C.B.P.F.-A0022/79

JUNE 1979

CANONICAL AND FEYNMAN FUNCTIONAL INTEGRAL QUANTIZATION OF ELECTRODYNAMICS IN TEMPORAL GAUGE

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ABSTRACT -

Dirac's method for singular Lagrangians is used to implement systematically the $A_0=0$ gauge for electromagnetic field interacting with a Dirac field and a point charge. The gauge may be fixed completely by imposing additional constrains or by means of a canonical transformation. Quantization by Feynman functional integral may be done without removing the residual gauge invariance if we understand that the functional integral must act over the corresponding gauge covariant states.

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I. INTRODUCTION

The lagrangians of eletrodynamics and Yang-Mills gauge theories are customarily written in singular form. Such constrained dynamical systems may be treated by a general method given by Dirac[1] to construct the corresponding Hamiltonian dynamics which may be then quantized. An alternative procedure for quantization is by the functional integral of Feynman[2] with appropriate modifications in the measure as suggested by Faddeev and Popov[3] in connection with Yang-Mills theory. In this connection it was pointed out recently by Gribov[4] that the most commonly used Coulomb gauge may fail to fix the gauge for sufficiently strong fields.

We study in this paper a systematic implementation of A_0=0 gauge[5] following Dirac's method[1][6]. The Hamiltonian dynamics can be formulated in a self consistent fashion both for abelian and non-abelian gauge theory. The canonical commutation relations acquire very simple forms. This permits us to construct a unitary operator to remove all dependence on 'longitudinal' mode in the Hamiltonian and the effective contribution of this mode comes out to be a self-interaction energy term. It is of course possible to do the same in the classical context by imposing additional constraint to fix the gauge completely. For clarity in exposition we will only consider the case of electromagnetic field in interaction with a Dirac field and an external point charge. The case of Yang-Mills theory will be reported in a subsequent publication.[7]

We show in Sec.5 that the quantization by Feynman functional integral in temporal gauge may be done without removing the residual gauge freedom and is shown to be consistent with the one done after fixing the gauge completely. An argument is given which suggests that for constrained dynamical systems (at least at a certain stage) the residual invariance is no problem for quantization by the functional integral only if we understand that the functional integral is defined over the corresponding covariant state vectors. In the appendix we evaluate some useful path integral.

II. HAMILTONIAN DYNAMICS OF INTERACTING ELECTROMAGNETIC FIELD.

We will discuss the dynamics corresponding to the following action functional

$$S = \int L(t)\, dt \tag{2.1}$$

where ($\hbar = c = 1$)

$$L(t) = -M\sqrt{1-\dot{\vec{r}}^{\,2}} + \int d^3x \left[\bar{\Psi}(i\gamma\cdot\partial - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \right.$$

$$\left. - e(\bar{\Psi}\gamma^{\mu}\Psi + j^{\mu})A_{\mu}\right] \tag{2.2}$$

Here $\Psi(\vec{x},t)$ is the Dirac field, $A_{\mu}(\vec{x},t)$ are electromagnetic potentials, $\vec{r}(t)$ are the position coordinates on the trajectory of a charged particle of rest mass M and $F_{\mu\nu} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})$. The four-current vector, $j^{\mu}=(\rho,\vec{j})$, where $\rho(\vec{x},t) = \delta^3(\vec{x}-\vec{r}(t))$, $\vec{j}(\vec{x},t) = \dot{\vec{r}}\,\delta^3(\vec{x}-\vec{r}(t))$, is the external current of the charged particle. The action is invariant under the local gauge transformation

$$\Psi \rightarrow e^{ie\omega(x,t)}\Psi$$

$$A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\,\omega(\vec{x},t) \qquad (2.3)$$

The transformation of gauge field may as well be written as

$$\vec{A}^{T} \rightarrow \vec{A}^{T}$$

$$A_{0} \rightarrow A_{0} + \dot{\omega}$$

$$\Lambda \rightarrow \Lambda - \omega \qquad (2.4)$$

where an overdot indicates time derivative and we decompose the vector field $\vec{A}$ into its longitudinal and transverse components: $\vec{A} = \vec{\nabla}\Lambda + \vec{A}_{T} = \vec{A}_{L} + \vec{A}_{T}$, $\vec{\nabla}.\vec{A}_{T}=0$, $\vec{\nabla}\times\vec{A}_{L}=0$.

The Euler-Lagrange equations are

$$\partial_\mu F^{\mu\nu} = -e(j^\nu + \bar{\Psi}\gamma^\nu \Psi),$$

$$(i\gamma.\partial - m)\Psi = -e\gamma.A\Psi,$$

$$\dot{p}^\mu = e F^{\mu\nu}\dot{r}_\nu . \qquad (2.5)$$

where $p^\mu = M\gamma(1,\dot{\vec{r}})$, $\gamma = (1-\dot{\vec{r}}^2)^{-1/2}$, $\vec{E} = (F^{10}, F^{20}, F^{30})$ and $\vec{B} = -(F^{23}, F^{31}, F^{12})$.

The Lagrangian in Eq. (2.2) does not contain any kinetic term corresponding to the component A^0 and consequently is constrained (singular). The Hamiltonian dynamics may be constructed by the method proposed by Dirac [1,6]. We define the dynamics on t=const. hyperplanes so that all the variations in what follows are taken at fixed time. The canonical momenta are

$$\Pi = \frac{\delta L}{\delta\dot{\Psi}} = i\,\Psi^+$$

$$\Pi_\mu = \frac{\delta L}{\delta\dot{A}^\mu} = F^{0\mu}$$

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{M\dot{\vec{r}}}{\sqrt{1 - \dot{\vec{r}}^2}} + e\,\vec{A}(\vec{r},t) \tag{2.6}$$

The second relation here may be written as

$$\vec{\Pi} = \dot{\vec{A}} + \vec{\nabla} A_0 \tag{2.7}$$

where, $\vec{\Pi} = (\Pi_1, \Pi_2, \Pi_3)$, $\vec{A} = (A^1, A^2, A^3)$ along with a primary constraint

$$\Pi_0 \simeq 0 \tag{2.8}$$

which is written as a "weak" relation in the sense of Dirac.

