilon)\, p_h\, \dot{u}^h_\varepsilon \phi'_\varepsilon(p_h\dot{u}^h_\varepsilon),\ p_h\ddot{u}^h_\varepsilon)$$

$$+ (\psi_\varepsilon(p_h u^h_\varepsilon)\ \frac{d}{dt}\phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t)]\ ,\ p_h\ddot{u}^h_\varepsilon)$$

$$= (\dot{f}\ , p_h\ddot{u}^h_\varepsilon) + \dot{F}(t)\ p_h\ddot{u}^h_\varepsilon(0,t).$$

Notice that we have

$$(\psi_\varepsilon(p_h u^h_\varepsilon)\ \frac{d}{dt}\ \phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t)]\ ,\ \frac{d}{dt}\ p_h\dot{u}^h_\varepsilon(t)) =$$

$$= (\psi_\varepsilon(p_h u^h_\varepsilon)\ \lim_{\Delta t\to 0}\ \frac{\phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t+\Delta t)] - \phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t)]}{\Delta t}\ ,$$

$$\lim_{\Delta t\to 0}\ \frac{p_h\dot{u}^h_\varepsilon(t+\Delta t) - p_h\dot{u}^h_\varepsilon(t)}{\Delta t}\ ) =$$

$$= \lim_{\Delta t\to 0}\ \frac{1}{\Delta t^2}\int_0^1 \psi_\varepsilon(p_h u^h_\varepsilon)\{\phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t+\Delta t)] -$$

$$- \phi'_\varepsilon[p_h\dot{u}^h_\varepsilon(t)]\}\ \{[p_h\dot{u}^h_\varepsilon(t+\Delta t)] - [p_h\dot{u}^h_\varepsilon(t)]\}dx$$

$$\geq 0\ ,$$

by the monotonicity of ϕ'_ε and the positivity of ψ_ε. Furthermore,

$$|\psi'_\varepsilon(\lambda)| \leq \gamma F K \ , \ |\phi'_\varepsilon(\lambda)| \leq 1 \ ,$$

for any argument λ, and thus (3.29) implies, after integrating from 0 to t:

$$\frac{1}{2}|p_h\ddot{u}^h_\varepsilon|^2(t) - \frac{1}{2}|p_h\ddot{u}^h_\varepsilon|^2(0) + \frac{1}{2}|Dp_h\dot{u}^h_\varepsilon|^2(t) +$$
$$+ \int_0^t |Dp_h\ddot{u}^h_\varepsilon|^2(\tau)d\tau + \frac{1}{2}[p_h\dot{u}^h_\varepsilon(1,t)]^2$$
$$+ \int_0^t [p_h\ddot{u}^h_\varepsilon(1,\tau)]^2 d\tau \leq \gamma F K \int_0^t |p_h\dot{u}^h_\varepsilon|(\tau)|p_h\ddot{u}^h_\varepsilon|(\tau)d\tau$$
$$+ \int_0^t |\dot{f}|(\tau)|p_h\ddot{u}^h_\varepsilon|(\tau)d\tau + \int_0^t |\dot{F}(\tau)| \ |p_h\ddot{u}^h_\varepsilon(0,\tau)|d\tau.$$

If we operate in this inequality through the trace theorem in $H^1(\Omega)$ and Cauchy-Schwarz inequality we reach

$$\frac{1}{2}|p_h\ddot{u}^h_\varepsilon|^2(t) + \frac{1}{6}|p_h\dot{u}^h_\varepsilon|_1^2(t) + \frac{1}{3}\int_0^t |p_h\ddot{u}^h_\varepsilon|_1^2(\tau)d\tau$$
$$\leq \frac{1}{2}|p_h\ddot{u}^h_\varepsilon|^2(0) + \alpha\int_0^t |p_h\ddot{u}^h_\varepsilon|_1^2(\tau)d\tau$$
$$+ C_\alpha^2 \int_0^t \{|p_h\dot{u}^h_\varepsilon|_1^2(\tau) + |\dot{f}|^2(\tau) +$$

