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SCATTERING OF NUCLEONS BY NUCLEI IN THE 30-MEV REGION*

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I. INTRODUCTION

Evidence, both experimental[1,2] and theoretical,[3] has been accumulating in recent years that the mean free path of nucleons in nuclear matter is least in the 20-60 Mev region. (The evidence

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is usually expressed in terms of the imaginary part of the optical potential which is inversely proportional to the mean free path.) It might therefore be thought that compound nucleus formation is a particularly probable event in this energy region and that therefore the inelastic scattering of nucleons by nuclei in this region would be well described by the statistical theory of compound nucleus decay.[4] This theory makes the following predictions:

(a) The differential scattering cross section at a given energy is symmetrical about 90º.

(b) The energy distribution of the scattering cross section at a given angle is Maxwellian.

(c) The absolute cross section for inelastic proton scattering is very small, particularly for heavy elements, since the presence of the Coulomb barrier favors the emission of neutrons from the compound nucleus.

When these predictions were tested by scattering 31-Mev protons inelastically from medium and heavy elements,[5] results in complete contraction to the above were found, i.e., (a) the differential cross sections were strongly peaked forward; (b) the energy distribution is nearly flat, and (c) the absolute cross section is about 250 mb, which is least ten times as large as it should be according to the statistical theory.

These properties are strongly reminiscent of two-body scattering and, while this does not of course mean that compound nucleus formation does not take place at all, it must mean that there are other processes which are of much greater importance.

The above are the conclusions drawn by Eisberg and Igo from their experiments. They thereupon produced a qualitative theory, according to which inelastic proton scattering was due to two-body processes in the diffuse rim of the nucleus, while interactions between the incident proton and the target core would involve the whole nucleus, and hence lead to compound nucleus formation and subsequent neutron emission.

A different explanation has been put forward by Hayakawa et al.[6] According to them the scattering is due to knock-on processes taking place throughout the whole nuclear volume. They take a semiclassical approximation, and trace the path of the incident proton into the nucleus, there being a coefficient of transmission and one of refraction at the nuclear boundary; the proton then scatters once inside the nucleus before reemerging with a second refraction at the nuclear boundary. In their application to the results of Eisberg and Igo they make several simplifying assumptions, which enable them to do at least part of the work analytically, and the result is in very fair agreement with experiment.

The present work is an attempt to decide between the two rival theories. To do this it was necessary to perform a calculation similar to that of Hayakawa et al., but omitting the above-mentioned simplifying assumptions. This is done in Sec. II and the result is in complete disagreement with the experimental results. In Sec. III we turn to the scattering by the diffuse rim and find that we must make a distinction here between the transition region in which the nuclear density drops from its maximum value to almost zero, and the part of the nuclear density which leaks out into the classically forbidden region. Our conclusion will be that it is only this latter which contributes significantly to the process under consideration.

II. KNOCK-ON SCATTERING

A. Method

As we are concerned here with an effect due to the whole nuclear volume, we shall assume the nucleus to have a constant density and sharp boundary. We further assume that the interaction between the incident proton and the target is given by an averaged constant potential $V = -V_o$, together with an interaction with individual nucleons of the target. The former leads to elastic and the latter to inelastic scattering, and we shall assume for this latter the usual impulse approximation, i.e., that the two nucleons can be considered independent of the rest of the nucleus, while they are interacting. In this way we can use the experimental two-body scattering data. It turns out that the choice of numerical values for the constants is not critical. We have taken the nuclear potential radius to be given by $R = 1.35 \times A^{1/3}$ in agreement with Saxon's results[1] (lengths will be measured in units of 10^{-13} cm), and the potential $V_o = 40$ Mev. We use this value for both protons and neutrons, i.e., we ignore the Coulomb effect for protons. This is not as bad as it seems, because recent experiments,[1,7] have shown that the purely nuclear part of the potential is a good deal deeper for protons than for neutrons, so that after the addition of the Coulomb potential to the proton well, the two are not too different from each other. The value chosen is also a compromise between the correct value for the target nucleons and that for the incident proton. Because of the velocity dependence of the potential,[1] these are not the same. It will be noticed that this potential is somewhat deeper than would be obtained from the degenerate gas model for the given radius, but there seems no good reason why this model should give the correct answer. We take the binding energy of the least bound nucleon to be 8 Mev.

Because of the short wavelengths of the protons at this energy ($\lambdabar = 1.2$ outside the nucleus and $\lambdabar = 0.8$ inside) we felt justified in using a semiclassical approach. By this we mean that we let the proton follow a Coulomb trajectory outside the nucleus, but calculate the refraction and transmission coefficients at the nuclear boundary quantum mechanically. Inside the nucleus the proton follows a straight path until it is scattered. After the scattering it emerges from the nucleus in the same way as it entered. (See Fig. 1.) The three scatterings need not of course take place in the same plane. The approach is basically the same as in reference 6, but there are important differences of detail.

Because of the complexity of the problem we have made no attempt at an analytic formulation, but have instead traced trajectories through the nucleus. This has been done systematically (see Fig. 2), not by a Monte Carlo method - i.e., we split the incident beam into contributions of equal flux, $j = 1-5$; scattering inside the nucleus was assumed to take place at equally probable scattering centers, $k = 1-5$ (because of the short mean free path the number of protons traversing the whole nucleus unscattered is negligible); the scattered beam was analyzed into scattering angles in steps of 30°, $l=1-6$, as well as azimuthally in steps of 90°, $m=1-4$; and lastly the energy of the scattered beam was analyzed into equally likely intervals, $r=1-6$. Thus a total of 3600 rays had to be traced through the nucleus. Fortunately it turned out that only a small fraction of these contributed to knock-on scattering, since most of the rays led to compound nucleus formation. The final results, which are shown in Fig. 6, are actually based on 347 rays. These results are given for the scattering by Sn, for which all the calculations in this section were made.

B. Detailed Calculation

For the meaning of the angles referred to see Fig. 1. The relationship between θ and β, given by the classical path equation, is

$$Z\alpha\,(1-\cos\beta) = 2ER\,\sin\theta(\sin\beta-\sin\theta), \qquad (2.1)$$

where Z is the atomic number, α the fine structure constant, and E the kinetic energy of the incident proton. The refraction is governed by Snell's law, which here follows simply from angular momentum conservation,

$$k_{\infty}\sin\theta = k_{o}\,\sin\alpha_{2}, \qquad (2.2)$$

where k_{∞} and k_{o} are wave numbers at infinity and inside the nucleus, respectively.[8] At this point we did not use the averaged value of the potential $V_{o} = 40$ Mev, but instead used the exact experimental value[1] at the incident energy of 31 Mev, since the angle of refraction is rather sensitive to variations in V_{o}.

