ON THE RELATION BETWEEN FIELDS AND POTENTIALS IN NON-ABELIAN GAUGE THEORIES

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§ 1. Introduction

T.S. Wu and C.N. Yang [1] have pointed out, by constructing a specific example, that in non-abelian gauge theories the field tensor $F_{\mu\nu}$ does not determine the potentials A_μ, not even locally and not even up to a gauge.

In their example one of the potentials corresponds to a magnetic monopole in three dimensions, the other being proportional to a pure gauge. One may then ask if their result is valid only in some pathological examples or in some other cases as well.

We intend to show in this paper that there exists a more general class of field configurations displaying that property. Specifically we will show that to any element U of the gauge group we can assign a one parameter family of fields:

$$F_{\mu\nu}^{(\alpha)} = \alpha(1-\alpha)\ (\partial_\mu U^{-1}\partial_\nu U - \partial_\nu U^{-1}\partial_\mu U),$$

each member being derived from two different potentials, not related by any gauge transformation, unless U satisfies an "integrability condition".

Further, for three dimensions and the group SU_2, we shall show that the example given in reference [1] is a particular case within a whole family of fields. Each class of the family being characterized by a unit vector field $\vec{\underset{\sim}{n}}(\vec{\underset{\sim}{r}})$ $(\vec{\underset{\sim}{n}}\cdot\vec{\underset{\sim}{n}}=1)$. For a given $\underset{\sim}{n}(\underset{\sim}{r})$, the gauge field $\underset{\sim}{B}$ (dual of F_{ij}) has the form

$$\underset{\sim}{B}^{(\alpha)} = \alpha(1-\alpha)\ \underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma \quad ; \quad \sigma = \underset{\sim}{\sigma}\cdot\underset{\sim}{n}$$

and can be derived from two different potentials not related by a gauge transformation. Wu and Yang's example is the class corresponding to the election $\underset{\sim}{n} = \underset{\sim}{r}/r$.

§ 2. Relevant theorems

Let us call

(1) $$\Phi_\mu = \Phi_\mu^k \chi_k \ ,$$

χ^k being the generator of the corresponding Lie group.

Let ϕ_μ and ψ_μ be two vacuum potentials, i.e.:

(2) $$\partial_\mu\phi_\nu - \partial_\nu\phi_\mu + [\phi_\mu, \phi_\nu] = 0$$

(3) $$\partial_\mu\psi_\nu - \partial_\nu\psi_\mu + [\psi_\mu, \psi_\nu] = 0 \ ,$$

then:

Theorem 1. The potentials $A_\mu^{(\alpha)}$ and $A_\mu^{(1-\alpha)}$ where

(4) $$A_\mu^{(\alpha)} = \alpha\ \phi_\mu + (1-\alpha)\psi_\mu \qquad (\alpha = \text{arbitrary constant})$$

$$A_\mu^{(1-\alpha)} = (1-\alpha)\ \phi_\mu + \alpha\ \psi_\mu$$

give exactly the same field $F_{\mu\nu}$.

Proof:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\nu , A_\mu]$$

$$= \alpha(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu) + (1-\alpha)(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu) + \alpha^2 [\phi_\mu , \phi_\nu]$$

$$+ (1-\alpha)^2 [\psi_\mu , \psi_\nu] + \alpha(1-\alpha)([\phi_\mu , \psi_\nu] + [\psi_\mu , \phi_\nu])$$

Eliminating the curls by using (2) and (3) we obtain

(5) $$F_{\mu\nu} = -\alpha(1-\alpha) [(\phi_\mu - \psi_\mu), (\phi_\nu - \psi_\nu)]$$

As (5) is invariant under the interchange $\alpha \rightleftarrows (1-\alpha)$, the theorem is proved.