$$+ |\dot{F}(\tau)|^2 \} d\tau \ ,$$

so that, if we choose $\alpha = \frac{1}{6}$, and take (3.28) into account, we get

$$(3.30) \qquad |p_h \ddot{u}_\varepsilon^h|^2(t) + |p_h \dot{u}_\varepsilon^h|_1^2(t) + \int_0^t |p_h \ddot{u}_\varepsilon^h|_1^2(\tau)d\tau$$

$$\leq 3|p_h \ddot{u}_\varepsilon^h|^2(0) + 6\ C_{\frac{1}{6}}^2 \{ \int_0^\infty |\dot{f}|^2(\tau)d\tau +$$

$$+ \int_0^\infty |\dot{F}(\tau)|^2 d\tau + 6\ C_{\frac{1}{6}}^1 \left[\int_0^\infty |f|^2(\tau)d\tau + \right.$$

$$\left. + \int_0^\infty |F(\tau)|^2 d\tau \right] \}.$$

We emphasize the independence of the constants $C_{\frac{1}{6}}^1$ and $C_{\frac{1}{6}}^2$ on ε, h and T.

But we still need a bound for $|p_h \ddot{u}_\varepsilon^h|(0)$. For this we evaluate (3.24) at $t = 0$ and take into consideration the initial conditions and hypothesis (3.4) of the theorem, obtaining

$$(p_h \ddot{u}_\varepsilon^h(0),\ p_h v_h) = (f(0),\ p_h v_h) \quad .$$

$$v_h \in V_h.$$

Choosing $v_h = \ddot{u}_\varepsilon^h(0)$, we get

$$(3.31) \qquad |p_h \ddot{u}_\varepsilon^h|(0) \leq |f|(0),$$

which, combined with (3.30) gives estimate II:

$$(3.32)\qquad |p_h\ddot{u}^h_\varepsilon|^2_{L^\infty(0,T;L^2(\Omega))} + |p_h\dot{u}^h_\varepsilon|^2_{L^\infty(0,T;H^1(\Omega))} + |p_h\ddot{u}^h_\varepsilon|^2_{L^2(0,T;H^1(\Omega))}$$

$$\leq 3|f|^2(0) + 6\, C^2_{\frac{1}{6}}\{\int_0^\infty |\dot{f}|^2 d\tau + \int_0^\infty |\dot{F}(\tau)|^2 d\tau + 6\, C^1_{\frac{1}{6}}\left[\int_0^\infty |f|^2 d\tau + \int_0^\infty |F(\tau)|^2 d\tau\right]\}.$$

Estimates I and II tell us that the Galerkin approximations are well defined on $[0,T]$, for any given $0<T\leq\infty$, and much more, that

(i) $p_h u^h_\varepsilon$ are all in a bounded set of $L^\infty(0,T;H^1(\Omega))$,

(ii) $p_h\dot{u}^h_\varepsilon$ are all in a bounded set of $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$,

(iii) $p_h\ddot{u}^h_\varepsilon$ are all in a bounded set of $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

For a finite T, $p_h u^h_\varepsilon$ remains in a bounded set of

$L^2(0,T;H^1(\Omega))$, this set now depending on T. Hence $p_h u_\varepsilon^h$ and $p_h \dot{u}_\varepsilon^h$ remain in a bounded set of $H^1(Q_T)$, where $Q_T = (0,T)\times\Omega$. If we invoke Rellich's theorem (see [3]) we can pass to the limit in the dimension $(h \downarrow 0)$ and conclude the existence of a function $u_\varepsilon \in L^2(Q_T)$ such that $\dot{u}_\varepsilon \in L^2(Q_T)$ and

$$(3.33) \qquad \begin{cases} p_h u_\varepsilon^h \to u_\varepsilon & \text{a.e. in } Q_T \ , \\ \\ p_h \dot{u}_\varepsilon^h \to u_\varepsilon & \text{a.e. in } Q_T \ , \end{cases}$$

where we really mean convergence of a certain subsequence.