To calculate the transmission coefficient, we assume the surface to be plane at the point of impact. It is clear that the approximation of a plane surface becomes worse as the angle of incidence α_1 increases, for, since the proton spends more time than it should in the strong part of the Coulomb field, the transmission coefficient will be smaller than it should be. We show below that it is possible to compensate partially for this.

For a plane surface making angle β with the incident beam, it would appear that the effective momentum at infinity is $k_{eff} = k_{\infty} \cos\beta$. However, when we then consider conservation of linear momentum parallel to the surface, we are led to the formula

$$k_{\infty} \sin\beta = k_0 \sin\alpha_2. \qquad (2.3)$$

This is in contradiction with the correct law (2.2), for which it is necessary to take

$$k_{eff} = k_{\infty} \cos\theta. \qquad (2.4)$$

This replacement of β by θ is equivalent to curving the surface away from the path and so make it more nearly spherical. The transmission coefficient T is then obtained in standard fashion[9] with the help of Coulomb wave function tables,[10] and the results are shown in Fig. 3. The largest value of θ for which a classical trajectory enters the nucleus is $\theta = 55^\circ$. It is clear then that the inclusion of a quantum mechanical leakage of the incident beam into the nucleus would not change the results significantly, since for angles larger than 55° the transmission coefficient is so small. It is also clear that at the energy considered the smoothing out of the nuclear boundary would not significantly increase the

the transmission coefficient, since it is already of the order 0.8 - 0.9 for most θ.

The method for calculating the differential cross-section $d^2\sigma/d\omega dE_f$ for the scattering of the incident proton by a target nucleon into solid angle $d\omega$ and final kinetic energy E_f (in the laboratory system) was first developed by Goldberger,[11] using the impulse approximation and assuming that the two-body scattering cross-section inside nuclear matter is the same as for free particles of the same relative momentum. At the energies concerned this is isotropic for p-p scattering and not far from isotropic for n-p scattering. The total two-body cross section is very nearly inversely proportional to the energy of the particles and the best fit to the data[12] is given by

$$\sigma^{np} = 8/E_L \text{ barns}, \quad \sigma^{np} = 3/E_L \text{ barns}, \qquad (2.5)$$

where E_L is the energy in the lab system in Mev. For the total cross section inside nuclear matter we can take either the value of E_L corresponding to the lab energy of the incident particle[11] or else take into account also the energy of the target particle.[6] The latter method is of course more correct, but it leads to considerably more complicated formulas. It is important to use it in estimating mean free paths, but at the energy with which we are dealing it gives almost identical results as the simpler method for the differential cross sections.[6] The result[13] is best expressed in terms of P_i, P_f and P_F, the momenta of the incident and one of the final particles, and of the most energetic target particle, respectively:

$$\frac{d^2\sigma}{d\omega dE_f} = \frac{3\bar{\sigma} P_f}{2\pi P_F^3 P_i^3} \times \begin{cases} Q - 2S, & Q - P_F \leq S, \\ (P_F^2 - S^2)/2Q, & Q - P_F \geq S, \end{cases} \tag{2.6}$$

where

$$Q = |\vec{P}_i - \vec{P}_f|, \quad S = (Q^2 - P_i^{\,2} + P_f^{\,2})/2Q,$$

and [see (2.5)]

$$\bar{\sigma} = (8N + 3Z)/A.$$

This is plotted for our case in Fig. 4. When (2.6) is integrated (the integration is troublesome, but not difficult) over all scattering angles, we obtain the energy distribution of the scattered particles,

$$\frac{d\sigma}{dE_f} = \frac{2\bar{\sigma}}{P_i^2} \times \begin{cases} 1, & 2P_F^2 \leq P_i^2\ , \\ 1 - \left[1 - (P_i^2 - P_f^2)/P_F^2\right]^{3/2}, & 2P_F^2 \geq P_i^2\ . \end{cases} \tag{2.7}$$

From this formula we obtain the division into energy intervals of equal probability, r = 1-6. The average energy loss $E_i - \bar{E}_f$ is given by

$$\int_{E_F}^{\bar{E}_f} \frac{d\sigma}{dE_f}\, dE_f = \frac{1}{2}\sigma\ , \tag{2.8}$$

where E_F is the kinetic energy corresponding to momentum P_F. It should be noted that the above formulation takes account of the exclusion principle, according to which the initial momentum of the target nucleon is less than P_F and the final momenta of the two particles are both greater than P_F .

The question next arises whether a particle could scatter twice before emerging from the nucleus, or whether after two scatterings the particle has lost so much energy that it will be bound in the compound nucleus. This problem is not so simple as it seems, because as long as the particle has sufficient kinetic energy, i.e., more than 40 Mev, to leave the nucleus at all, then the less kinetic energy it has, the longer is its mean free path and so the greater its chance of leaving the nucleus. This argument is correct for neutrons and more detailed calculations, using (2.8), show that about half the neutrons which are knocked-on in the first collision and then suffer a further collision, leave the nucleus after that. Because of the Coulomb barrier this is not true for protons at this energy, and in fact we can assume that the number of protons leaving the nucleus after two collisions is negligible.

We therefore take it that protons emerging from the first collision proceed along a straight path to the nuclear boundary, being attenuated on the way according to the usual exponential law associated with the correct mean free path. The transmission and refraction coefficients at the nuclear boundary are calculated as before, but it is clear that the lower the energy of the emerging protons, the less valid becomes the approximation of a sharp nuclear boundary. For a more realistic model the transmission coefficients, as given in Fig. 5, would all be somewhat larger. However, and this is the crucial result of this calculation, the angle at which the transmission coefficient drops abruptly is practically independent of the sharpness of the nuclear boundary. This is the angle at which classically total internal reflection occurs, i.e., it is a function simply of the component of linear momentum of the particle perpendicular to the boundary, and so is not a function of the sharpness of the boundary. Now for about 90% of the rays traced through the nucleus, the angle α_3 turns out to be larger than this critical angle. It is for this reason that the knock-on cross section comes out so small.

We can now see why the previous calculation[6] obtained so much larger a cross section. Apart from some other approximations, which are more defensible or at least have less effect on the final result, it was assumed there that particles after the scattering take on the whole the shortest possible path out of the nucleus. This means of course that $\alpha_3 \simeq 0$ for all rays, so that the reduction in the cross section due to total internal reflection never arises.

