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ABSTRACT

A model is presented in which electromagnetic and weak type short-range forces are related to geometric properties of the space-time.

A dependence of the coupling constant with the range of the interaction is shown to exist as a consequence of our geometrization scheme.

I - INTRODUCTION

We present here a model by means of which a new category of forces is introduced by a modification of the riemanian nature of space-time. It represents interaction between two particles through an intermediate current that violates parity conservation (in the special case we will treat here) and shows a dependence of the coupling constant on the range of the interaction. Weak and electromagnetic processes are examples of a hierarchy of forces contained in our model.

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Section II introduces the mathematical entities that will be used to represent the structure of space-time. The geometrical properties are given by a Restricted Cartan Geometry (RCG) endowed with an affine connection which has eight degrees of freedom, instead of 24. The non-metrical part of the connection is a functional of the torsion, the anti--symmetric part of the connection. It will be characterized by two four-vectors T^{μ} and Z^{μ}, that are respectivelly the trace and the pseudo-trace of the torsion, as shown in the text.

In Section III we develop the equation of motion for spin one-half field in RCG. The two geometric vector fields interact with the spinor particle through a combination of violating and conserved current. We examine the special case in which both fields degenerate into one. Then, as we will show, only a parity-violating term survives.

Section IV shows how a hierarchy of forces are contained in our model. The relationship between our geometric approach and gauge theories are sketched out and some general remarks are made.

II - CARTAN RESTRICTED GEOMETRY

The geometrical properties of space-time are contained in the symmetrical metric tensor $g_{\mu\nu}$ and in the independent affine connection $\Gamma^{\alpha}_{\mu\nu}$. Calling $\{^{\alpha}_{\mu\nu}\}$ the Christoffel symbol, we write

$$\Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\} + C^{\alpha}_{\mu\nu} \tag{1}$$

The tensor $C^{\alpha}_{\mu\nu}$ is given in terms of the antisymmetric part of the connection, called torsion $\tau^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ through the expression

$$2C^{\alpha}_{\mu\lambda} = \tau^{\alpha}_{\mu\lambda} - \tau^{\varepsilon}_{\nu\lambda}\, g_{\varepsilon\mu}\, g^{\nu\alpha} - \tau^{\varepsilon}_{\nu\mu}\, g_{\varepsilon\lambda} g^{\nu\alpha} \tag{2}$$

This geometry has been used previously in many occasions and recently some authors(1) have given emphasis to a discussion of the influence of spin on properties of space-time based on Cartan geometry. However, we will use it here in a new and very different context.

By reasons which will appear clear later on, we restrict the torsion, that has 24 degrees of freedom, by freezing 16 of them. This will be achieved by setting its irreducible trace-less part equal to zero. We are then left with only two vector fields to characterize the full torsion. Indeed, we write

$$\tau^{\alpha}_{\mu\nu} = \frac{\lambda}{3}\left[\delta^{\alpha}_{\nu} T_{\mu} - \delta^{\alpha}_{\mu} T_{\nu}\right] + \lambda\, \eta^{\alpha}_{\mu\nu\rho}\, Z^{\rho} \tag{3}$$

in which the trace T^{μ} and the pseudo-trace Z^{μ} are defined respectivelly by

$$T_{\mu} = \tau^{\alpha}_{\mu\alpha}$$

$$Z_{\lambda} = \frac{1}{6}\, \eta_{\lambda\alpha}{}^{\mu\nu} \tau^{\alpha}_{\mu\nu}$$

$\eta_{\mu\nu\rho\sigma} = \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}$; ($\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita Symbol) and λ is a constant. The reason to introduce such constant will be made clear later on.

A geometry which has this property will be called a Cartan Restricted Geometry (CRG). The full properties of CRG will be discussed elsewhere.

Let us turn now to dynamics. CRG is an extension of Riemannian geometry. The continuum space-time is the limit of Cartan geometry when the torsion tensor is null everywhere. The equation of motion for the geometric vector fields T^{μ} and Z^{μ} are to be constrained by this limit on the geometry. In other words, when torsion goes to zero the dynamics of space-time must be given by Einstein's equations.