We define canonical equal time Poisson brackets for any two functionals f and g by

$$\{f\ ,\ g\} = \frac{\partial f}{\partial r^k(t)} \cdot \frac{\partial g}{\partial P_k(t)} - \frac{\partial f}{\partial P_k(t)} \cdot \frac{\partial g}{\partial r^k(t)}$$

$$+ \int d^3x \left\{ \frac{\delta f}{\delta A^\mu(\vec{x},t)} \; \frac{\delta g}{\delta \Pi_\mu(\vec{x},t)} - \frac{\delta f}{\delta \Pi_\mu(\vec{x},t)} \; \frac{\delta g}{A^\mu(\vec{x},t)} \right.$$

$$+ \frac{\delta f}{\delta\Psi_\alpha(\vec{x},t)} \frac{\delta g}{\delta\Pi_\alpha(\vec{x},t)} - \frac{\delta f}{\delta\Pi_\alpha(\vec{x},t)} \frac{\delta g}{\delta\Psi_\alpha(\vec{x},t)} \Bigg\} \tag{2.9}$$

where $\mu = 0,1,2,3$ are space-time indices while $\alpha,\beta = 1,2,3,4$ are spinor indices and k runs over space indices 1,2,3. The standard non-vanishing brackets follow to be

$$\{A^\mu(\vec{x},t), \Pi_\nu(\vec{y},t)\} = \delta^\mu{}_\nu \, \delta^3(\vec{x}-\vec{y})$$

$$\{\Psi_\alpha(\vec{x},t), \Pi_\beta(\vec{y},t)\} = \delta_{\alpha\beta} \, \delta^3(\vec{x}-\vec{y})$$

$$\{r^k(t), P_\ell(t)\} = \delta^k{}_\ell \tag{2.10}$$

We take as our preliminary Hamiltonian

$$H' = H_c + \int v(\vec{x},t) \, \Pi^0(\vec{x},t) \, d^3x \tag{2.11}$$

where H_c is the canonical Hamiltonian

$$H_c = \int (\Pi_\mu \dot{A}^\mu + \Pi \dot{\Psi}) \, d^3x + \vec{P}\cdot\dot{\vec{r}} - L$$

$$= \sqrt{[\vec{P} - e\,\vec{A}(\vec{r}(t))]^2 + M^2}$$

$$+ \int d^3x \left\{ \frac{1}{2} \vec{\Pi}^2 + \frac{1}{4} F_{ij} F^{ij} + \right.$$

$$+ \Psi^+ [- i\,\vec{\alpha}.(\vec{\nabla} - ie\vec{A}) + \beta\, m]\Psi - \vec{\Pi}.\vec{\nabla}A_0 +$$

$$\left. + e\,(\rho + \Psi^+ \Psi)\, A_0 \right\} \qquad (2.12)$$

and v is an arbitrary functional. The equations of motion are given by

$$\dot{f} = \{f, H'\} + \frac{\partial f}{\partial t}$$

where the second term on the right hand side is taken at constant $\Psi, \Pi, \Pi_\mu, A^\mu, \vec{r}, \vec{P}, \vec{x}$. For the constraint $\Pi^0 \simeq 0$ to hold for all times, we require

$$\dot{\Pi}_0 = \{\Pi_0, H'\} = - \frac{\delta H'}{\delta A_0} \simeq 0 \qquad (2.13)$$

This leads to a secondary constraint

$$\chi \equiv \vec{\nabla}.\vec{\Pi} + e(\rho + \psi^{+}\psi) \simeq 0 \qquad (2.14)$$

which is just the Gauss' law equation also contained in Eq.(2.5). We may verify that other equations of motion are consistent with the Lagrange equations and $v=\dot{A}_0$. The requirement

$$\dot{\chi} = \{\chi, H'\} + \frac{\partial\chi}{\partial t} = \{\chi, H_c\} + \frac{\partial\chi}{\partial t} \simeq 0 \qquad (2.15)$$

is satisfied and does not lead to new constraint. From the fact that $\{\Pi^0, \chi\}=0$, we have two first class constraints $\Pi^0 \simeq 0$ and $\chi \simeq 0$.

A more general Hamiltonian may be taken to be

$$H'' = H' + \int u(\vec{x},t)\ \chi(\vec{x},t)\ d^3x \qquad (2.16)$$

we find

$$\dot{A}_0 = v$$

$$\dot{\vec{A}} = \frac{\delta H_c}{\delta \vec{\Pi}} + \frac{\delta}{\delta \vec{\Pi}} \int u \, \chi \, d^3x$$

$$= \vec{\Pi} - \vec{\nabla}(A_0 + u) \tag{2.17}$$

We remark that the first class constrains Π^0 and χ are generators of infinitesimal gauge transformation. Writing

$$\delta F = \{F, \varepsilon G(t)\} \tag{2.18}$$

we find

$$\varepsilon G(t) = \int \left[(\partial^\mu \omega) \, \Pi_\mu + i \, e \, \omega \, \Pi_\alpha \Psi_\alpha \right] d^3x$$

$$= \int \left[\omega \chi + \dot{\omega} \, \Pi_0 \right] d^3x \tag{2.19}$$

III. CHOICE OF GAUGE CONSTRAINT. RESIDUAL GAUGE INVARIANCE.

The functionals u, v, are completely arbitrary. We may remove Π_0 completely by imposing additional constraints in view of the gauge invariance of theory. We may write Eq.(2.14) on using Eq.(2.7) as