The total scattering angle Θ is easily obtained for cases m = 1 or 3, when the two refracting angles and the scattering angles are in the same plane. For m = 2 or 4, the geometry is more complicated, but nearly all the work can be done graphically. Lastly it must be remembered that for cases where protons knock on protons the cross section must be counted twice. The final results are given in Fig. 6, where the error brackets indicate the uncertainties due to the small number of rays used. The total cross section is about 15 mb and so more than ten times smaller than the experimental one. Knock-on processes are therefore no more important than compound nucleus processes for inelastic proton scattering at this energy.

The situation would be very different if we were discussing (p,n) reactions. Then the total internal reflection effect would be very much less important, double and maybe even triple scattering would have to be included in the knock-on cross section, and the compound nucleus cross section would of course be at least an order of magnitude larger. It is clear that a discussion of the (p,n) reaction along the lines of this paper would be a formidable undertaking.

III. DIFFUSE RIM SCATTERING

As the knock-on process does not lead to a large cross section, it is natural next to investigate the scattering by the diffuse rim of the nucleus. For this purpose we compare the nuclear potential[1] and the nuclear mass distribution which we take to be proportional to the nuclear charge distribution.[14] Although this may not be absolutely correct, it is certainly a good approximation.[15] We then see (Fig. 7) that even the surface region in which the density drops from its maximum value to almost zero is still well within the nuclear force field, and so the remarks we made about total internal reflection apply to this region too. To obtain scattering effects of any size we must therefore consider as targets those nucleons which have leaked out into the classically forbidden region and so are temporarily almost unbound. In this region the Coulomb barrier is highest, and so the kinetic energy of the incident protons least. This of course leads to increased cross sections. That this region extends outwards to a fairly considerable distance ($\sim 1.7 \times A^{1/3}$) has been shown by inelastic cross section measurements at high energies.[16]

The momentum distribution of particles in the diffuse rim can be otained by a Fourier analysis of the tail of the wave function. In this region the wave function is of course given by $\psi(r) = Ae^{-\alpha r}/r$, where A is a normalization constant, and $\alpha = (2MB)^{1/2}/\hbar$, where B is the binding energy of the least bound nucleons. This we take to be 8 Mev. Now as far as the inelastically scattered protons are concerned, the scattering is due entirely to the tail of the wave function, and we therefore Fourier-analyze the function

$$\psi(r) = 0, \quad r < R_0; \quad \psi(r) = Ae^{-\alpha r}/r, \quad r > R_0, \qquad (3.1)$$

where $R_0 \simeq 1.4 \times A^{1/3}$. This is in accord with the impulse approximation, and the validity of this approach has been discussed by Austern et al.[17] The result is

$$\psi(k) = \frac{4\pi A}{k(\alpha^2+k^2)}(\alpha \sin kR_0 - k \cos kR_0)e^{-\alpha R_0}. \qquad (3.2)$$

The scattering cross section is now obtained by a method very similar to that of Goldberger.[11] There are two differences: (a) the momentum distribution of the target particles is not uniform up to a maximum, as is the case in a Fermi gas, but is given by $\rho(k) = |\psi(k)|^2$, and (b) the Pauli principle is not operative in the diffuse rim. Instead we have the requirement that the final state of the target nucleus has more energy than the initial one, which is the ground state.

Because of the algebraic complexity of $\psi(k)$, we have replaced $k^2\rho(k)$, which is the expression that occurs in the integral for the scattering cross section, by a Gaussian function $k^2 \exp(-k^2/K^2)$. For $K^2 = 1/2\ \alpha^2$, this fits the exact expression well up to $k \simeq 2\alpha$. For larger values of k the Gaussian rapidly falls below the exact expression, but this is probably realistic physically, since the higher momenta are due to the sharp cutoff of the wave function at $r = R_o$.

The normalization constant A is obtained from the experiments on electron scattering.[14] These show that for a medium-sized nucleus about three nucleons are outside a radius $R_o = 1.4 \times A^{1/3}$. However, not all three will be effective for our purpose. Clearly nucleons at the back of the nucleus will not contribute to the scattering, and scattering from the front of the nucleus is likely to lead to compound nucleus formation. We shall therefore consider the number N of effective nucleons in the rim to be close to N=1.

The differential scattering cross section can now be written down. It is given by

$$\frac{d^2\sigma}{d\Omega dE_f} = \frac{N \bar{\sigma} P_f}{\pi^{-3/2} QKP_i} \exp(- S^2/Q^2), \qquad (3.3)$$

where the notation is that of Eq.(2.6). The parameters K and N are not of course inflexibly given by the theory outlined so far, and in fact the actual values given above were chosen to give the best fit to the experimental results. These and the corresponding theoretical curves are plotted in Fig. 8. The agreement is very reasonable, especially when it is remembered that at small angles it was particularly difficult in the experiment considered to disentangle experimentally the inelastic from the elastic scattering and that we have ignored the Coulomb effect on the scattering angle. This again is most serious at small angles, being of the order 5-10°. The Coulomb effect on the energy of the scattered protons has of course been taken into account. The discrepancy at the lower end of the spectrum is due to the neglect of particles that have leaked through the Coulomb barrier.

Once the two parameters have been fixed for a particular target nucleus and incident energy, we should be able to use the same parameters for different nuclei and different energies. By this we mean that K should be fitted as before, while N should be proportional to $A^{1/3}$, since the number of effective nucleons is given by the rim area rather than by the whole surface area. There are no experimental results at different energies in the region 30-50 Mev, where our theory may be considered valid, but there are results[18] for Ta, Au, and Pb at 31-Mev incident energy. The nuclides Au and Pb are not sufficiently different from each other in A and Z to give significantly different results on our theory, although they certainly give results different from those for Sn. In Fig.9 we have compared the theoretical curves with the experiments for Pb, since the experimental results for Au are incomplete. The agreement is quite satisfactory except that the experimental cross sections are somewhat lower than the calculated ones. This may be due to the magic nature of the Pb nuclide, which would tend to reduce N and increase the binding energy of the least bound nucleons. The cross sections for Au that are available are actually slightly larger than the corresponding ones for Pb. The experimental cross sections for Ta are nearly identical with those for Sn, in spite of the fact that the A value of Ta is of course much nearer to that of Au. This may be due to the considerable asphericity of the Ta nucleus.

IV. CONCLUSION

We can summarize the results as follows. We have shown conclusively that knock-on processes taking place throughout the nuclear volume contribute only in a minor way to the inelastic proton scattering cross section. The same is undoubtedly true also of processes taking place via compound nucleus formation. It is difficult to escape the conclusion that the main effect must be due to scattering by almost free nucleons in the extreme rim of the nucleus.