Corollary 1. By taking $\psi_\mu \equiv 0$ in theorem 2, we see that $A_\mu^{(\alpha)} = \alpha\,\phi_\mu$ and $A_\mu^{(1-\alpha)} = (1-\alpha)\phi_\mu$ give exactly the same field

(6) $$F_{\mu\nu} = \alpha(1-\alpha) [\phi_\mu , \phi_\nu] = \alpha(1-\alpha) (\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)$$

Theorem 2. Theorem 1 is invariant under any gauge transformation.

Proof. Under a gauge transformation U:

(7) $$A'_\mu = U^{-1} A_\mu U + U^{-1} \partial_\mu U ,$$

and substituting in (4) we get:

$$A'^{(\alpha)}_{\mu} = U^{-1} (\alpha\phi_{\mu} + (1-\alpha)\psi_{\mu}) U + U^{-1}\partial_{\mu} U =$$

$$= \alpha U^{-1}\phi_{\mu}U + (1-\alpha)U^{-1}\psi_{\mu}U + \alpha U^{-1}\partial_{\mu}U + (1-\alpha)U^{-1}\partial_{\mu}U$$

$$= \alpha \phi'_{\mu} + (1-\alpha)\psi'_{\mu}$$

and

$$F'_{\mu\nu} = -\alpha(1-\alpha) \left[(\phi'_{\mu} - \psi'_{\mu}) , (\phi'_{\nu} - \psi'_{\nu}) \right] \text{ Q.E.D.}$$

Corollary 2. It is always possible to find a gauge transformation which brings theorem 1 into the form of Corollary 1.

Proof. ψ_{μ} being a vacuum potential it has the form:

(8) $$\psi_{\mu} = V^{-1} \partial_{\mu} V \quad \text{for some } V.$$

By performing the inverse gauge transformation (V^{-1}) ψ_{μ} is taken to zero, which together with theorem 2 completes the proof.

Corollary 3. The gauge of Corollary 2 determines a group element U such that

(9) $$\phi_{\mu} = U^{-1} \partial_{\mu} U$$

(10) $$A^{(\alpha)}_{\mu} = \alpha U^{-1} \partial_{\mu} U$$

The field $F_{\mu\nu}$ common to both potentials $(A^{(\alpha)}_{\mu} , A^{(1-\alpha)}_{\mu})$ takes the form

(11) $$F^{(\alpha)}_{\mu\nu} = \alpha(1-\alpha)(\partial_{\mu}U^{-1}\partial_{\nu}U - \partial_{\nu}U^{-1}\partial_{\mu}U) = F_{\mu\nu}{}^{(1-\alpha)}$$

Proof. From Corollary (1):

$$F_{\mu\nu}^{(\alpha)} = -\alpha(1-\alpha)(U^{-1}\partial_\mu U\, U^{-1}\partial_\nu U - U^{-1}\partial_\nu U\, U^{-1}\partial_\mu U) \tag{12}$$

Using now

$$U^{-1}\partial_\mu U + \partial_\mu U^{-1} U = 0 \tag{13}$$

the corollary follows immediately.

Theorem 3. According to the definition

$$j_\nu = \partial^\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \tag{14}$$

we have, in the gauge of corollary 2

$$j_\nu^{(\alpha)} = \partial^\mu F_{\mu\nu}^{(\alpha)} + \alpha[\phi_\mu, F_{\mu\nu}^{(\alpha)}] \tag{15}$$

Then, from corollary (1) we can deduce the following theorem:

$$j_\nu^{(\alpha)} - j_\nu^{(1-\alpha)} = \alpha(1-\alpha)(2\alpha-1)[\phi_\mu, [\phi_\mu, \phi_\nu]] \tag{16}$$

$$j_\nu^{(\alpha)} + j_\nu^{(1-\alpha)} = 2\alpha(1-\alpha)\{\partial_\mu[\phi_\mu, \phi_\nu] + \frac{1}{2}[\phi^\mu, [\phi^\mu, \phi_\nu]]\} \tag{17}$$

$$j_\nu^{(\alpha)} + j_\nu^{(1-\alpha)} = 8\alpha(1-\alpha) j_\nu^{(\frac{1}{2})} \tag{18}$$

The latter formula follows from (15) (for $\alpha = \frac{1}{2}$), and (17).

