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EIGENVALUE TREATMENT OF COSMOLOGICAL MODELS

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ABSTRACT

From the decomposition of Weyl tensor into its electric and magnetic parts, we formulate the eigenvalue problem for cosmological models, and we use quasi-Maxwellian form of Einstein's equation to propagate it along a time-like congruence. Three related theorems are presented.

It is well known(1) that Weyl tensor can be decomposed into symmetric, trace-free electric $E_{\alpha\beta}$ and magnetic $H_{\alpha\beta}$ parts, relative to a velocity field V^{α}. Accordingly, we construct the Petrov tensor

$$F_{\alpha\beta} = E_{\alpha\beta} + iH_{\alpha\beta} \tag{1}$$

with the properties

$$F_{\alpha\beta}\, V^{\beta} = 0$$

$$F^{\alpha}{}_{\alpha} = 0 \quad ; \quad F_{\alpha\beta} = F_{\beta\alpha} \tag{2}$$

This tensor is defined in the 3-space orthogonal to V^{α} and carries all the information contained in Weyl tensor. Then we define the eigenvalue problem for $F_{\alpha\beta}$:

$$(F_{\alpha\beta} - \lambda g_{\alpha\beta})\, Z^{\beta} = 0 \tag{3}$$

From (2) it follows that $\vec{V}$ is an eigenvector of $F_{\alpha\beta}$ with null eigenvalue. If the eigenvalue associated to a generic eigenvector $\vec{L}$ is non-null, then $\vec{L}$ rests on the 3-space orthogonal to $\vec{V}$. In what follows we assume that $\vec{V}$ is time-like and the unique eigenvector with null eigenvalue. The solutions $\{\lambda_{(i)}, \vec{Z}_{(i)}\}$ of the eigenvalue problem (3) provide a classification of gravitational fields, first made by A. Petrov[(2)*].

Now the eigenvalue problem is of algebraic nature and purely local. Having in mind that gravitational field invariants are constructed with eigenvalues of $F_{\alpha\beta}$, we are led to ask about the modifications of properties associated to Petrov eigenvectors/eigenvalues that can be measured by an observer along his path. To follow the eigenvalue problem along a path on the space-time manifold we use quasi-Maxwellian form of Einstein's equation relative to $V^{\alpha}(1)$, projected on the eigen vectors $\{ Z^{\alpha}_{(i)}\}$. A straightforward calculation gives

$$\left(\lambda_{(i)} Z^{\alpha}_{(i)}\right)_{\|\alpha} - \lambda_{(i)} \dot{Z}^{\alpha}_{(i)} V_{\alpha} - F_{\alpha\beta} Z^{\alpha\|\beta}_{(i)} - i F_{\alpha}{}^{\mu} \sigma_{\mu\beta} V_{\lambda} \eta^{\lambda\beta\alpha}{}_{\varepsilon} Z^{\varepsilon}_{(i)} +$$

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* Remark that by choosing time-like observers with $V^{\alpha} = \delta^{\alpha}_{0}$, $F^{\alpha}{}_{\beta}$ reduces to the form M+iN where M,N are 3x3 real matrices such that

$$M = \begin{pmatrix} C_{2323} & C_{2313} & C_{2312} \\ C_{3123} & C_{3131} & C_{3112} \\ C_{1223} & C_{1231} & C_{1212} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} C_{2310} & C_{2320} & C_{2330} \\ C_{3110} & C_{3120} & C_{3130} \\ C_{1210} & C_{1220} & C_{1230} \end{pmatrix}.$$

Locally, this is just Petrov's choice of matrices for classifying gravitational fields.

$$- i(3\lambda_{(i)} + \rho + p)\omega_\alpha Z^\alpha_{(i)} - \frac{1}{3}\rho_{|\alpha} Z^\alpha_{(i)} = 0 \qquad (4a)$$

$$\dot{\lambda}_{(i)} Z^\sigma_{(i)} Z_{(j)\sigma} + \lambda_{(i)} \dot{Z}^\sigma_{(i)} Z_{(j)\sigma} - Z^\mu_{(j)} \dot{Z}_{(i)\mu} \lambda_{(j)} + \frac{i}{2} \lambda_{(i)|\nu} Z^\alpha_{(i)} \eta_{\sigma\lambda}{}^\nu{}_\alpha V^\lambda Z^\sigma_{(j)} +$$

$$+ \frac{i}{2} \lambda_{(i)} Z^\alpha_{(i)\|\nu} \eta^\sigma{}_\lambda{}^\nu{}_\alpha V^\lambda Z_{\sigma(j)} - \frac{i}{2} F_\alpha{}^\mu Z_{(i)\mu\|\nu} \eta^{\sigma\lambda}{}_\nu{}^\alpha V_\lambda Z_{\sigma(j)} +$$