To obtain a quantitative estimate of this rim scattering with at least some pretence at quantum mechanical rigor is extremely difficult and has in fact not been attempted. The present semiclassical calculation is based on an estimate of the momentum distribution of the nucleons in the extreme outer region and the number of nucleons in this region at any time. There are thus two free parameters, and the above conclusions are fortified by the reasonableness of the values of these with which we can fit the experimental results. We are aware of, but have not solved the difficulty that lies in a search after the momentum distribution of part of a quantum mechanical system. All we can say is that the experiments do seem to be measuring such a thing.

Fortunately, neither the size of the two parameters, nor the functional form of the momentum distribution of the target nucleons is given very critically by the theory. Thus, for instance, almost identical results can be obtained with a distribution which is uniform up to a momentum corresponding to a kinetic energy of 6-7 Mev and then vanishes. It is, however, not at all easy to see in what way to refine the theory so as to obtain more detailed information. Not only is it difficult to obtain numerical estimates of corrections to the theory, but we would certainly also have to include then the contributions from knock-on scattering and the very inexactly known compound nucleus processes. Perhaps the most important conclusion that we can draw is that the experiment investigates a very particular region of the nucleus, i.e., the extreme limit of the mass distribution, to the practical exclusion of all others.

We should like to thank Professor V.F. Weisskopf for many helpful discussions.

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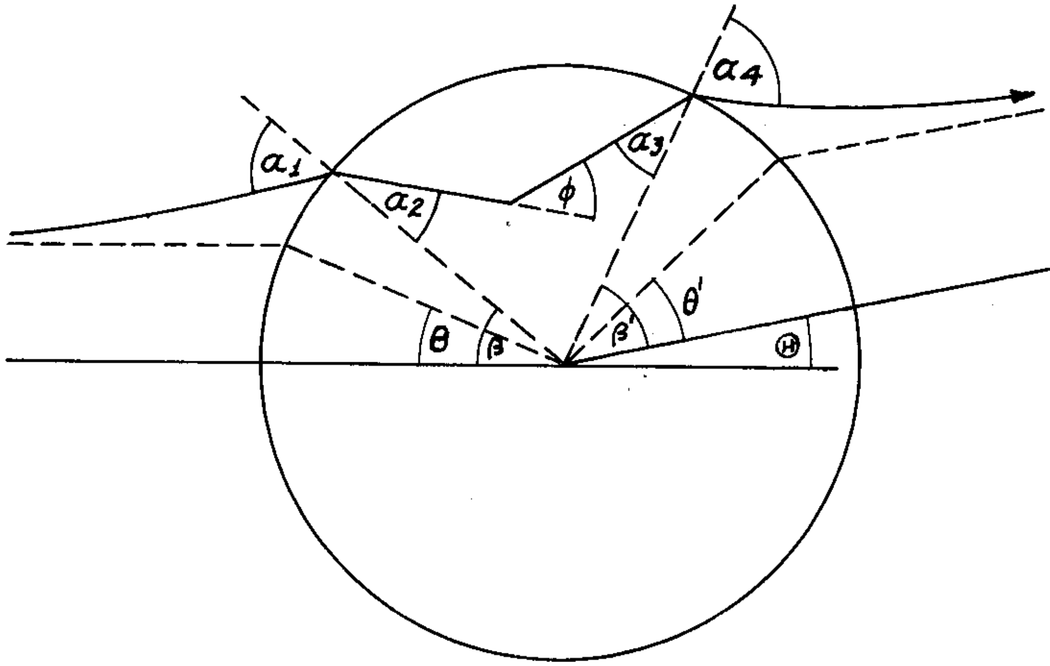

Fig. 1. The trajectory of a proton through the nucleus, Θ is the total scattering angle. (The diagram is drawn for the special case, in which the initial and final trajectories are coplanar.)

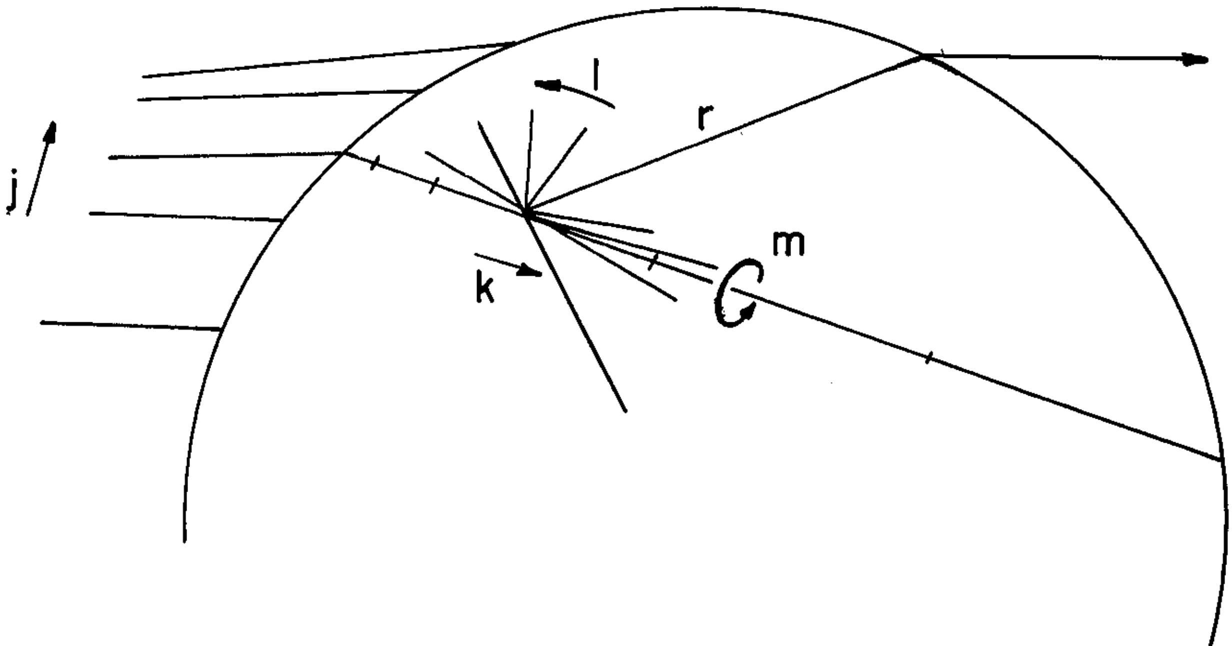

Fig. 2. The scanning scheme for knock-on scattering.