Corollary 4. If $j_\nu^{(\frac{1}{2})}$ is zero, then for any α ,

$$j_\nu^{(\alpha)} = - j_\nu^{(1-\alpha)} .$$

Theorem 4. If $j_\nu^{(\alpha)}$ is zero, for some particular value α_0, then, for any other α it is proportional to the divergence of the field $F_{\mu\nu}^{(\alpha)}$.

Proof. From (15), with $j_\nu^{(\alpha_0)} = 0$ and (6), we deduce

$$[\phi_\mu, [\phi_\mu , \phi_\nu]] = - \frac{1}{\alpha_0} \partial^\mu [\phi_\mu , \phi_\nu] . \tag{19}$$

Replacing now (19) in (15) (remembering (6)), we have:

$$j_\nu^{(\alpha)} = \partial^\mu F_{\mu\nu}^{(\alpha)} - \frac{\alpha}{\alpha_0} \partial^\mu F_{\mu\nu}^{(\alpha)} = (1-\frac{\alpha}{\alpha_0}) \partial^\mu F_{\mu\nu}^{(\alpha)}$$

which proves the theorem.

§ 3. Gauge independence of $A^{(\alpha)}$, $A^{(1-\alpha)}$

Up to this point we have given some general theorems on different potentials giving the same fields. One can now rise a natural question: Could it be that these potentials are not essentially different, but in fact one of them is the gauge transform of the other?

In order to answer this question let us take again

$$A_\mu^{(\alpha)} = \alpha \phi_\mu \quad \text{where } \phi_\mu \text{ is a vacuum potential.}$$

Suppose now that $A^{(\alpha)}$ and $A^{(1-\alpha)}$ are related by a gauge transformation, i.e.

$$(20) \qquad V^{-1}\,\alpha\,\phi_\mu V + V^{-1}\,\partial_\mu V = (1-\alpha)\phi_\mu$$

where V depends on α. As the corresponding fields are equal, it follows that

$$(21) \qquad V\,F^{(1-\alpha)}_{\mu\nu}\,V^{-1} = F^{(1-\alpha)}_{\mu\nu} = F^{(\alpha)}_{\mu\nu}$$

It follows from (20), with $\alpha = 0$ that

$$(22) \qquad \phi_\mu = V_0^{-1}\,\partial_\mu V_0$$

where $V_0 = V(\alpha)|_{\alpha=0}$.

Using (11) in (21) and (22) we get

$$(23) \qquad V_0(\partial_\mu V_0^{-1}\partial_\nu V_0 - \partial_\nu V_0^{-1}\partial_\mu V_0)V_0^{-1} = \partial_\mu V_0^{-1}\partial_\nu V_0 - \partial_\nu V_0^{-1}\partial_\mu V_0$$

and, as $V_0\partial_\mu V_0^{-1} = -(\partial_\mu V_0)V_0^{-1}$

$$(24) \qquad \partial_\mu V_0\partial_\nu V_0^{-1} - \partial_\nu V_0\partial_\mu V_0^{-1} = \partial_\mu V_0^{-1}\partial_\nu V_0 - \partial_\nu V_0^{-1}\partial_\mu V_0$$

which is a necessary condition for the potentials to be connected by a gauge transformation.

We will show now that there exist vacuum potentials which do not fulfill (24). We shall particularize with SU(2), for which the general form of V_0 is

(25) $$V_0 = e^{if\sigma} \quad , \quad \text{where } \sigma = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} \quad , \quad |\underset{\sim}{n}| = 1$$

and $\sigma_{1,2,3}$ are the Pauli matrices.