In view of (i)—(iii) we can further extract a subsequence, still indexed by the same h, such that

$$(3.34) \qquad \begin{cases} p_h u_\varepsilon^h \to u_\varepsilon & \text{weakly* in } L^\infty(0,T;H^1(\Omega)), \\ \\ p_h \dot{u}_\varepsilon^h \to \dot{u}_\varepsilon & \begin{cases} \text{weakly* in } L^\infty(0,T;H^1(\Omega)) \\ \text{weakly in } L^2(0,T;H^1(\Omega)), \end{cases} \\ \\ p_h \ddot{u}_\varepsilon^h \to \ddot{u}_\varepsilon & \begin{cases} \text{weakly* in } L^\infty(0,T;L^2(\Omega)) \\ \text{weakly in } L^2(0,T;H^1(\Omega)). \end{cases} \end{cases}$$

Hence, if we take the limit $h \downarrow 0$ in

$$\int_0^T (p_h \ddot{u}_\varepsilon^h \ , \ p_h r_h v) \ \alpha(t)dt + \int_0^T (Dp_h u_\varepsilon^h \ , \ Dp_h r_h v)\alpha(t)dt$$

$$+ \int_0^T (Dp_h \dot{u}_\varepsilon^h \ , \ Dp_h r_h v)\alpha(t)dt + \int_0^T [p_h u_\varepsilon^h(1,t) +$$

$$+ p_h \dot{u}_\varepsilon^h(1,t)\big] \, p_h r_h v(1)\, \alpha(t)dt \;+$$

$$+ \int_0^T (\psi_\varepsilon(p_h u_\varepsilon^h)\phi'_\varepsilon(p_h \dot{u}_\varepsilon^h),\; p_h r_h v)\alpha(t)dt$$

$$= \int_0^T (f\, ,\, p_h r_h v)\alpha(t)dt + \int_0^T F(t) p_h r_h v(0)\alpha(t)dt,$$

$$\forall\; v \in H^1(\Omega)\;,\; \alpha \in L^1(0,T;\mathbb{R})\;,$$

we obtain

$$(3.35) \qquad \int_0^T (\ddot{u}_\varepsilon, v)\;\alpha(t)dt + \int_0^T (Du_\varepsilon, Dv)\;\alpha(t)dt$$

$$+ \int_0^T (D\dot{u}_\varepsilon, Dv)\;\alpha(t)dt + \int_0^T \big[u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)\big] v(1)\alpha(t)dt$$

$$+ \int_0^T (\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon(\dot{u}_\varepsilon), v)\;\alpha(t)dt \;=$$

$$= \int_0^T (f,v)\alpha(t)dt + \int_0^T F(t)v(0)\alpha(t)dt,$$

$$\forall\; v \in H^1(\Omega)\;,\; \alpha \in L^1(0,T;\mathbb{R})\;,$$

since $p_h r_h v \to v$ strongly in $H^1(\Omega)$ and

$$\lim_{h\to 0} \phi'_\varepsilon(p_h \dot{u}_\varepsilon^h)\psi_\varepsilon(p_h u_\varepsilon^h) = \phi'_\varepsilon(\dot{u}_\varepsilon)\psi_\varepsilon(u_\varepsilon) \quad , \quad \text{a.e. in } Q_T.$$

On the other hand, for any function $\alpha \in C^1(0,T)$ with $\alpha(0) = 1$ and $\alpha(T) = 0$, and $v \in H^1(\Omega)$, we have

$$0 = -(p_h u_\varepsilon^h(0), v) = \int_0^T (p_h \dot{u}_\varepsilon^h , v)\alpha \, dt + \int_0^T (p_h u_\varepsilon^h , v)\alpha' \, dt$$

$$\to \int_0^T (\dot{u}_\varepsilon , v)\alpha dt + \int_0^T (u_\varepsilon , v)\alpha' dt = -(u_\varepsilon(0), v) ,$$

$$0 = -(p_h \dot{u}_\varepsilon^h(0), v) = \int_0^T (p_h \ddot{u}_\varepsilon^h, v)\alpha dt + \int_0^T (p_h \dot{u}_\varepsilon^h , v)\alpha' dt$$

$$\to \int_0^T (\ddot{u}_\varepsilon , v)\alpha dt + \int_0^T (\dot{u}_\varepsilon , v)\alpha' dt = -(\dot{u}_\varepsilon(0), v),$$

as $h \downarrow 0$, that is, $u_\varepsilon(0) = \dot{u}_\varepsilon(0) = 0$.