$$\nabla^2 (A_0 + \dot{\Lambda}) + e(\rho + \Psi^+ \Psi) \simeq 0 \tag{3.1}$$

It follows that, in the presence of interactions, $(A_0 + \dot{\Lambda})$ must be non-vanishing. For free electromagnetic field we may make the convenient choice $A_0=0$ and $\vec{\nabla} \cdot \vec{A} = 0$ leaving us with two independent (transverse components) degrees of freedom. Moreover, the gauge is completely fixed. For interacting case we must retain at least three degrees of freedom of the gauge field. The non-transverse component, gives rise to Coulomb interaction energy and the transverse modes of electromagnetic field come out as the quanta of the field in quantized theory. In the Coulomb gauge, for example, $\left(\vec{\nabla} \cdot \vec{A}\right) = 0$, we require $\nabla^2 \Lambda = 0$ and $\nabla^2 \omega = 0$. If we assume that the gauge potentials A^μ vanish over the surface of a large sphere we may set Λ = const=0. By imposing similar boundry conditions on ω we may, essentially, fix the gauge completely. In general, however, we are left with a residual gauge invariance corresponding to space-independent transformations $A^0 \to A^0 - \dot{\omega}(t)$. This becomes quite transparent in the choice of temperal gauge $A_0=0$. In view of the Gauss' law, in this case, $\Lambda(\vec{x},t)$ must dependend on both space and time coordinates and the residual gauge freedom corresponds to time-independent gauge transformations. We conclude that, in the presence of interactions, we must work with at least three degrees of freedom of the gauge field and may live any residual gauge invariance without fixing the gauge completely.

It is clear that a simple choice to start in our context would be

$$A_0 \simeq 0 \quad , \dot{A}_0 \simeq 0 = v \tag{3.2}$$

From $\{A_0, \chi\} = 0$, $\{A_0, \Pi_0\} \neq 0$, $\{A_0, H'\} = 0$ we infer that $\chi \simeq 0$ continues to be a first class constraint while $\Pi_0 \simeq 0$ and $A_0 \simeq 0$ are now second class. Defining Dirac brackets [1]

$$\{f,g\}^* = \{f,g\} + \int d^3z \quad \{f,A_0(\vec{z},t)\}\{\Pi_0(\vec{z},t),g\} \tag{3.3}$$

$$-\{f,\Pi_0(\vec{z},t)\}\{A_0(\vec{z},t),g\}$$

we verify $\{f,\Pi_0\}^* = \{f,A_0\}^* = 0$, so that, we may set $\Pi_0=0$

and $A_0=0$ as strong relations. The equation of motion takes the form

$$\frac{df}{dt} = \{f,H\}^* + \frac{\partial f}{\partial t} \tag{3.4}$$

where

$$H=\int d^3x \left\{\frac{1}{2}\vec{\Pi}^2 + \frac{1}{4}F_{ij}F^{ij} + \psi^+ \left[-i\,\vec{\alpha}.(\vec{\nabla} - ie\vec{A}) + \beta\, m\right]\Psi \right.$$

$$\left. + \sqrt{\left[\vec{P} - e\vec{A}(\vec{r},(t))\right]^2 + M^2}\right\} \tag{3.5}$$

In addition, we are left with the Gauss law $\chi \simeq 0$ as a first class constraint. We remark that if we had naively implemented temporal gauge in the canonical Hamiltonian we would not be able to derive Gauss' law. In fact, for only a static external source $(\rho,\vec{0})$ the interaction term is totally absent.

We also observe that we may rewrite the Dirac bracket

$$\{f,g\}^* = \int d^3x \left[\frac{\delta f}{\delta \vec{A}} \cdot \frac{\delta g}{\delta \vec{\Pi}} - \frac{\delta f}{\delta \vec{\Pi}} \cdot \frac{\delta g}{\delta \vec{A}} + \frac{\delta f}{\delta \Psi_\alpha}\frac{\delta g}{\delta \Pi_\alpha} - \right.$$

$$- \frac{\delta f}{\delta \Pi_\alpha} \frac{\delta g}{\delta \Psi_\alpha} \Bigg] + \frac{\partial f}{\partial \vec{r}} \cdot \frac{\partial g}{\vec{P}} - \frac{\partial f}{\partial \vec{P}} \cdot \frac{\partial g}{\partial \vec{r}} \; .$$

and the nonvanishing brackets are

$$\{A^k(\vec{x},t) \, , \, \Pi_\ell(\vec{y},t)\}^* = \delta^k_\ell \, \delta^3(\vec{x}-\vec{y})$$

$$\{\Psi_\alpha(\vec{x},t) \, , \, \Pi_\beta(\vec{y},t)\}^* = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

$$\{r^k(t) \, , \, P_\ell(t)\}^* = \delta^k_\ell \qquad (3.7)$$

Thus the independent canonical variables in the temporal gauge have canonical brackets equal to their Dirac brackets. There are no fields appearing on the right hand side in Eq.(3.7). In the Coulomb gauge we impose $\vec{\nabla} \cdot \vec{A} \simeq 0$, $\vec{\nabla} \cdot \dot{\vec{A}} \simeq 0$

so that $\chi \simeq 0$ becomes a second class while $\Pi^0 \simeq 0$ continues to be first class. Dirac brackets may be defined with respect to $\chi \simeq 0$ and $\vec{\nabla}.\vec{A} \simeq 0$ and we verify, for example, that $\{\Psi, \Pi_k\}^* \neq 0$ and contains field Ψ on the right hand side. In quantized theory in such cases we get ghost loops in the Feynman rules. In fact the Dirac brackets in Coulomb gauge are rather cumbersome. However, the Coumlomb self energy interaction term in the Hamiltonian is readily seperated out in this gauge since we have now strong relation $\chi=0$.

It is important to note that the canonical brackets in Eq.(3.7) are very simple compared to those in Coulomb gauge. Further constraint[(†)] may be imposed to fix the gauge completely. It is, however, more convenient to achieve this via a canonical transformation. We will do this in the following section for quantized theory by making a corresponding unitary transformation.

(†) See section 4.