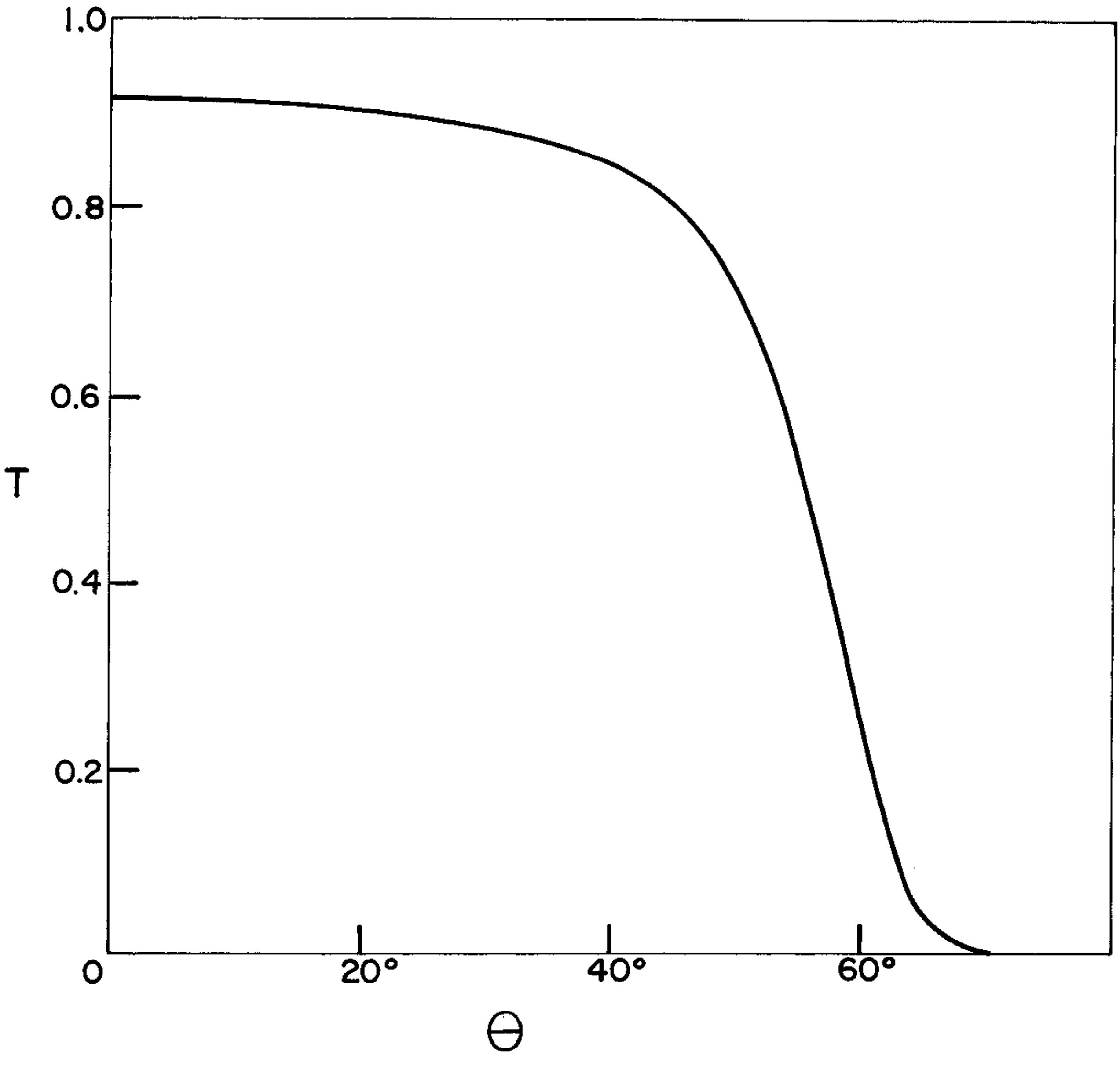

Fig. 3. Coulomb barrier transmission coefficient T as a function of angle Θ.

