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Quantum Bound on the Specific Entropy in Strong-Coupled Scalar Field Theory

M. Aparicio Alcalde[1], *G. Menezes*[2] *and N. F. Svaiter*[3]
Centro Brasileiro de Pesquisas Físicas-CBPF
Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil

Abstract

Using the Euclidean path integral approach with functional methods, we discuss the $(g_0\,\varphi^p)_d$ self-interacting scalar field theory, in the strong-coupling regime. We assume the presence of macroscopic boundaries confining the field in a hypercube of side L. We also consider that the system is in thermal equilibrium at temperature β^{-1}. For spatially bounded free fields, the Bekenstein bound states that the specific entropy satisfies the inequality $\frac{S}{E} < 2\pi R$, where R stands for the radius of the smallest sphere that circumscribes the system. Employing the strong-coupling perturbative expansion, we obtain the renormalized mean energy E and entropy S for the system up to the order $(g_0)^{-\frac{2}{p}}$, presenting an analytical proof that the specific entropy also satisfies in some situations a quantum bound. Defining $\varepsilon_d^{(r)}$ as the renormalized zero-point energy for the free theory per unit length, the dimensionless quantity $\xi = \frac{\beta}{L}$ and $h_1(d)$ and $h_2(d)$ as positive analytic functions of d, for the case of high temperature, we get that the specific entropy satisfies $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{h_2(d)}\,\xi$. When considering the low temperature behavior of the specific entropy, we have $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{\varepsilon_d^{(r)}}\xi^{\,1-d}$. Therefore the sign of the renormalized zero-point energy can invalidate this quantum bound. If the renormalized zero point-energy is a positive quantity, at intermediate temperatures and in the low temperature limit, there is a quantum bound.
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[1]e-mail: aparicio@cbpf.br
[2]e-mail: gsm@cbpf.br
[3]e-mail: nfuxsvai@cbpf.br

1 Introduction

There have been a lot of activities discussing classical and quantum fields in the presence of macroscopic boundaries. These subjects raise many interesting questions, since the boundaries introduce a characterized size in the theory. For example, in the field-theoretical description of critical phenomena, the confinement of critical fluctuations of an order parameter is able to generate long-range forces between the surfaces of a film. This is known as statistical mechanical Casimir effect [1] [2] [3]. These long-range forces in statistical mechanical systems is characterized by the excess free energy due to the finite-size contributions to the free energy of the system [4]. It should be noted that the critical-statistical mechanical Casimir effect is still waiting for a quantitative satisfactory experimental verification. By the other hand, the electromagnetic Casimir effect, where neutral perfectly conducting parallel plates in vacuum attract each other, has been tested experimentally with high accuracy. The origin of the electromagnetic Casimir effect, as pointed out by Casimir [5], is the fact that the introduction of a pair of conducting plates into the vacuum of the electromagnetic field alters the zero-point fluctuations of the field and thereby produces an attraction between the plates [6] [7] [8]. A still open question is how the sign of the Casimir force depends on the topology, dimensionality of the bounding geometry or others physical properties of the system [9] [10] [11] [12]. We should emphasize that the problem of the sign of the renormalizad zero-point energy of free fields described by Gaussian functional integrals is crucial for the subject that we are interested to investigate in this paper.

Another basic question that has been discussed in this scenario, when quantum fields interact with boundaries, is about the issue that these systems may be subjected to certain fundamental bounds. 't Hooft [13] and Susskind [14], combining quantum mechanics and gravity, introduced the holographic entropy bound $S \leq \pi\, c^3\, R^2/\hbar\, G$ [15]. This bound relates information not with the volume, but with the area of surfaces. Since in a local quantum field theory on classical spacetime, we expect that the number of degrees of freedom of a system must be proportional to the volume of the system, the holographic principle implies a radical reduction in the number of degrees of freedom we use to describe physical systems. Therefore we face here a basic difficulty of combining the principles of quantum field theory with the holographic principle. To solve this puzzle one need to explain how does locality emerges in a framework where Nature is described by a holographic principle.