From (25), we have:

(26) $$V_0 = \cos f + i\,\sigma \operatorname{sen} f$$

$$V_0^{-1} = \cos f - i\,\sigma \operatorname{sen} f,$$

and

(27) $$\partial_\mu V_0 = i\, V_0\, \sigma\, \partial_\mu f + i \operatorname{sen} f\, \partial_\mu \sigma$$

$$\partial_\mu V_0^{-1} = -\, i\, V_0^{-1}\, \sigma\, \partial_\mu f - i \operatorname{sen} f\, \partial_\mu \sigma$$

from which we deduce

(28) $$(\partial_\mu V_0 \partial_\nu V_0^{-1} - \partial_\nu V_0 \partial_\mu V_0^{-1}) - (\partial_\mu V_0^{-1} \partial_\nu V_0 - \partial_\nu V_0^{-1} \partial_\mu V_0) =$$

$$= 4\, i \operatorname{sen}^2 f(\partial_\mu f \partial_\nu \sigma - \partial_\nu f \partial_\mu \sigma) \ ,$$

showing that the condition (24) is only satisfied if (25) fulfills

(29) $$\partial_\mu f \partial_\nu \sigma - \partial_\nu f \partial_\mu \sigma = 0.$$

This is a necessary condition for the existence of a gauge transformation.

In the special case of three dimensions and taking $\underset{\sim}{n} = \frac{\underset{\sim}{r}}{r}$ i.e. $= \frac{\underset{\sim}{\sigma} \cdot \underset{\sim}{r}}{r} = \sigma_r$, we can write (20) in the form

(30) $$\underset{\sim}{\nabla} f \wedge \underset{\sim}{\nabla} \sigma_r = 0.$$

As $\underset{\sim}{\nabla} \sigma_r = \frac{1}{r} (\underset{\sim}{\sigma} - \sigma_r \frac{\underset{\sim}{r}}{r})$:

(31) $$\underset{\sim}{\nabla} f \wedge \underset{\sim}{\sigma} - \underset{\sim}{\nabla} f \wedge \underset{\sim}{r} \frac{\sigma_r}{r} = 0$$

By a scalar multiplication with $\underset{\sim}{r}$, we immediately deduce

$$\underset{\sim}{r} \wedge \underset{\sim}{\nabla} f = 0$$

and so, in (31) we must have

(32) $$\underset{\sim}{\nabla} f = 0$$

i.e.;

(33) $$f = \text{constant}.$$

When (33) is satisfied, the gauge transformation relating $\underset{\sim}{A}^{(\alpha)} = \alpha\, e^{-if\sigma_r} \underset{\sim}{\nabla}\, e^{if\sigma_r}$, and $\underset{\sim}{A}^{(1-\alpha)}$, is

(34) $$V = e^{iv\sigma_r} \quad ; \quad tgv = (1-2\alpha)\, tgf.$$

§ 4. The case of three dimensions

We have just seen that in three dimensions (and SU_2) the integrability condition (29) is only satisfied with a constant f in which case $\underset{\sim}{A}^{(\alpha)}$ and $\underset{\sim}{A}^{(1-\alpha)}$ are not physically different due to the existence of the gauge transformation (34).

Let us take $f = \frac{\pi}{2}$, for which (25) gives $V_0 = i\sigma$, and

$$\underset{\sim}{A}^{(\alpha)} = \alpha\, \sigma\, \underset{\sim}{\nabla}\, \sigma \quad ; \quad \sigma = \underset{\sim}{\sigma} \cdot \underset{\sim}{n} \tag{35}$$

It is easy to see directly that $V_0 = i\sigma$ transforms $\underset{\sim}{A}^{(\alpha)}$ in $\underset{\sim}{A}^{(1-\alpha)}$.

The field corresponding to (35) is: ($\underset{\sim}{B}$ is the dual of F_{ij}) :

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A} + \underset{\sim}{A} \wedge \underset{\sim}{A}$$

$$\underset{\sim}{B} = \alpha(1-\alpha)\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma \tag{36}$$

In order to construct another potential (not equivalent to (35)) for the field (36) we will first prove the following

Lemma:

$$\underset{\sim}{K} = \underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma\ \sigma \tag{37}$$

is an ordinary vector (free from Pauli matrices) having zero divergence.