With this result and those expressed in formulas (3.34) and (3.35) we conclude that the limit function u_ε is a "weak" solution of problem (3.13)—(3.17), (3.19), the adjective being defined by (3.35).

We need then, following our proof program, to show now that u_ε is really a "strong" solution in the sense of (3.19) or (3.18).

For this we take $v - \dot{u}_\varepsilon$ for v in (3.35), add $[J_\varepsilon(u_\varepsilon; v) - J_\varepsilon(u_\varepsilon; \dot{u}_\varepsilon)]$ under the sign $\int_0^T$ to both sides, and use the convexity property of ϕ_ε. We reach:

$$(3.36) \qquad \int_0^T \{(\ddot{u}_\varepsilon , v - \dot{u}_\varepsilon) + (u_{\varepsilon x} , v_x - \dot{u}_{\varepsilon x}) +$$

$$+ (\dot{u}_{\varepsilon x} , v_x - \dot{u}_{\varepsilon x}) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)] \, [v(1) - \dot{u}_\varepsilon(1,t)]$$

$$+ J_\varepsilon(u_\varepsilon\,;v) - J_\varepsilon(u_\varepsilon\,;\dot{u}_\varepsilon)\}\alpha(t)dt$$

$$\geq \int_0^T \{(f,v-\dot{u}_\varepsilon) + F(t)\ [v(0)-\dot{u}_\varepsilon(0,t)]\}\alpha(t)dt,$$

$$\forall\ v \in H^1(\Omega)\ ,\ \alpha \in L^1(0,T;\mathbb{R}),\ \alpha \geq 0.$$

Now let $s \in (0,T)$ be fixed arbitrarily for the moment. We take the family $\mathcal{O}_k$ of neighborhoods of s:

$$\mathcal{O}_k = (s - \frac{1}{k}\ ,\ s + \frac{1}{k})\ ,$$

and let $\alpha(t)$ be defined by

$$\alpha(t) = \begin{cases} 0 & \text{if } t \notin \mathcal{O}_k\ , \\ 1 & \text{if } t \in \mathcal{O}_k\ . \end{cases}$$

Then (3.36) yields

$$\int_{\mathcal{O}_k} \{(\ddot{u}_\varepsilon\,, v) + (u_{\varepsilon x}\,, v_x) + (\dot{u}_{\varepsilon x}\,, v_x) +$$

$$+ [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\ v(1) + J_\varepsilon(u_\varepsilon\,;v) - (f,v)$$

$$- F(t)\ v(0)\}\ dt - \int_{\mathcal{O}_k} \{(\ddot{u}_\varepsilon\,, \dot{u}_\varepsilon) +$$

$$+ (u_{\varepsilon x}\,, \dot{u}_{\varepsilon x}) + (\dot{u}_{\varepsilon x}\,, \dot{u}_{\varepsilon x}) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\dot{u}_\varepsilon(1,t)$$

$$+ J_\varepsilon(u_\varepsilon ; \dot{u}_\varepsilon) - (f , \dot{u}_\varepsilon) - F(t)\, \dot{u}_\varepsilon(0,t)\}dt \geq 0,$$

from which again follows — denoting the measure of $\mathcal{O}_k$ by $|\mathcal{O}_k|$ —

$$(3.37) \qquad (|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} \ddot{u}_\varepsilon \, dt , v) + (|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} u_{\varepsilon x} \, dt , v_x) +$$

$$+ (|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} \dot{u}_{\varepsilon x} \, dt , v_x) + |\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} \left[u_\varepsilon(1,t)+\dot{u}_\varepsilon(1,t)\right] dt \; v(1)$$

$$+ \gamma K F \, (|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} \psi_\varepsilon(u_\varepsilon)\, dt, \phi_\varepsilon(v)) -$$

$$- (|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} f \, dt , v) - |\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} F \, dt \; v(0) -$$