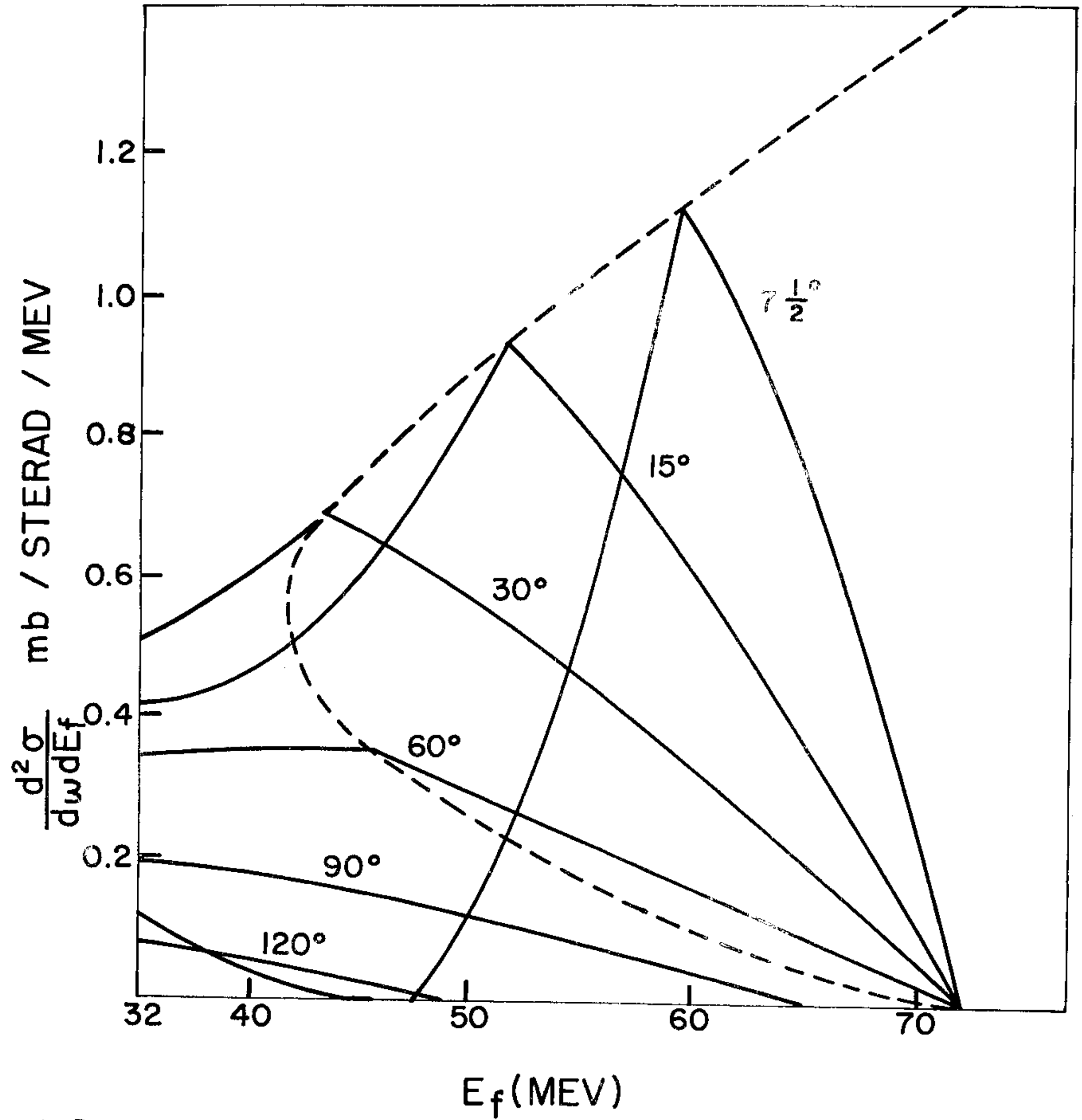

Fig. 4. Differential scattering cross section for single scattering of 31-Mev photons in Sn. The sharp corners in the curve are a result of the exclusion principle. The energy is measured from the bottom of the well.

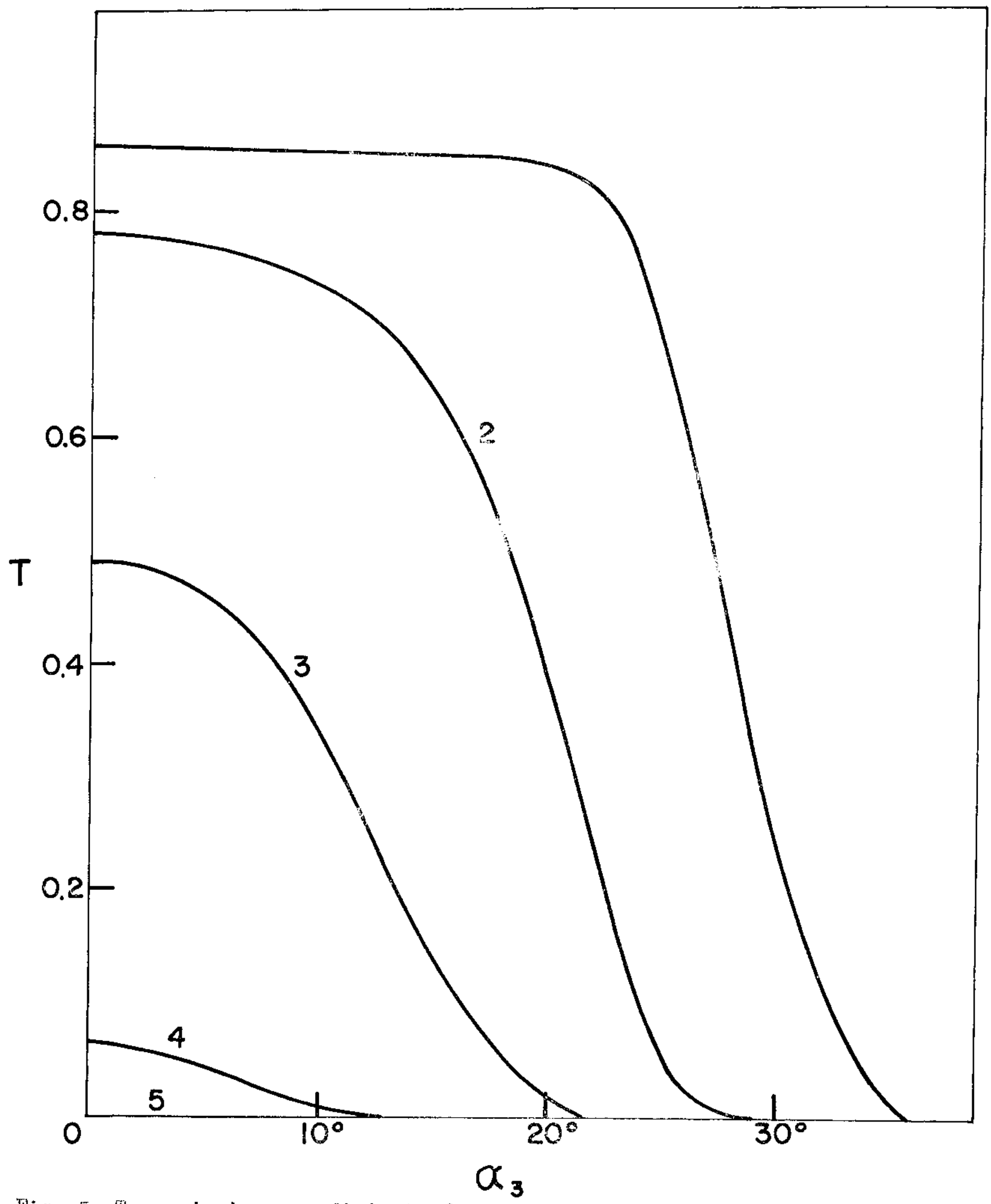

Fig. 5. Transmission coefficients for particles leaving the nucleus for energies r = 1-6, corresponding to kinetic energies 62, 55, 50, 46, 42, <40 Mev inside nuclear matter.