In order to be able to establish such contact with Einstein's theory, the most natural assumption for the action of the fields is given by

$$A = \frac{1}{K} \int \sqrt{-g}\,.\ R\, d^4x + \int \sqrt{-g}\, \mathcal{L}\,(T^{\mu}) d^4x + \int \sqrt{-g}\, \mathcal{L}(Z^{\mu}) d^4x \qquad (4)$$

To deal with a specific model let us assume a Maxwellian-type Lagrangian for both geometric vector fields:

$$\mathcal{L}\,(T^{\mu}) = \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \qquad (5a)$$

$$\mathcal{L}\,(Z^{\mu}) = \frac{1}{4} H_{\mu\nu} H^{\mu\nu} \qquad (5b)$$

where

$$B_{\mu\nu} \equiv \nabla_{\nu} T_{\mu} - \nabla_{\mu} T_{\nu} = \overset{*}{\nabla}_{\nu} T_{\mu} - \overset{*}{\nabla}_{\mu} T_{\nu} + \eta_{\mu\nu\varepsilon\rho}\, T^{\varepsilon} Z^{\rho} \qquad (6a)$$

$$H_{\mu\nu} \equiv \nabla_{\nu} Z_{\mu} - \nabla_{\mu} Z_{\nu} = \overset{*}{\nabla}_{\nu} Z_{\mu} - \overset{*}{\nabla}_{\mu} Z_{\nu} + \frac{1}{2} (Z_{\nu} T_{\mu} - Z_{\mu} T_{\nu}) \qquad (6b)$$

The covariant derivative of a generic vector Q^{μ} is equal to

$$\nabla_{\mu} Q_{\nu} \equiv \frac{\partial Q_{\nu}}{\partial x^{\mu}} - \left\{ {\varepsilon \atop \mu\nu} \right\} Q_{\varepsilon} - C^{\varepsilon}_{\mu\nu} Q_{\varepsilon} \equiv \overset{*}{\nabla}_{\mu} Q_{\nu} - C^{\varepsilon}_{\mu\nu}\, Q_{\varepsilon} \qquad (7)$$

(we remark that any geometrical quantity with an asterisk is to be evaluated in the associated Riemann space).

Lagrangian (5) shows that the geometric vector fields have, as usual, dimensionality of $(\text{energy})^{1/2}$ $(\text{length})^{-1/2}$. So, in order to the connection (1) to have the correct dimensionality $(\text{length})^{-1}$, the constant λ must have the dimensionality $[\lambda] = (\text{energy})^{-1/2}$ $(\text{length})^{-1/2}$. This is the reason to introduce such constant λ in expression (3).

We will make use of a relation between the scalar of curvature R in the RCG and the corresponding scalar $\overset{*}{R}$ in Riemanian associated space, given by

$$\sqrt{-g}\ R = \sqrt{-g}\ \overset{*}{R} - 2(\sqrt{-g}\ T^{\alpha})_{|\alpha} + \frac{2}{3}\sqrt{-g}\ T^{\alpha}T_{\alpha} - \frac{3}{2}\sqrt{-g}\ Z^{\alpha}Z_{\alpha}$$

in which a simple bar denotes usual derivative.

Using (1) and the above relation into (4) and developping the expression for the scalar of curvature, a straightforward calculation gives

$$A = \frac{1}{K}\int\sqrt{-g}\ \overset{*}{R}d^4x + \frac{2}{3}\frac{\lambda^2}{K}\int\sqrt{-g}\ T^{\alpha}T_{\alpha}d^4x - \frac{3}{2}\frac{\lambda^2}{K}\int\sqrt{-g}\ Z^{\alpha}Z_{\alpha}d^4x + \frac{1}{4}\int\sqrt{-g}\ H_{\mu\nu}H^{\mu\nu}d^4x + \frac{1}{4}\int\sqrt{-g}\ B^{\mu\nu}B_{\mu\nu}d^4x \qquad (8)$$

we see that due to the generalized Einstein's action (the term proportional to the scalar of curvature) in the RCG the geometric vector fields acquire a mass proportional do $\frac{\lambda}{\sqrt{K}}\ \frac{\hbar}{c}$.

III - THE SPINOR FIELD

The equation of motion of a spinor field in a curved space-time endowed with an affine connection $\Gamma^{\alpha}_{\mu\nu}$ is given by the Lagrangian

$$\mathcal{L}_{\psi} = \hbar c\ i\ \bar{\psi}\ \gamma^{\mu}\nabla_{\mu}\psi - mc^2\ \bar{\psi}\psi \qquad (9)$$

The covariant derivative $\nabla_\mu \psi$ is defined by

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi \tag{10}$$

In this expression Γ_μ is the generalized Fock-Ivanenko internal connection:

$$\begin{aligned}\Gamma_\nu = \frac{1}{8} \Bigg[\frac{\partial \gamma_\mu}{\partial x^\nu} \gamma^\mu - \gamma_\mu \frac{\partial \gamma^\mu}{\partial x^\nu} + \{^{\ \varepsilon}_{\mu\nu}\} (\gamma^\nu \gamma_\varepsilon - \gamma_\varepsilon \gamma^\nu) \\ + C^{\varepsilon}_{\mu\nu} (\gamma^\nu \gamma_\varepsilon - \gamma_\varepsilon \gamma^\nu) \Bigg] \equiv \overset{*}{\Gamma}_\nu + \frac{1}{8} C^{\varepsilon}_{\mu\nu} (\gamma^\nu \gamma_\varepsilon - \gamma_\varepsilon \gamma^\nu)\end{aligned} \tag{11}$$