IV. CANONICAL QUANTIZATION IN TEMPORAL GAUGE: GAUGE-FIXING CONDITIONS

The canonical quantization is obtained by replacing Dirac brackets by commutator a anticommutator between corresponding operators in a self consistent manner

$$\{f,g\}^* \longrightarrow \frac{1}{i\hbar}\,[\hat{f},\hat{g}] \quad \text{or} \quad \frac{1}{i\hbar}\,\{\hat{f},\hat{g}\} \tag{4.1}$$

while the first class constraints are imposed as conditions on physical states vectors. Appealing to the quantization of free fields we are led to the following non-vanishing standard commutators,

$$[\hat{A}^k(\vec{x},t)\,,\ \hat{\Pi}_\ell(\vec{y},t)] = i\ \delta^k_\ell\ \delta^3(\vec{x}-\vec{y})$$

$$\{\hat{\Psi}_\alpha(\vec{x},t)\,,\ \hat{\Pi}_\beta(\vec{y},t)\} = i\ \delta_{\alpha\beta}\ \delta^3(\vec{x},\vec{y})$$

$$[\hat{r}^k(t)\,,\ \hat{P}_\ell(t)] = i\ \delta^k_\ell \tag{4.2}$$

while

$$\hat{\chi}\,\underline{\Psi} \equiv (\vec{\nabla}\cdot\vec{\Pi} + e\,[\rho + \hat{\psi}^{+}\psi])\,\underline{\Psi} = 0 \tag{4.3}$$

$$\hat{H}\,\underline{\Psi} = E\,\underline{\Psi} \tag{4.4}$$

From Eq.(4.2) we may derive

$$\left[\hat{A}^{Tk}(\vec{x},t)\,,\ \hat{\Pi}^{T}_{\ell}(\vec{y},t)\right] = i\left(\delta^{k}_{\ell} + \frac{\partial^{k}\partial_{\ell}}{\nabla^{2}}\right)\delta^{3}(\vec{x}-\vec{y}) \tag{4.5a}$$

$$\left[\vec{\nabla}\cdot\hat{\vec{\Pi}}(\vec{x},t)\,,\ \hat{\Lambda}(\vec{y},t)\right] = i\ \delta^{3}(\vec{x}-\vec{y}) \tag{4.5b}$$

$$\left[\hat{\Lambda}(\vec{x},t)\,,\ \hat{\vec{\Pi}}^{T}(\vec{y},t)\right] = 0 \tag{4.5c}$$

$$\left[\vec{\nabla}\cdot\hat{\vec{\Pi}}(\vec{x},\vec{t})\,,\ \hat{\vec{A}}^{T}(\vec{y},t)\right] = 0 \tag{4.5d}$$

The Hamiltonian in Eq.(3.5) contains terms involving longitudinal mode. We will show now that their contribution amounts to Coulomb self-energy interaction term leaving only the tranverse modes as quanta of the gauge field. We perform a unitary transformation by the operator(†)

(†) We represent the operators $\hat{\vec{r}}$ and $\hat{\vec{\mathbf{P}}}$ by $\hat{\vec{r}} = \vec{r}$ and $\hat{\vec{P}} = \frac{1}{i}\vec{\nabla}_{\vec{r}}$.

$$U = e^{ie \int [\rho(\vec{x},t) + \hat{\Psi}^{+}(\vec{x},t)\ \hat{\Psi}(\vec{x},t)]\ \hat{\Lambda}(\vec{x},t)\ d^3x}$$

$$= e^{ie\ \hat{\Lambda}(\vec{r},t)}\ e^{ie \int \hat{\Psi}^{+}\ \hat{\Psi}\ \hat{\Lambda}\ d^3x} \qquad (4.6)$$

We show easily

$$U\ \hat{\vec{P}}\ U^{-1} = \hat{\vec{P}} - e\ \vec{\nabla}_{\vec{r}}\ \hat{\Lambda}\ (\vec{r},t) \qquad (4.7a)$$

$$U\ \vec{\nabla}\ .\ \hat{\vec{\Pi}}\ U^{-1} = \vec{\nabla}\ .\hat{\vec{\Pi}} + e(\rho + \hat{\Psi}^{+}\ \hat{\Psi}) \qquad (4.7b)$$

$$U\ \hat{\Psi}\ (\vec{x},t)\, U^{-1} = e^{-ie\ \hat{\Lambda}(\vec{x},t)}\ \Psi(x,t) \qquad (4.7c)$$

$$U\ \hat{\Psi}^{+}\hat{\Psi}\ U^{-1} = \hat{\Psi}^{+}\hat{\Psi} \qquad (4.7d)$$

$$U(\vec{\nabla}\ \hat{\Psi}\ (\vec{x},t))\ U^{-1} = e^{-ie\ \hat{\Lambda}(\vec{x},t)} \left[\vec{\nabla} - ie\ \vec{\nabla}\hat{\Lambda}(\vec{x},t)\right]\ \hat{\Psi}(\vec{x},t) \qquad (4.7e)$$

From Eqs.(4.3) and (4.7b) it follows

$$U\,\vec{\nabla}\cdot\hat{\vec{\Pi}}\;U^{-1}\,\underline{\Psi} = 0$$

or

$$(\vec{\nabla}\cdot\hat{\vec{\Pi}})\;\underline{\tilde{\Psi}} = 0 \qquad (4.8)$$

where $\underline{\Psi} = U\,\underline{\tilde{\Psi}}$ (4.9)