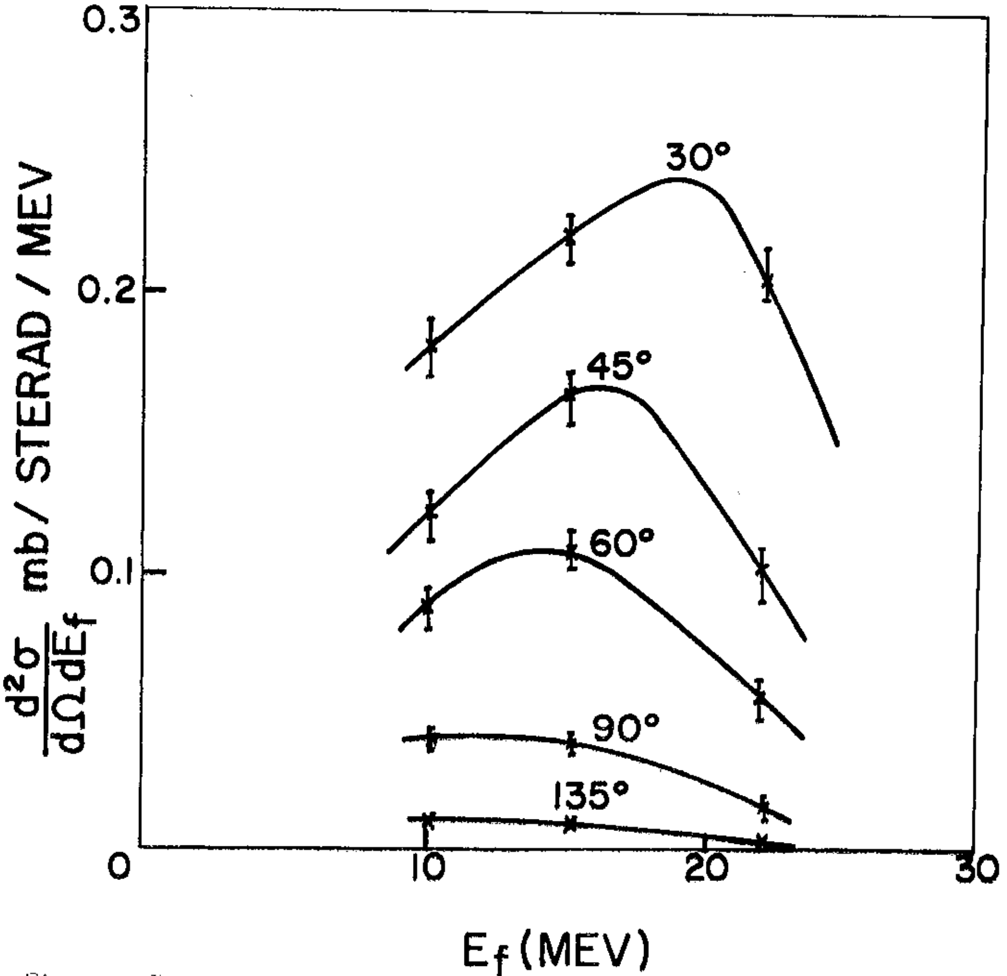

Fig. 6. Differential cross section for knock-on scattering of 31-Mev protons by Sn.

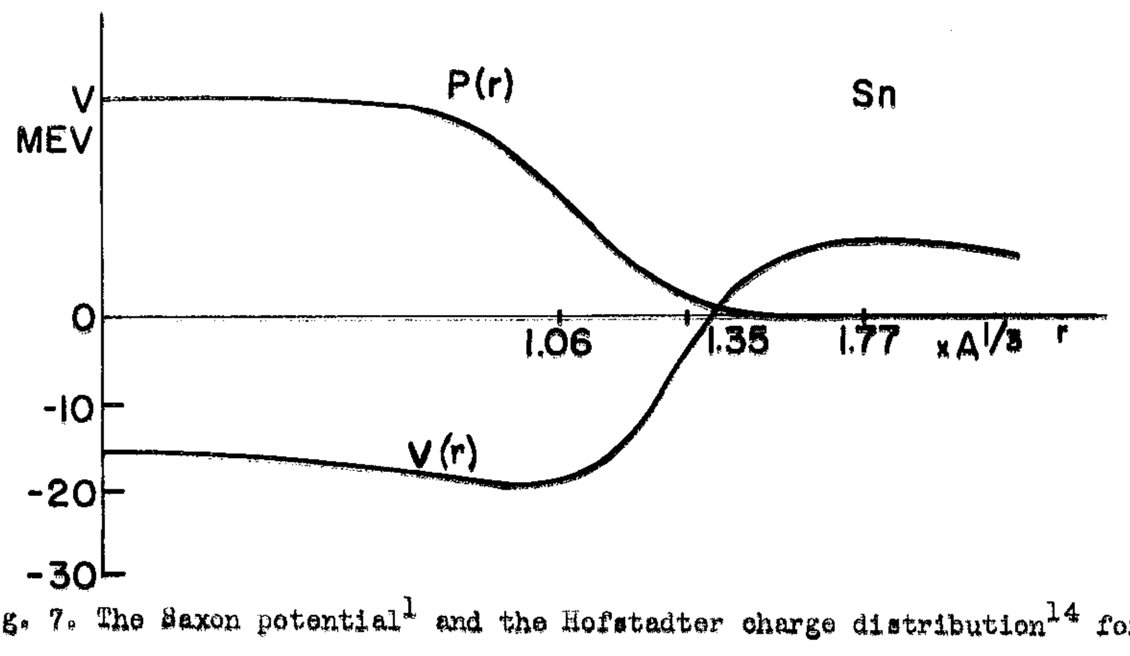

Fig. 7. The Saxon potential[1] and the Hofstadter charge distribution[14] for Sn.