$$- |\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} \{(\ddot{u}_\varepsilon , \dot{u}_\varepsilon) + (u_{\varepsilon x} , \dot{u}_{\varepsilon x}) + (\dot{u}_{\varepsilon x} , \dot{u}_{\varepsilon x})$$

$$+ \left[u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)\right] \dot{u}_\varepsilon(1,t) + J_\varepsilon(u_\varepsilon ; \dot{u}_\varepsilon) - (f , \dot{u}_\varepsilon)$$

$$- F(t)\, \dot{u}_\varepsilon(0,t)\} \, dt \geq 0.$$

But, in general, for a scalar-valued or vector-valued integrable function G, we have, by a theorem of Lebesgue's,

$$|\mathcal{O}_k|^{-1} \int_{\mathcal{O}_k} G(t)dt \to G(s) \quad , \quad k \to \infty \quad ,$$

for almost all s.

Therefore, we conclude from (3.37) that, except possibly for s in a set of measure zero, we have

$$(\ddot{u}_\varepsilon(s), v-\dot{u}_\varepsilon(s)) + (u_{\varepsilon x}(s), v_x-\dot{u}_{\varepsilon x}(s)) + (\dot{u}_{\varepsilon x}(s), v_x-\dot{u}_{\varepsilon x}(s))$$

$$+ [u_\varepsilon(1,s) + \dot{u}_\varepsilon(1,s)]\,[v(1) - \dot{u}_\varepsilon(1,s)] + J_\varepsilon(u_\varepsilon(s); v)$$

$$- J_\varepsilon(u_\varepsilon(s); \dot{u}_\varepsilon(s)) \geq (f(s), v-\dot{u}_\varepsilon(s)) + F(s)\,[v(0)-\dot{u}_\varepsilon(0,s)],$$

which is (3.18).

Step 4. We now are able to exhibit a function u with the desired properties (3.5)—(3.9), by passing to the limit $\varepsilon \to 0$ in the defining conditions for u_ε. In view of estimates I and II (the bounds are independent of ε) we can extract from $\{u_\varepsilon\}$ a subsequence, still denoted by $\{u_\varepsilon\}$, such that there exists $u \in L^\infty(0,T;H^1(\Omega))$ with

$$(3.38)\quad \begin{cases} u_\varepsilon \to u & \text{weakly* in } L^\infty(0,T;H^1(\Omega)), \\ \dot{u}_\varepsilon \to \dot{u} & \begin{cases}\text{weakly* in } L^\infty(0,T;H^1(\Omega)), \\ \text{weakly in } L^2(0,T;H^1(\Omega)),\end{cases} \\ \ddot{u}_\varepsilon \to \ddot{u} & \begin{cases}\text{weakly* in } L^\infty(0,T;L^2(\Omega)), \\ \text{weakly in } L^2(0,T;H^1(\Omega)),\end{cases} \\ u_\varepsilon \to u & \text{a.e. in } Q_T, \\ \dot{u}_\varepsilon \to \dot{u} & \text{a.e. in } Q_T, \end{cases}$$

by the same argument used in step 3 for the study of the convergence $p_h u_\varepsilon^h \to u_\varepsilon$. The function u is the candidate for solution.

Conditions (3.5) , (3.6) are satisfied, and the initial conditions (3.7), (3.8) are obtained for u with exactly the same argument in step 3 for u_ε. To conclude the proof of the theorem we need then, finally, to show that u satisfies (3.9).

Relation (3.18) yields, for any $v \in L^1(0,T;H^1(\Omega))$,

$$\int_0^T \{(\ddot{u}_\varepsilon , v) + (u_{\varepsilon x} , v_x) + (\dot{u}_{\varepsilon x} , v_x) + [u_\varepsilon(1,t) + \dot{u}_\varepsilon(1,t)]\, v(1,t) - (f,v-\dot{u}_\varepsilon) - F(t)\,[v(0,t) - \dot{u}_\varepsilon(0,t)]\}dt + \int_0^T J(u_\varepsilon ; v)dt \geq$$