The commutation relation in Eq.(4.5b) implies that in coordinate representation $\hat{\Lambda} = \Lambda$, $\vec{\nabla}\cdot\hat{\vec{\Pi}} = i\frac{\delta}{\delta\Lambda}$ ~~so that~~ we get $\frac{\delta\underline{\tilde{\Psi}}}{\delta\Lambda} = 0$. The functional $\underline{\tilde{\Psi}}$, apart from its dependence on Ψ, depends only on the transverse components $\vec{A}^T$ of the gauge field and U carries all the dependence on Λ of the state functional $\underline{\Psi}$. From Eqs. (4.7) and (4.8) we find

$$\hat{\tilde{H}}\,\underline{\tilde{\Psi}} \equiv U^{-1}\,\hat{H}\,U\,\underline{\tilde{\Psi}} \equiv (\hat{H}_{Coul.} + \hat{H}_T)\,\underline{\tilde{\Psi}} = E\,\underline{\tilde{\Psi}} \qquad (4.10)$$

where

$$\hat{H}_T = \sqrt{[\hat{\vec{P}} - e\,\hat{\vec{A}}^T(\vec{r},t)]^2 + M^2} +$$

$$+\int d^3x \left\{ \frac{1}{2} \hat{\vec{\Pi}}_T{}^2 + \frac{1}{4} \hat{F}_{ij} \hat{F}^{ij} + \hat{\tilde{\Psi}}^+ \left[-i\vec{\alpha} \cdot (\vec{\nabla} - ie \hat{\vec{A}}_T) + \beta m \right] \hat{\tilde{\Psi}} \right\} \quad (4.11)$$

depends only on gauge invariant transverse components of gauge field and the gauge invariant $\hat{\tilde{\Psi}} = e^{ie\hat{\Lambda}}\tilde{\Psi}$ and

$$\hat{H}_{Coul.} = -\frac{e^2}{2} \int \int d^3x \, d^3y \, \left[\rho(\vec{x},t) + \hat{\tilde{\Psi}}^+(\vec{x},t) \, \hat{\tilde{\Psi}}(\vec{x},t) \right]$$

$$< \vec{x} \, | \frac{1}{\nabla^2} | \, \vec{y} > \left[\rho(\vec{y},t) + \hat{\tilde{\Psi}}^+(\vec{y},t) \, \hat{\tilde{\Psi}}(\vec{y},t) \right] \quad (4.12)$$

is the Coulomb self-energy interaction term obtained on integration over, the longitudinal component of $\vec{\Pi}$ and using Eq.(4.8). The transverse part $\hat{H}_T$ is supposed to take care of all corrections to the instantaneous Coulomb term, such as, for example, that the total effects are retarded and act no faster than the speed of light. We may also verify that the form in Eq. (4.9) is also satisfatory in so far as no new conditions arise after quantization and we have a self consistent quantization procedure in temporal gauge.

The residual time independent gauge transformations are generated by the operstor $\hat{\chi}(\vec{x},t)$ so that $\delta\hat{F} = -i[\hat{F}, \epsilon\hat{G}(t)]$. The state vector transforms as

$$\underline{\Psi} \longrightarrow e^{-i \int \hat{j}_0(\vec{x},t)\,\omega(\vec{x})\,d^3x}\underline{\Psi} \qquad (4.13)$$

where

$$\hat{j}_0(\vec{x},t) = e\,\left[\rho(\vec{x},t) + \hat{\tilde{\Psi}}^{+}(\vec{x},t)\,\hat{\tilde{\Psi}}(\vec{x},t)\right] \qquad (4.14)$$

For obsevables we have

$$\Theta(\hat{\vec{A}}^{[\omega]}, \hat{\vec{\Pi}}^{[\omega]}, \hat{\Psi}^{[\omega]}, \hat{\Psi}^{[\omega]+}) = \Theta(\hat{\vec{A}}, \hat{\vec{\Pi}}, \hat{\Psi}, \hat{\Psi}^{+}) \qquad (4.15)$$

where $\hat{B}^{[\omega]}$ indicates the gauge transformed of operator $\hat{B}$. We may discuss the selection rules for matrix elements of observables as usual and show the appropriate gauge covariance properties of the matrix elements of non-gauge invariant operators.

We note that the discussion in the present section could as well have been in terms of classical field theory context. A canonical transformation of variables would replace the unitary transformation of quantized theory. The Eq.(4.5b) implies that in the context of discussion in Sec.III, we may impose, in $A_0=0$ gauge, further constraint, say,

$$(\Lambda - c) \simeq 0 \qquad (4.16)$$

in order to fix the gauge completely. Here $c=c(t)$ is a space independent constant. In fact Λ and χ constitute a canonical pair in temporal gauge as evidenced by their poisson bracket. The Gauss law constraint $\chi \simeq 0$ now becomes a second class constraint and we may define new Dirac brackets using their iterative property [6]

$$\{f,g\}^{**} = \{f,g\}^{*} + \int d^3z \; [\{f,\chi(\vec{z},t)\}^{*} \{\Lambda(\vec{z},t),g\}^{*} - \{f,\Lambda(\vec{z},t)\}^{*} \{\chi(\vec{z},t),g\}^{*}] \tag{4.17}$$

We may then write these constraints as strong relations, viz

$$\Lambda = c \quad ,$$

$$\vec{\nabla}.\vec{\Pi} = -e(\rho + \Psi^{+} \Psi) \tag{4.18}$$

where $\vec{\Pi}$ is now given by Eq.(2.17),

and obtain the final Hamiltonian $\tilde{H}=H_{Coul.}+ H_T$ in the form analogous to that given in Eqs.(4.11) and (4.12). The gauge is thus fixed completely. The final form is independent of c. We may now quantize the theory by replacing new Dirac brackets by commutators or anticommutators between operators. In view of Eq.(4.17) the standard commutators are the same as written above.