Proof.

$$K_i = \varepsilon_{ijk}\, \partial_j\, \sigma\, \partial_k\, \sigma\, \sigma$$

$$K_i = \varepsilon_{ijk}\, \partial_j\, n_a\, \partial_k\, n_b\, (\delta_{ab} + i\varepsilon_{abc}\, \sigma_c)\sigma_d\, n_d$$

The δ_{ab}-term does not contribute as

$$n_a\, \partial_i\, n_a = 0. \tag{38}$$

We are left with:

$$K_i = i\varepsilon_{ijk}\, \partial_j\, n_a\, \partial_k\, n_b\, \varepsilon_{abc}\, n_d(\delta_{dc} + i\varepsilon_{dce}\, \sigma_e)$$

This time the term in ε_{dce} does not contribute as

$$\varepsilon_{abc}\varepsilon_{dce} = \delta_{ae}\delta_{bd} - \delta_{ad}\delta_{be}. \tag{39}$$

Then

$$K_i = i\, \varepsilon_{ijk}\partial_j n_a \partial_k n_b n_c \varepsilon_{abc}\ , \tag{40}$$

proving the first part of our lemma.

We now take the divergence of K_i.

$$\partial_i K_i = i\, \varepsilon_{ijk}\, \partial_j\, n_a\, \partial_k\, n_b\, \partial_i\, n_c\, \varepsilon_{abc}$$

In this expression,

$$f_{abc} = \varepsilon_{ijk}\, \partial_j\, n_a\, \partial_k\, n_b\, \partial_i\, n_c \tag{41}$$

is a completely antisymmetric tensor of the third rank. Due to (38), this tensor is totally orthogonal to $\underset{\sim}{n}$

$$n_a\, f_{abc} = 0$$

The tensor (41) belongs to a two-dimensional sub-space (orthogonal to $\underset{\sim}{n}$). But we know that a completely antisymmetric tensor cannot exist when the rank is greater than the number of dimension. So f_{abc} is identically zero and the lemma is proved.

As an immediate consequence of the lemma, we have:

$$\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma = i\,\sigma\, \underset{\sim}{\nabla} \wedge \underset{\sim}{a} \tag{42}$$

for some vector field $\underset{\sim}{a}(\underset{\sim}{r})$.

It is easy to see that

$$\{\sigma\, , \underset{\sim}{\nabla}\sigma\} = \underset{\sim}{\nabla}\, \sigma^2 = 0. \tag{43}$$

So that, from (42) we deduce

$$\underset{\sim}{\nabla}\sigma \wedge (\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma) = (\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma) \wedge \underset{\sim}{\nabla}\sigma \tag{44}$$

We are now in a position to prove the following

Theorem 5. The potentials $\underset{\sim}{A}^{(\alpha)}$, $\underset{\sim}{A}^{(1-\alpha)}$ (given by (35)) and

$$\overline{\underset{\sim}{A}}^{(\alpha)} = \frac{1}{2}\,\sigma\,\underset{\sim}{\nabla}\sigma - i\left(\alpha-\frac{1}{2}\right)^2 \sigma\,\underset{\sim}{a} \quad (\underset{\sim}{a} \text{ satisfying } (42)), \tag{45}$$

give exactly the same field (36).