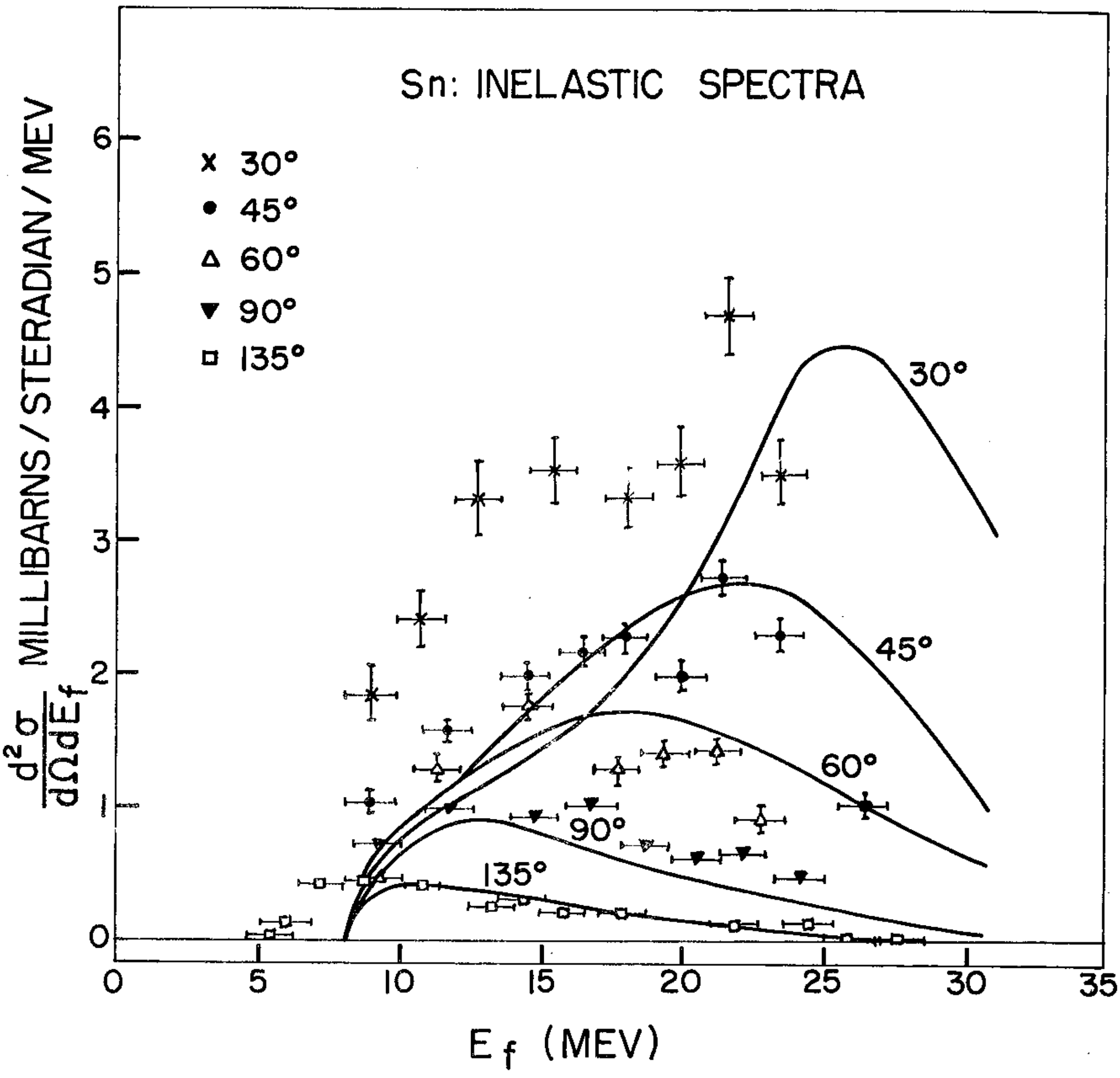

Fig. 8. Differential cross section for diffuse-rim scattering of 31-Mev protons by Sn. The experimental points are due to Eisberg and Igo.[5]

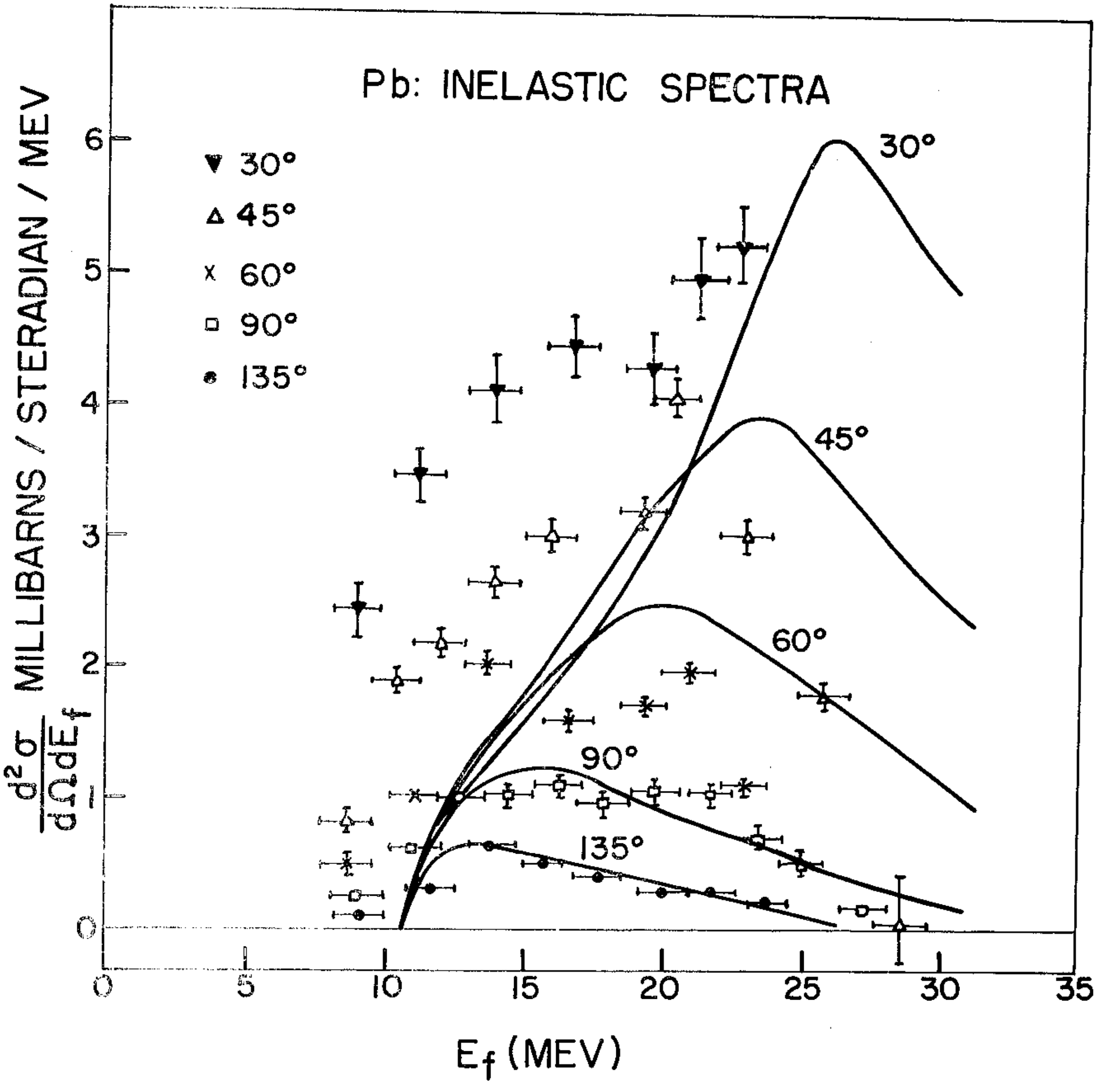

Fig. 9. Differential cross section for diffuse-rim scattering of 31-Mev protons by Pb.