$$+ \frac{i}{2} \lambda_{(j)|\nu} Z^\alpha_{(j)} \eta_{\sigma\lambda}{}^\nu{}_\alpha V^\lambda Z^\sigma_{(i)} + \frac{i}{2} \lambda_{(j)} Z^\alpha_{(j)\|\nu} \eta^{\sigma\lambda\nu}{}_\alpha V_\lambda Z_{(i)\sigma} +$$

$$- \frac{i}{2} F^{\alpha\mu} Z_{(i)\mu\|\nu} \eta^{\sigma\lambda\nu}{}_\alpha V_\lambda Z_{(i)\sigma} + \theta\lambda_{(i)} Z^\sigma_{(j)} Z_{\sigma(i)} +$$

$$- \frac{1}{2} \omega^{\sigma\nu} \left[\lambda_{(j)} \dot{Z}_{(j)\nu} Z_{(i)\sigma} - \lambda_{(i)} Z_{(j)\sigma} Z_{(i)\nu}\right] - \frac{1}{3}\theta \lambda_{(i)} Z^\alpha_{(i)} Z_{(j)\alpha} +$$

$$- i \dot{V}^\alpha \eta^\sigma{}_{\lambda\alpha\beta} V^\lambda \left(Z_{(i)\sigma} Z^\beta_{(j)} \lambda_{(j)} - Z_{(j)\sigma} Z^\beta_{(i)} \lambda_{(i)} \right) - \frac{1}{6}\theta \lambda_{(j)} Z_{(i)\nu} Z^\nu_{(j)} +$$

$$- \frac{1}{6} \theta\lambda_{(i)} Z_{\nu(i)} Z^\nu_{(j)} - \frac{1}{2} \lambda_{(j)} Z_{(j)\nu} Z_{(i)\rho} \sigma^{\nu\rho} - \frac{1}{2} \lambda_{(i)} Z_{\varepsilon(j)} Z_{\nu(i)} \sigma^{\varepsilon\nu} +$$

$$- \eta^{\sigma\nu\rho\tau} \eta^{\varepsilon\lambda\alpha\beta} V_\rho V_\lambda \sigma_{\beta\nu} F_{\tau\alpha} Z_{\varepsilon(j)} Z_{\sigma(i)} = - \frac{1}{4}(\rho+p)\sigma^{\varepsilon\rho} Z_{\varepsilon(j)} Z_{\rho(i)} \qquad (4b)$$

$$\text{for}(i \neq j)$$

where ($\|$) denotes covariant derivative and $\dot{\phi} = \phi_{\|\alpha} V^\alpha$. $\sigma^{\mu\nu}$ and θ are respectively the shear and expansion of the congruence of cruves with 4-velocity V^α; and ω^μ is the vorticity vector.

Let us restrict our discussion here to the case in which the Weyl tensor is of electric type for the observer $\vec{V}$ ($H_{\alpha\beta} = 0$). This restricts us to Petrov types I and D only[3]. Then we use equations (4) to prove the following theorems:

Theorem 1: For a shear-free observer with four velocity $\vec{V}$, if two eigenvalues of $E_{\alpha\beta}$ coincide at a given point, they shall coincide along $\vec{V}$.

Theorem 2: If an eigenvector $\vec{L}$ of $E_{\alpha\beta}$ is a Killing vector, then the variation of the density ρ along L measures the variation of the corresponding eigenvalue along $\vec{L}$.

Theorem 3: Under the assumptions

(i) $\dot{\vec{\omega}} = \xi \vec{\omega}$

(ii) the eigenvalue of $\sigma_{\alpha\beta}$ and $E_{\alpha\beta}$ coincide

(iii) expansion is constant (steady state model)

if one of the eigenvalues of shear is null, then the geometry is Petrov type D (if not trivial).

These theorems follow directly from an examination of equations (4a,b) and the equation of evolution of the shear. Theorem 1 tells that in the absence of shear the Petrov type (I or D) is an invariant for the observer $\vec{V}$ along his world line. Actually, we can exhibit simple models in which the shear has the role of changing Petrov type (I $\rightarrow$ D $\rightarrow$ I etc) along $\vec{V}$. Theorem 2 tells us that in the absence of matter, the eigenvalue is constant along the corresponding Killing eigenvector. Other results can be obtained and they constitute a guide to classify cosmological models and eliminate possibilities when integrating cosmological solutions. They can also be useful in treating perturbation of the models.

The above procedure has also been used to investigate a general situation for any Petrov-type solution. We will publish our results elsewhere.

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