Proof. Let us compute the field due to $\overline{\underset{\sim}{A}}^{(\alpha)}$.

$$\underset{\sim}{\nabla} \wedge \overline{\underset{\sim}{A}}^{(\alpha)} = \frac{1}{2}\,\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma - i\left(\alpha-\frac{1}{2}\right)^2 \underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{a} - i\left(\alpha-\frac{1}{2}\right)^2 \sigma\underset{\sim}{\nabla} \wedge \underset{\sim}{a}$$

Using (42),

$$\underset{\sim}{\nabla} \wedge \overline{\underset{\sim}{A}}^{(\alpha)} = \left[\frac{1}{2} - \left(\alpha-\frac{1}{2}\right)^2\right] \underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma - i\left(\alpha-\frac{1}{2}\right)^2 \underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{a} \tag{46}$$

$$\overline{\underset{\sim}{A}}^{(\alpha)} \wedge \overline{\underset{\sim}{A}}^{(\alpha)} = -\frac{1}{4}\,\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma - \frac{i}{2}\left(\alpha-\frac{1}{2}\right)^2 \left(\sigma\underset{\sim}{\nabla}\sigma \wedge \vec{\underset{\sim}{a}}\sigma + \underset{\sim}{a}\sigma \wedge \sigma\underset{\sim}{\nabla}\sigma\right) \tag{47}$$

Both terms in the last parenthesis of (47) are equal and opposite to $\underset{\sim}{\nabla}\sigma \wedge \vec{\underset{\sim}{a}}$. Then, when we add (46) and (47) we find:

$$\overline{\underset{\sim}{B}}^{(\alpha)} = \left(\frac{1}{4} - \left(\alpha - \frac{1}{2}\right)^2\right)\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma$$

$$\overline{\underset{\sim}{B}}^{(\alpha)} = \alpha(1-\alpha)\underset{\sim}{\nabla}\sigma \wedge \underset{\sim}{\nabla}\sigma = i\alpha(1-\alpha)\sigma\,\underset{\sim}{\nabla} \wedge \underset{\sim}{a} \tag{48}$$

which coincides with (36) and proves the theorem.

It is not difficult to calculate the currents corresponding to (35) and (45). We only give the final answer :

(49) $\quad \mathbf{j}^{(\alpha)} = i\,\alpha\,(1-\alpha)(1-2\alpha)\boldsymbol{\nabla}\sigma \wedge \boldsymbol{\nabla} \wedge \mathbf{a} + i\alpha(1-\alpha)\sigma\, \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \mathbf{a}$

(50) $\quad \bar{\mathbf{j}}^{(\alpha)} = i\,\alpha(1-\alpha)\sigma\, \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \mathbf{a}$

It is easy to see that $V_0 = i\sigma$ transforms $\mathbf{j}^{(\alpha)}$ in $\mathbf{j}^{(1-\alpha)}$ (= $\sigma\, \mathbf{j}^{(\alpha)}\, \sigma$). On the other hand, no possible gauge transformation relating (35) and (45) can exist; for such a transformation would commute with σ (as $\mathbf{B}$ is invariant); but then it would also commute with $\bar{\mathbf{j}}^{(\alpha)}$ (cf. (50)), leaving it invariant.

When $\boldsymbol{\nabla} \wedge \mathbf{a}$ is curless

(51) $\quad \bar{\mathbf{j}}^{(\alpha)} = 0 \qquad (\text{if } \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \mathbf{a} = 0)$

while

(52) $\quad \mathbf{j}^{(\alpha)} = i\,\alpha\,(1-\alpha)(1-2\alpha)\boldsymbol{\nabla}\sigma \wedge \boldsymbol{\nabla} \wedge \mathbf{a} = -\,\mathbf{j}^{(1-\alpha)}$

The example of reference [1] belongs to this class with $\boldsymbol{\nabla} \wedge \mathbf{a} = \frac{\mathbf{r}}{r^3}$. We would like to point out that (45) is a kind of "magnetic" potential referred to the $\vec{n}$ isotopic direction, and for which $\mathbf{a}$ plays the role of the usual vector potential. In fact we have

$$\bar{\mathbf{A}}^{(\alpha)} = \frac{1}{2}\,\sigma\,\boldsymbol{\nabla}\sigma - i\left(\alpha-\frac{1}{2}\right)^2 \sigma\,\mathbf{a} = \bar{\mathbf{A}}^{(0)} + i\alpha(1-\alpha)\sigma\,\mathbf{a}$$