It is instructive to show the details of $A^0 \approx 0$, $\chi' \equiv \Lambda - c(t) \approx 0$ gauge. We note

$$\{\chi', A^0\}^* \approx 0$$

$$\{\chi'(\vec{x},t), \chi(\vec{y},t)\}^* = \{\Lambda(\vec{x},t), \vec{\nabla}.\vec{\Pi}(\vec{y},t)\}^* = -i\,\delta^3(\vec{x}-\vec{y}) \tag{4.19}$$

The first relation assures the iterative property for Dirac brackets. The second one shows that det $||\{\chi', \chi\}^*||$=const. and is independent of fields. Contrary to the non-abelian [4,3] theory the Coulomb gauge, in abelian case, is ghost-free. In view of the non-vanishing determinant above we may solve unambiguously for arbitrary functional u appearing in the Hamiltonian using Eq.(2.17). We obtain in our gauge

$$\nabla^2 u = \vec{\nabla}\,.\,\vec{\Pi} \tag{4.20}$$

Substituting in the general Hamiltonian of Eq.(2.16) this result, $\Lambda=c(t)$, $A^0=0$, $\dot{A}^0=v=0$ and making use of Eq.(4.18) we readly obtain the Hamiltonian in Eq.(4.11) which involve only physical transverse degrees of freedom of the gauge field. Also

$$\vec{\Pi}^T = \vec{\Pi} - \vec{\nabla}\frac{1}{\nabla^2}(\vec{\nabla}.\vec{\Pi}) = \dot{\vec{A}}^T + \vec{\nabla}u - \vec{\nabla}u = \dot{\vec{A}}^T \qquad (4.21)$$

There is still another interesting gauge-fixing choice possible and which may be generalized to non-abelian Yang-Mills theory for any gauge group [7]. It is simply

$$A_0 \simeq 0 \quad , \; A^3 \simeq 0 \qquad (4.22)$$

We note

$$\{A_3(\vec{x},t)\,,\; \chi(\vec{y},t)\}^* = \partial_3^y \,\delta^3(\vec{x}-\vec{y}) \qquad (4.23)$$

The gauge is thus ghost-free and det $||\{A_3, \chi\}^*||$ = det(∂_3^y) is a 'constant' independent of the fields. This is also true for any form can be readily reduced to the form in Eq.(4.11), and Eq.(4.12) for the abelian case under discussion. Only the kinetic terms require some comments. They may be straightened, say, by using the unitary transformations written in Eqs.(4.7a) and (4,7c)

$$U^{-1}\,\hat{\Psi}^{+}\left[-i\vec{\alpha}\,.(\vec{\nabla}-ie\hat{\vec{A}})\right]\hat{\Psi}\,U=\hat{\Psi}^{+}\left[-i\vec{\alpha}.(\vec{\nabla}-ie(\hat{\vec{A}}-\vec{\nabla}\hat{\Lambda})\,)\right]\hat{\Psi}$$

$$=\hat{\Psi}^{+}\left[-\,i\vec{\alpha}\,.(\vec{\nabla}-ie\hat{\vec{A}}_{T})\right]\hat{\Psi}$$

and

$$U^{-1}(\hat{\vec{P}}-e\,\hat{\vec{A}})\;U=(\hat{\vec{P}}-e\,\hat{\vec{A}}_{T}) \qquad (4.26)$$

Yang-Mills theory. Since we must also require $\dot{A}_3=0$ Eq.(2.17) gives

$$\Pi_3 = \partial_3 u \tag{4.24}$$

and Hamiltonian is written in terms of two independent degrees of freedom as follows [7]

$$H= \sqrt{[\vec{P}-e\vec{A}(\vec{r}(t))]^2 + M^2} + \int d^3x \left\{ \frac{1}{2} \bar{\Pi}^2 + \frac{1}{4} F_{ij} F^{ij} + \right.$$

$$+ \Psi^+ \left[-i \vec{\alpha}.(\vec{\nabla}- ie\bar{\mathbf{A}}) + \beta m \right]\Psi - \frac{1}{2} \int d^3y \left[\bar{\nabla} . \bar{\Pi} + e(\rho + \Psi^+\Psi)\right] (\vec{y},t)$$

$$\left. . \; K(\vec{x},\vec{y}) \left[\bar{\nabla} . \bar{\Pi} + e(\rho + \Psi^+\Psi)\right] (\vec{x},t) \right\} \tag{4.25}$$

Here $K(\vec{x},\vec{y}) = G(x^3,y^3)\, \delta^2(\bar{x}-\bar{y})$ and $G(t,t')$ is the Green's function satisfying $\frac{\partial^2 G}{\partial t^2} = \delta(t-t')$. The two physical components are written as $\bar{A}=(A^1,A^2)$, $\bar{\Pi} = (\Pi_1,\Pi_2)$ while $\bar{\nabla} =(\partial_1,\partial_2)$. This

V. QUANTIZATION BY FEYNMAN INTEGRAL:

The quantization using Feynman functional integral[2] trajectories in phase space is also straightforward. We may do it, in temporal gauge even without having removed the residual gauge freedom corresponding to time independent gauge transformations. This is possible since $\hat{\tilde{H}}$ and $\hat{H}$ are related by a unitary transformation and consequently the evolution operator satisfies

$$(\underline{\overline{\Psi}}', e^{-i\hat{H}t} \underline{\overline{\Psi}}) = (\underline{\tilde{\overline{\Psi}}}', e^{-i\hat{\tilde{H}}t} \underline{\tilde{\overline{\Psi}}}) \qquad (5.1)$$

where $\tilde{H} = \tilde{H}\,[\vec{r}, \vec{P}, \vec{\Pi}_T, \vec{A}_T, \Psi^+, \Psi]$ and

$$H = H\,[\vec{r},\vec{P}, \vec{\Pi},\vec{A}, \Psi^+, \Psi].$$

The generating functional for the S-matrix over the Hilbert space of state vectors $\{\underline{\tilde{\overline{\Psi}}}\}$ is expressed formally as the following phase space functional integral

$$\tilde{Z} = \int e^{i\tilde{S}} \prod_x \left\{ \Pi d\vec{A}^T \, \Pi d\,\vec{\Pi}^T \prod_\alpha d\,\overline{\Psi}_\alpha(x)\, d\,\Psi_\alpha(x) \right\} \prod_t \prod_k \frac{d\,r^k\, d\,P_k(t)}{(2\Pi)}$$

(5.2)

where

$$\tilde{S} = \int_{t'}^{t''} dt \left[\vec{P}.\dot{\vec{r}} + \int d^3x \; (\vec{\Pi}_T.\dot{\vec{A}}_T + \Pi\dot{\Psi}) - \tilde{H}\right] \tag{5.3}$$

The functional integral over the state vectors $\{\underline{\Psi}\}$ according to the prescription of Ref.[3] and noting that det $\| \{A^0, \Pi^0\} \| =$ const., is given by

$$Z = \int e^{iS} \prod_x \left(\prod_k dA^k \quad d\,\Pi_k \prod_\alpha d\,\bar{\Psi}_\alpha \, d\,\Psi_\alpha \right) \prod_t \prod_{k=1}^{3} \frac{d\,r^k \, d\,P_k}{(2\Pi)} \tag{5.4}$$

where

$$S = \int_{t'}^{t''} dt \left[\vec{P}.\dot{\vec{r}} + \int d^3x (\vec{\Pi}.\dot{\vec{A}} + \Pi\,\dot{\Psi}) - H\right] \tag{5.5}$$

and H is given in Eq.(3.5).