Another of these proposed bounds relates the entropy S and the energy E of the quantum system, respectively, with the size of the boundaries that confine the fields. This is known as the Bekenstein bound which is given by $S \leq 2\pi\, E\, R/\hbar\, c$, where R stands for the radius of the smallest sphere that circumscribes the system [16] [17] [18] [19] [20] [21]. Such bound was originally motivated by considerations of gravitational entropy, a consistency condition between black hole thermodynamics and ordinary statistical physics that could guarantee that the generalized second law of thermodynamic is respected, which states that the sum of the black-hole entropy and the entropy of the matter outside the black-hole does not decreases. For example, in a Schwarschild black-hole in a four-dimensional spacetime, the Bekenstein entropy, which is proportional to the area of the spherical symmetric system, exactly saturates the bound. As was stressed by 't Hooft, a black-hole is the most entropic physical system one can put inside a spherical surface [13]. When gravity is negligible, the bound must be valid for a variety of systems.

Although analytical proofs of this quantum bound on specific entropy for free fields has been proposed in the literature, there are many examples pointing out that the Bekenstein bound is not valid in many situations [22] [23] [24] [25] [26] [27]. An argument used in one of these examples is based the fact that the renormalized zero-point energy of some free quantum field could be negative [22] [23]. Some authors claim that if we take into account the boundaries responsible for the Casimir energy, it is possible to compensate their negative energy yielding a positive total energy which respects the Bekenstein bound, although this is far from a simple problem [28]. Deutsch claims that the quantum bound is inapplicable as it stands to non-gravitating systems, since an absolute value of energy can not be observed, and also that for sufficient low temperatures, a generic system in thermal equilibrium also violates the entropy bound. Unruh pointed out that for system with zero modes, the specific entropy can not satisfies any bound.

We may observe that in all of these discussions a quite important situation has not been discussed in the literature, at least as far we known. A step that remains to be derived is the validity of the bound for the case of interacting fields, which are described by non-Gaussian functional integrals, at least up to some order of the perturbation theory. Nonlinear interactions can change dramatically the energy spectrum of the system and this might lead to the overthrow of the bound [29] [30]. The difficulties that appear in the implementation of this program are well known. A regularization and renormalization procedure can in principle be carry out in any order of the perturbative expansion [31]. See for example the Refs. [32] [33] [34] [35], where the perturbative renormalization were presented in first and second order of the loop expansion in the $\lambda\varphi^4$ self-interacting scalar field theory. We might attempt to show for a given self-interaction field theory in which situations the specific entropy satisfies a quantum bound.

The aim of this paper is to study the $(g_0\,\varphi^p)_d$ self-interacting scalar field theory in the strong-coupling regime. We assume the presence of macroscopic boundaries that confine the field in a hypercube of side L and also that the system is in thermal equilibrium with a reservoir. We present an analytic proof that, up to the order $(g_0)^{-\frac{2}{p}}$, the specific entropy also satisfies in some situations a quantum bound. Defining $\varepsilon_d^{(r)}$ as the renormalized zero-point energy for the free theory per unit length, $\xi = \frac{\beta}{L}$ and $h_1(d)$ and $h_2(d)$ as positive analytic functions of d, for the case of high temperature, we get that the specific entropy satisfies the inequality $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{h_2(d)}\,\xi$. When considering the low temperature behavior of the specific entropy, we have $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{\varepsilon_d^{(r)}\,\xi^{\,d-1}}$. We are establishing a bound in the strong-coupled system in the following cases: in the high temperature limit and if the renormalized zero point-energy is a positive quantity, at intermediate temperatures and also in the low temperature limit.

In the weak-coupling perturbative expansion, the information about the boundaries can be implemented over the free two-point Schwinger function $G_0(m_0; x-y)$ of the system. In the strong-coupling perturbative expansion, we have to deal with the problem of how the boundary conditions can be imposed. Let us briefly discuss the strong-coupling expansion in Euclidean field theory at zero temperature. The basic idea of the approach is the following: in a formal representation for the generating functional of complete Schwinger functions of the theory $Z(V,h)$, we treat the Gaussian part of the action as a perturbation with respect to the remaining terms of the functional integral, i.e., in the case for the $(g_0\,\varphi^p)_d$ theory, the local self-interacting part, in the functional integral. In the generating functional of complete Schwinger functions, V is the volume of the Euclidean space where the fields are defined and $h(x)$ is an external source. We are developing our perturbative expansion around the independent-value generating functional $Q_0(h)$ [36] [37] [38] [39] [40]. In the zero-order approximation, different points of the Euclidean space are decoupled since the gradient terms are dropped [41] [42] [43] [44] [45] [46] [47] [48] [49].