(3.39)

$$\frac{1}{2}|\dot{u}_\varepsilon|^2(T) + \frac{1}{2}|u_{\varepsilon x}|^2 (T) + \frac{1}{2} u_\varepsilon^2(1,T) + \int_0^T |\dot{u}_{\varepsilon x}|^2 (t)dt + \int_0^T \dot{u}^2 (1,t)dt + \int_0^T J_\varepsilon(u_\varepsilon ; \dot{u}_\varepsilon)dt.$$

But, by the weak* lower semi-continuity property of the norm in a Banach space and (3.38),

$$\liminf_{\varepsilon\to 0} \left[\frac{1}{2}|\dot{u}_\varepsilon|^2(T) + \frac{1}{2}|u_{\varepsilon x}|^2 (T) + \frac{1}{2} u_\varepsilon^2(1,T) + \int_0^T |\dot{u}_{\varepsilon x}|^2 (t)dt + \int_0^T \dot{u}_\varepsilon^2(1,t)dt\right] \geq$$

$$\geq \left[\frac{1}{2}|\dot{u}|^2(T) + \frac{1}{2}|u_x|^2(T) + \frac{1}{2}u^2(1,T) + \int_0^T |\dot{u}_x|^2(t)dt\right.$$

$$+ \int_0^T \dot{u}^2(1,t)dt = \int_0^T \{(\ddot{u}\,,\dot{u}) + (u_x\,,\dot{u}_x) + (\dot{u}_x\,,\dot{u}_x) +$$

$$+ [u(1,t) + \dot{u}(1,t)]\,\dot{u}(1,t)\}dt.$$

By (3.38) and the definition of ψ_ε and ϕ_ε ,

$$\int_0^T \{J_\varepsilon(u_\varepsilon\,;v) - J(u\,;v)\}dt = \int_0^T\int_0^1 \{\psi_\varepsilon(u_\varepsilon)\phi_\varepsilon(v) -$$

$$- \gamma F K\, H(x+u-1)\ (x+u-1)|v|\}dx\ dt\ =$$

$$= \int_0^T\int_0^1 \{\psi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u)\}\phi_\varepsilon(v)\ dx\ dt\ +$$

$$+ \int_0^T\int_0^1 \{\phi_\varepsilon(v) - |v|\}\psi_\varepsilon(u)\ dx\ dt\ +$$

$$+ \int_0^T\int_0^1 \{\psi_\varepsilon(u) - \gamma F K H(x+u-1)\ (x+u-1)\}|v| dx\ dt$$

goes to zero with ε , since ϕ_ε and ψ_ε are Lipschitz continuous with constants independent of ε . Also by (3.38) and the mentioned property of ϕ_ε and ψ_ε ,

$$\int_0^T \{J_\varepsilon(u_\varepsilon\,;\dot{u}_\varepsilon) - J(u\,;\dot{u})\}dt = \int_0^T\int_0^1 \{\psi_\varepsilon(u_\varepsilon)\phi_\varepsilon(\dot{u}_\varepsilon) -$$

$$- \gamma FKH(x+u-1)\,(x+u-1)|\dot{u}|\}dx\,dt =$$

$$= \int_0^T \int_0^1 \{[\psi_\varepsilon(u_\varepsilon) - \psi_\varepsilon(u)]\phi_\varepsilon(\dot{u}_\varepsilon) + [\phi_\varepsilon(\dot{u}_\varepsilon) - \phi_\varepsilon(\dot{u})]\psi_\varepsilon(u) +$$

$$+ [\psi_\varepsilon(u) - \gamma KFH(x+u-1)\,(x+u-1)]\phi_\varepsilon(\dot{u}) +$$

$$+ \gamma KF[\phi_\varepsilon(\dot{u}) - |\dot{u}|]H(x+u-1)\,(x+u-1)\}dx\,dt$$

$$\to 0 \quad , \quad \text{as} \quad \varepsilon \to 0.$$

Therefore (3.39) gives, through (3.38),

$$(3.40) \qquad \int_0^T \{(\ddot{u}, v-\dot{u}) + (u_x, v_x - \dot{u}_x) + (\dot{u}_x, v_x - \dot{u}_x)$$