For simplicity in discussion to demonstrate the relevant points we will consider the case of electromagnetic field interacting only with an external non-relativistic charged particle(†). We may easily do the functional integration over P_k and $\vec{\Pi}$ to obtain from Eq. (5.4) the Feynman path integral

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(†) We replace the square root term in Eq.(3.5) by $\frac{1}{2M}(\vec{P}-e\,\vec{A})^2$. We remark that we may choose the relativistic Lagrangean in a alternative from: $L(t) = \frac{1}{2}\alpha\,\dot{\vec{r}}^2 - \frac{1}{2}\left(\alpha + \frac{M^2}{\alpha}\right) - j.A$ where $\alpha(t)$ is an auxiliary variable.

representation

$$Z = \int e^{i\bar{S}} \left(\prod_x \prod_k d A^k \right) \left(\prod_t \prod_\ell dr^\ell (t) \right) \tag{5.6}$$

where

$$\bar{S} = \int_{t'}^{t''} dt \left[\frac{1}{2} M \dot{\vec{r}}^2 + \int \left\{ \frac{1}{2} \dot{\vec{A}}^2 - \frac{1}{4} F_{ij} F^{ij} + e \vec{j}.\vec{A} \right\} d^3x \right] \tag{5.7}$$

From the corresponding integration in Eq.(5.2) we obtain

$$\tilde{\bar{S}} = \int_{t'}^{t''} dt \left[\frac{1}{2} M \dot{\vec{r}}^2 + \int \left\{ \frac{1}{2} \dot{\vec{A}}_T^2 - \frac{1}{4} F_{ij} F^{ij} + e \vec{j}.\vec{A}_T \right\} d^3x - H_{Coul.} \right] \tag{5.8}$$

To perform the path integration in Eq.(5.7) over longitudinal component of the gauge field it is convenient to use normal mode coordinates [2]. We expande $\vec{A}(\vec{x},t)$ in Fourier series

$$\vec{A}(\vec{x},t) = \sum_{\vec{k}} \sum_{\lambda=\pm 1} \vec{q}_{k,\lambda}(t)\ \phi_{k,\lambda}(\vec{x}) \tag{5.9}$$

The field is suppose to be enclosed in a large box of volume $\Omega = (2L)^3$ and satisfies periodic boundry conditions. Here

$$\phi_{k,\lambda}(\vec{x}) = \sqrt{\frac{2}{\Omega}} \begin{cases} \cos(\vec{k}.\vec{x}) & , \lambda = 1 \quad , \quad \vec{k}\neq 0 \\ \sin(\vec{k},\vec{x}) & , \lambda = -1 \quad , \quad \vec{k}\neq 0 \end{cases}$$

$$\phi_0(\vec{x}) = \sqrt{\frac{1}{\Omega}} \quad , \quad \vec{k}=0 \tag{5.10}$$

and

$\vec{k} = \left\{\pm n_1 \frac{\Pi}{L}, \pm n_2 \frac{\Pi}{L}, n_3 \frac{\Pi}{L}\right\}$, where n_i are positive integers $0,1,2,\ldots$. The coefficients $\vec{q}_{k\lambda}(t)$, for each $(\vec{k},\lambda)$, represent three normal-mode coordinates. For $k\neq 0$ we may write

$$\vec{q}_{k\lambda}(t) = \frac{\vec{k}}{k}\, q_{k\lambda}(t) + \vec{q}^{\,T}_{k\lambda}(t) \tag{5.11}$$

where $\vec{k}.\vec{q}^{T(t)}_{k\lambda}=0$. Also

$$\vec{A}_L \equiv \vec{\nabla}\Lambda = \sum_{k,\lambda}{}' \frac{\vec{k}}{k} q_{k\lambda}(t) \; \phi_{k\lambda}(\vec{x})$$

$$\Lambda(\vec{x},t) = \sum_{k,\lambda}{}' \left(\frac{\lambda}{k}\right) q_{k\lambda}(t) \; \phi_{k,-\lambda}(\vec{x}) \tag{5.12}$$

where prime indicates that $\vec{k}\neq 0$. It follows that

$$\bar{S} = \int_{t'}^{t''} dt \left[\frac{1}{2} M \dot{\vec{r}}^2 + \frac{1}{2} \sum_{k,\lambda} (\dot{q}^2_{k\lambda} + \dot{\vec{q}}^{T^2}_{k\lambda} - k^2 \vec{q}^{T^2}_{k\lambda}) + \right.$$

$$\left. + e \, \dot{\vec{r}}.\vec{A}(\vec{r},t)\right] \tag{5.13}$$

and

$$Z = \int e^{i\bar{S}} \prod_t \prod_{k,\lambda} dq_{k\lambda}(t) \; d\vec{q}^T_{k\lambda}(t) \prod_\ell d \, r^\ell(t) \tag{5.14}$$