The fundamental problem of the strong-coupling expansion is how to give meaning to the independent-value generating functional and to this representation for the Schwinger functional. One attempt is to replace the Euclidean space by a lattice made by hypercubes. A naive use of the continuum limit of the lattice regularization where one simply makes use of the central limit theorem for the independent-value generating functional leads to a Gaussian theory. A solution to this problem was presented by Klauder long time ago [37] [40] [49]. The modification which allows to avoid this limitation is a change in the definition of the measure in the functional integral. In the usual approach one adopts a measure, which possess local translational invariance. Instead we can use the non-translational invariant measure, $[d\phi] = \prod_x \frac{d\,\phi(x)}{|\phi(x)|}$.

There is another point that must be considered. There are some analytical and numerical evidences based fundamentally in the results of Frohlich [50] and Aizenman and Graham [51], suggesting that the $(\lambda\varphi^4)_{d\geq 4}$ field theory constructed as a scaling limit of ferromagnetic lattice field theory, is non-interacting. If the only non-perturbative solution of $(\lambda\varphi^4)_{d=4}$ field theory is the trivial one, how can it have a non-trivial renormalized perturbative series [52]? Gallavottti [53] concludes that could be a different regularization procedure which would converges to a non-trivial solution. The key point is that with an appropriate replacement in the path integral measure $[d\phi] = \prod_x d\phi(x)$, which possess local translational invariance, by the non-translational invariant measure made the scalar model non-trivial.

Let us remark that, in the strong-coupling regime, assuming that the source is constant, we can perform the perturbative expansion around a independent-value generating function, up to the order $(g_0)^{-\frac{2}{p}}$, and it is possible to split $\ln Z(V,h)$ in two contributions: one that contains only the independent-value generating function and other that contains the spectral zeta-function. Therefore, in order to obtain the thermodynamic quantities, one must give a operational meaning to the independent-value generating function, and, as discussed before, implement the boundary conditions. To implement boundary conditions, we use the spectral zeta-function method [54] [55] [56] [57] [58]. Quite recently a very simple application of this formalism was presented [59] [60].

The organization of the paper is as follows: In section II we discuss the strong-coupling expansion for the $(g_0\,\varphi^p)_d$ theory. In section III we discuss he free energy and the spectral zeta-function of the system. In section IV we show that it is possible to obtain in some situations a quantum bound in the considered model. Finally, section V contains our conclusions. In the appendix A we present the Klauder's result, as the formal definition of the independent-value generating functional derived for scalar fields in a d-dimensional Euclidean space. In the appendix B we proof that the spectral zeta-function $\zeta_D(s)$ evaluated in the extended complex plane at $s = 0$ vanishes. To simplify the calculations we assume the units to be such that $\hbar = c = k_B = 1$.

2 The strong-coupling perturbative expansion for scalar $(g_0\,\varphi^p)_d$ theory

Let us consider a neutral scalar field with a $(g_0\,\varphi^p)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time allowed by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. In a d-dimensional Euclidean space, the self-interaction contribution to the action is given by

$$S_I(\varphi) = \int d^dx\, \frac{g_0}{p\,!}\,\varphi^p(x). \tag{1}$$

The basic idea of the strong-coupling expansion at zero temperature is to treat the Gaussian part of the action as a perturbation with respect to the remaining terms of the action in the functional integral. Let us assume a compact Euclidean space with or without a boundary, where the volume of the Euclidean space is V. Let us suppose that there exists an elliptic, semi-positive, and self-adjoint differential operator O acting on scalar functions on the Euclidean space. The usual example is $O = (-\Delta + m_0^2)$, where Δ is the d-dimensional Laplacian. The kernel $K(m_0;\, x, y) \equiv\, K(m_0;\, x - y)$ is defined by