As $\bar{\mathbf{A}}^{(0)}$ is a vacuum potential (see (48)) it can be eliminated by an appropriate gauge transformation V. The new potential is

$$\bar{\mathbf{A}}^{(\alpha)\,'} = i\alpha(1-\alpha)\bar{\sigma}\,\mathbf{a} \;;\; \text{where } \bar{\sigma} = V^{-1}\sigma V.$$

The corresponding field and current are (see (48) and (50))

$$\bar{\underset{\sim}{B}}^{(\alpha)'} = i\alpha(1-\alpha)\bar{\sigma}\, \underset{\sim}{\nabla} \wedge \underset{\sim}{a}$$

$$\bar{\underset{\sim}{j}}^{(\alpha)'} = i\alpha(1-\alpha)\bar{\sigma}\, \underset{\sim}{\nabla} \wedge \underset{\sim}{\nabla} \wedge \underset{\sim}{a}$$

From which it follows that $\bar{\sigma}$ is a constant matrix.

Further, under a gauge transformation $e^{iv\bar{\sigma}}$

$$\underset{\sim}{A}'' = e^{-iv\bar{\sigma}}\, \bar{\underset{\sim}{A}}'\, e^{iv\bar{\sigma}} + e^{-iv\bar{\sigma}}\, \underset{\sim}{\nabla}\, e^{iv\bar{\sigma}} = \bar{\underset{\sim}{A}}' + i\bar{\sigma}\underset{\sim}{\nabla} v$$

(53) $$\underset{\sim}{A}'' = i\,\bar{\sigma}\,\underset{\sim}{a}'$$

where $\underset{\sim}{a}' = \underset{\sim}{a} - \underset{\sim}{\nabla} \dfrac{v}{\alpha(1-\alpha)}$,

while $\bar{\underset{\sim}{B}}$ and $\bar{\underset{\sim}{j}}$ remain invariant.

§ 5. Example in four dimensions

Let us consider the following example:

(54) $$U = e^{i \operatorname{arctg} \frac{x_0}{r} \sigma_r} = \frac{x_0 + i\underset{\sim}{\sigma}\cdot\underset{\sim}{r}}{\sqrt{x^2}}$$

From which

(55) $$A_\mu^{(\alpha)} = \alpha\, U^{-1}\, \partial_\mu U = -2i\alpha\, \frac{\sigma_{\mu\nu} x_\nu}{x^2}$$

with $\sigma_{ij} = \frac{1}{2}\, \varepsilon_{ijk}\, \sigma_k$ and $\sigma_{i4} = \frac{1}{2}\, \sigma_i$.

The field corresponding to (55) is:

(56) $$F_{\mu\nu}^{(\alpha)} = \frac{4i\alpha(1-\alpha)}{x^4}\, (x_\mu \sigma_{\nu\rho} x_\rho - x_\nu \sigma_{\mu\rho} x_\rho + x^2 \sigma_{\mu\nu})$$

for which the current is:

$$j_\mu^{(\alpha)} = 8i\alpha\,(1-\alpha)(1-2\alpha)\,\frac{\sigma_{\mu\rho}x_\rho}{x^4} = -\,j_\mu^{(1-\alpha)} \tag{57}$$

We see that $j_\mu^{(\frac{1}{2})} = 0$, so corollary 4 is in force. In this example two equal and opposite currents give raise to the same field $F_{\mu\nu}^{(\alpha)}$. In this case it is possible to show directly (without recourse to (24)) that a gauge transformation relating $A^{(\alpha)}$ and $A^{(1-\alpha)}$ cannot exist. In fact, such a transformation V should commute with $F_{\mu\nu}$ and anticommute with (57). The last condition is easily shown to imply the anticommutativity of V with all three Pauli matrices.

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