Using (3) we find

$$C^{\alpha}_{\mu\nu} = \frac{\lambda}{3} \left[T^\alpha g_{\mu\nu} - \delta^\alpha_\mu T_\nu \right] + \frac{\lambda}{2} \eta^{\ \alpha}_{\mu\nu\rho} Z^\rho \tag{12}$$

From the whole set constituted by geometries of the restricted type belonging to RCG we choose a special and important sub-class in which the two geometric vector fields are characterized by one unique direction. Let us choose, for convenience, the constant of proportionality between them to be given by $i \frac{9}{4} C_\omega^{-1}$ and define the unique independent vector W^μ by the expression

$$T_\mu = i \frac{9}{4} \frac{1}{C_\omega} Z_\mu = \frac{i}{3} W_\mu \tag{13}$$

In this restricted class the motion of the spinor ψ will be given by

$$c \hbar i \gamma^\mu \overset{*}{\nabla}_\mu - mc^2 \psi + c \hbar \lambda W^\mu \gamma_\mu (1 + c_\omega \gamma_5) \psi = 0 \tag{14}$$

Equation (14) can be interpreted in two ways: either as an equation in RCG with only one preferred direction (trace rotated of $e^{i\frac{\pi}{2}}$ with respect to the pseudo-trace) or as a spin one-half particle in a Riemannian space interacting with a vectorial field W^{μ} through the Lagrangian

$$\mathcal{L}_I = g_\omega W^{\mu} \bar{\psi} \gamma_\mu (1 + c_\omega \gamma_5) \psi \qquad (15)$$

The coupling constant g_ω has the value

$$g_\omega = \hbar c \lambda \qquad (16)$$

Remark that the pseudo-trace is real and the trace is pure imaginary. This gives a complex structure for the connection on the region of the order of the length of the interaction. Outside this domain, the structure of space is real and riemannian.

IV - HIERARCHY OF FORCES AND CONCLUSION

The above scheme permit us to envisage a geometrical model for weak interactions. Further, a whole set of vector interactions appear which, in the special case treated here, all violates parity conservation.

Let us remark that the degree of parity violation measured by c_ω is related to a conformal property. Indeed, the Cabibbo angle that represents modifications on the weight of the axial vector current relative to the vector current due to others interactions, appears here as a consequence of making a scale transformation on the trace with respect to the pseudo-trace.

Now, an inspection on the action (8) and on the value (16) for the coupling constant shows that there is a relation between the mass of the vector field, μ_ω and the coupling constant given by

$$g_\omega = \mu_\omega \hbar c \sqrt{K} \qquad (17)$$

This expression shows a dependence of the strength of the force on the range of the interaction. This is a peculiar property of the kind of forces that are contained in our geometrical model.

In order to compare our results with the theory of weak interactions let us take for the mass of the vector field the value 10^4 times the proton mass which seems to be representative of the value of the mass of the intermediate vectorial meson.

This gives for the coupling constant g_ω the value

$$g_\omega \sim \sqrt{K} \qquad (18)$$

We remark that some 30 years ago W. Pauli has considered expression (18) in a pure numerical basis and gave no explanation for this somehow curious discovery. It appeared here as a consequence of our model and by taking the lenght of weak process to be of the order of 10^{-17}cm, which seems to be a reasonably good assumption.

Note however, that we have a continuous hierarchy of forces given by the dependence of the coupling constant on the range of the interaction. A simple inspection on this dependence shows that unless the value of the range of the interaction is of the order of 10^{-17}cm or less, the coupling constant will be very small, i.e, there is no effective long range interaction in our model.

At the value of $\mu^{-1} \sim 10^{-31}$ cm the coupling constant g_ω has aproximatively the value 10^{-2}. This force is competitive with electromagnetic forces for very near charges. However, this length is ridiculously small by comparison with the classical radius of the electrons - and certainly it has no important effect on the interaction between electrons. Neverthless, one can envisage the possible role it could have at the interior of the electron and even the influence on its stability. For this to be true, the electron should consist of very small centers of concentration of charges. We will come to this problem in the future.

Let us make another remark about the limit for long range forces. When the mass of the vector W^{μ} goes to zero the strenght of the force tends to infinity, as shown by relation (17). So, it is impossible to make a smooth transition from our model to a current - current Lagrangian formalism.

Finally, it is not difficult to show that the general case in which T^{μ} and Z^{μ} do not define a unique direction, contains the unification of short-range weak and electromagnetic type-forces . Indeed, by a simple relative rotation of the basic geometric vector fields, both a parity conserved and a parity non-conserved vector interaction appears.

REFERENCES

(1) Hehl, F.W. - GRGJ - 4, 333(1973) and references therein.