The longitudinal modes are zero frequency modes of the gauge field. We may write the interaction term involving longitudinal mode in a more convenient form as follows

$$\int_{t'}^{t''} \dot{\vec{r}}\cdot\vec{A}(\vec{r},t)\,dt = \int_{t'}^{t''} dt \int d^3x\ \vec{j}\cdot\vec{\nabla}\,\Lambda(\vec{x},t) = \int_{t'}^{t''} dt \int d^3x\,\Lambda(\vec{x},t)\frac{\partial\rho(\vec{x},t)}{\partial t}$$

$$= \int_{t'}^{t''} dt\ \Sigma'_{k,\lambda}\left(\frac{\lambda}{k}\right) q_{k\lambda}(t)\,\dot{\rho}_{k,-\lambda}(t)$$

$$= \Sigma'_{k,\lambda}\left(\frac{\lambda}{k}\right)\Bigg\{ \left[q''_{k\lambda}\,\rho''_{k,-\lambda} - q'_{k,\lambda}\,\rho'_{k,-\lambda}\right] -$$

$$- \int_{t'}^{t''} dt\ \dot{q}_{k\lambda}\,\rho_{k,-\lambda}\Bigg\} \qquad (5.15)$$

Here $\rho_{k\lambda} = \phi_{k\lambda}(\vec{r}(t))$ is the Fourier coefficient of $\rho(\vec{x},t)$ and $q' = q(t')$, $q'' = q(t'')$ etc. The path integral over each $q_{k\lambda}$ may be easily evaluated(†) to obtain the factor

$$K_L\left(\{q''_{k\lambda}\},t'';\{q'_{k\lambda}\},t'\right) = \frac{1}{\sqrt{2i\pi(t''-t')}}\,e^{iS_L} \qquad (5.16)$$

(†) See appendix.

where

$$S_L = e\left[\Lambda(\vec{r}'',t'') - \Lambda(\vec{r}',t')\right] - \frac{e^2}{2}\int_{t'}^{t''} dt \sum_{k,\lambda}{}' \left(\frac{\rho^2_{k,-\lambda}}{k^2}\right)$$

$$+ \sum_{k,\lambda}{}' \frac{1}{2(t''-t')}\left[q''_{k\lambda} - q'_{k\lambda} - \frac{\lambda e}{k}\int_{t'}^{t''} \rho_{k,-\lambda}\, dt\right]^2 \qquad (5.17)$$

The second term is the infinite self energy of point charge which also appears in Eq.(4 . 12) . For many point particles we will also get mutual interaction energy term. From the following observation $\left(\underline{\Psi} = e^{ie\Lambda}\,\tilde{\underline{\Psi}}\right)$

$$e^{ie\,\Lambda(\vec{r}'',t'')} = \left[\int K_L\left(\{q''_{k\lambda}\}, t''; \{q'_{k\lambda}\}, t'\right) . e^{ie\,\Lambda(\vec{r}',t')} \prod_{k,\lambda} d\,q'_{k\lambda}\right] e^{i\frac{e^2}{2}\int_{t'}^{t''} dt\, \Sigma'\left(\frac{\rho^2_{k,-\lambda}}{k^2}\right)} \qquad (5.18)$$

we infer that the longitudinal modes in temporal gauge serve two purposes. It isolates the factor corresponding to the Coulomb energy interaction term which in case of $\tilde{Z}$ is present explicitly as seen from Eq. (5.8). Their presence is also essential to secure that when the Feynman propagator functional acts on a state satisfying Gauss' law it results in a state satisfying the same law. We also conclude that the functional integral as defined in Eq. (5.4) is consistent with that defined in Eq.(5.2). The residual (gauge) invariance freedom is no problem for quantization by functional integral only if we undertand that the functional integral must act on corresponding (gauge) covariant states. Similar situation should occur for the case of other constrained systems. In the case of Yang-Mills theory the canonical quantization is as straightforward as discussed in the present case. However, functional integral quantization is quite involved . This work will be reported in a subsequent publication.

APPENDIX:

The path integral is of the type

$$\int_{x'}^{x''} \exp\left\{ i \int_{t'}^{t''} \left[\frac{1}{2} m \dot{x}^2 - g(t)\, \dot{x} \right] dt \right\} \prod_t dx(t)$$

According to Feyman [2] this is given by

$$\lim_{N \to \infty} \left(\sqrt{\frac{m}{2i\varepsilon\pi}} \right)^N \int_{-\infty}^{\infty} \exp \left\{ \left(\frac{im}{2\varepsilon} \right) \sum_{k=1}^{N} \left[(x_k - x_{k-1})^2 - \frac{2g_k \varepsilon}{m} (x_k - x_{k-1}) \right] \right\} \cdot$$

$$\cdot\ dx_1\, dx_2 \ldots dx_{N-1}$$

$$= e^{-\frac{i}{2m} \int_{t'}^{t''} g(t)^2 dt} \cdot \lim_{N \to \infty} \left(\sqrt{\frac{m}{2i\varepsilon\pi}} \right)^N \cdot \int_{-\infty}^{\infty} \exp \left\{ \left(\frac{im}{2\varepsilon} \right) \sum_{k=1}^{N} \left[x_k - x_{k-1} - \frac{\varepsilon g_k}{m} \right]^2 \right\} dx_1 \ldots dx_{N-1}$$

This is readily evaluated to give

$$= \sqrt{\frac{m}{2i\pi(t''-t')}}\, e^{iK}$$

where

$$K = -(1/2m)\int_{t'}^{t''} g^2(t)\,dt \;+\; (m/\,2(t''-t'))\left[x' - x'' + (1/m)\int_{t'}^{t''} g(t)\,dt\right]^2$$

It follows also

$$\int_{x'}^{x''} \exp\left\{i\int_{t'}^{t''}\left[(1/2)\,m\,\dot{x}^2 + J(t)\,x(t)\right]dt\right\}\prod_t dx(t)$$

$$= \sqrt{m/2i\pi(t''-t')}\;\exp\, i\,(K + g''\,x'' - g'\,x')$$

where $g(t) = \int_{t_0}^{t} J(t)\,dt + g(t_0)$.

The path integral containing a term $J(t)\,x(t)^2$ may be easily handled by the same trick.

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