$$+ [u(1,t) + \dot{u}(1,t)]\,[v(1,t) - \dot{u}(1,t)] + J(u\,;v) -$$

$$- J(u\,;\dot{u}) - (f, v-\dot{u}) - F(t)[v(0,t) - \dot{u}(0,t)]\}dt \geq 0 \ ,$$

$$\forall\ v \in L^1(0,T;H^1(\Omega)).$$

From (3.40) we pass to the pointwise inequality (3.9) by the same procedure applied to relation (3.36) in step 3, now with

$$v(t) = \begin{cases} \dot{u}(t) & \text{if} \quad t \notin \mathcal{O}_k \ , \\ v & \text{if} \quad t \in \mathcal{O}_k \ , \end{cases}$$

where v is any function in $H^1(\Omega)$.

Thus the proof of theorem 3.1 is brought to an end.

Remark. Since estimates I and II are also true for $T = \infty$, we actually have:

(3.41)
$$\begin{cases} u \in L^\infty(0,\infty;H^1(\Omega)) \ , \\ \dot{u} \in L^\infty(0,\infty;H^1(\Omega)) \cap L^2(0,\infty;H^1(\Omega)), \\ \ddot{u} \in L^\infty(0,\infty;L^2(\Omega)) \cap L^2(0,\infty;H^1(\Omega)). \end{cases}$$

Now we turn ourselves to the question of the asymptotic behavior of the motion of the pile when t increases without bound. The following stability result was obtained:

Theorem 3.2. Under the conditions on the data assumed in theorem 3.1, the motion is stable and tends to rest as $t \to \infty$, in the sense that

(3.42) $|u|_1(t)$ is bounded on $[0, +\infty)$,

(3.43) $|\dot{u}|_1(t) \to 0$ as $t \to \infty$.

Remark. Claim (3.42) says that the $L^2(\Omega)$-norms of the displacement and strain fields (a kind of "average" of the field on the pile) remain bounded during all times, and (3.43) says that the velocity field and the rate of deformation of the pile, measured for each t in the same $L^2(\Omega)$-norm, decay to zero as we proceed into the remote future. The claims allow us also to infer the behavior of the stress field, in average, in that time region.

Proof. The first claim is a consequence of the basic estimate (3.28):

$$|u|_{L^\infty(0,\infty;H^1(\Omega))} \leq \sqrt{6}\, C_{\frac{1}{6}}^{\frac{1}{1}} \left[|f|^2_{L^2(0,\infty;L^2(\Omega))} + |F|^2_{L^2(0,\infty;\mathbb{R})} \right]^{\frac{1}{2}} .$$

For the second, observe that

$$\dot{u}(x, t_2) - \dot{u}(x, t_1) = \int_{t_1}^{t_2} \frac{d}{d\tau} \dot{u}(x, \tau) d\tau \quad , \quad t_1 , t_2 > 0 ,$$

so that

$$|\dot{u}(t_2) - \dot{u}(t_1)|_1 \leq \int_{t_1}^{t_2} |\ddot{u}|_1 (\tau) d\tau \ ,$$

that is,

$$||\dot{u}|_1 (t_2) - |\dot{u}|_1 (t_1)| \leq \int_{t_1}^{t_2} |\ddot{u}|_1 (\tau) d\tau .$$

Since by (3.41) $\ddot{u} \in L^2(0,\infty;H^1(\Omega))$, given $\varepsilon > 0$ arbitrary, there exists $\delta > 0$ such that, if $|t_1 - t_2| < \delta$ then

$$\int_{t_1}^{t_2} |\ddot{u}|_1(\tau)d\tau \leq |t_1 - t_2|^{\frac{1}{2}} \left[\int_{t_1}^{t_2} |\ddot{u}|_1^2(\tau)d\tau\right]^{\frac{1}{2}} < \varepsilon ,$$

hence

$$||\dot{u}|_1(t_2) - |\dot{u}|_1(t_1)| < \varepsilon.$$

This tells us that $|\dot{u}|_1(t)$ is uniformly continuous on $[0 , +\infty)$. Therefore (3.43) is true, because by (3.41) $|\dot{u}|_1$ is square-integrable on $[0 , + \infty)$.

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