$$K(m_0; x - y) = \left(-\Delta + m_0^2\right)\delta^d(x - y). \tag{2}$$

Using the fact that the functional integral which defines $Z(V, h)$ is invariant with respect to the choice of the quadratic part, let us consider a modification of the strong-coupling expansion. We split the quadratic part in the functional integral which is proportional to the mass squared in two parts; one in the derivative terms of the action, and the other in the independent value generating functional. The Schwinger functional can be defined by a new formal expression for the functional integral given by

$$Z(V, h) = \exp\left(-\frac{1}{2}\int d^dx \int d^dy \frac{\delta}{\delta h(x)} K(m_0, \sigma; x - y)\frac{\delta}{\delta h(y)}\right) Q_0(\sigma, h), \tag{3}$$

where $Q_0(\sigma, h)$, the new independent value functional integral, is given by

$$Q_0(\sigma, h) = \mathcal{N}\int [d\varphi]\, \exp\left(\int d^dx\left(-\frac{1}{2}\,\sigma\, m_0^2\,\varphi^2(x) - \frac{g_0}{p\,!}\,\varphi^p(x) + h(x)\varphi(x)\right)\right), \tag{4}$$

and the modified kernel $K(m_0, \sigma; x - y)$ that appears in Eq. (3), is defined by

$$K(m_0, \sigma; x - y) = \left(-\Delta + (1 - \sigma)m_0^2\right)\delta^d(x - y), \tag{5}$$

where σ is a complex parameter defined in the region $0 \leq \text{ Re }(\sigma) < 1$.

The factor $\mathcal{N}$ is a normalization that can be found using that $Q_0(\sigma, h)|_{h=0} = 1$. Observe that the non-derivative terms which are non-Gaussian in the original action do appear in the functional integral that defines $Q_0(\sigma, h)$. At this point it is convenient to consider $h(x)$ to be complex. Consequently $h(x) = \text{Re}(h) + i\,\text{Im}(h)$. In the paper we are concerned with the case $\text{Re}(h) = 0$.

Since we are assuming a spatially bounded system in equilibrium with a thermal reservoir at temperature β^{-1}, the strong-coupling expansion can be used to compute first the partition function defined by $Z(\beta, \Omega, h)|_{h=0}$, where h is a external source and we are defining the volume of the $d-1$ manifold, $V_{d-1} \equiv \Omega$. From the partition function we define the free energy of the system given by $F(\beta, \Omega) = -\frac{1}{\beta} \ln Z(\beta, \Omega, h)|_{h=0}$. This quantity can be used to derive the mean energy $E(\beta, \Omega)$, defined as

$$E(\beta, \Omega) = -\frac{\partial}{\partial \beta} \ln Z(\beta, \Omega, h)|_{h=0}, \tag{6}$$

and the canonical entropy $S(\beta, \Omega)$ of the system in equilibrium with a reservoir with a finite size given by

$$S(\beta, \Omega) = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z(\beta, \Omega, h)|_{h=0}. \tag{7}$$

In the next section we will show that in a particular situation it is possible, up to the order $(g_0)^{-\frac{2}{p}}$ to split $\ln Z(\beta, \Omega, h)$ into two parts: the first one that contains only the independent-value generating function and the second one that has the information on the boundary condition and it is given by derivative of the spectral zeta-function.

3 The independent-value generating functional and the spectral zeta-function

We are interested in global quantities. For simplicity we are assuming that the external source $h(x)$ is constant. In this situation we call $\ln Z(V, h)$ as a generating function. At zero temperature, in the leading-order approximation (up to the to the order $(g_0)^{-\frac{2}{p}}$) we can write

$$\ln Z(V, h) = -\frac{1}{2Q_0(\sigma, h)} \frac{\partial^2}{\partial h^2} Q_0(\sigma, h) \int d^d x \int d^d y \, K(m_0, \sigma;\, x - y). \tag{8}$$

Since we are introducing boundaries in the domain where the field is defined, the spectrum of the operator $D = (-\Delta + (1 - \sigma) m_0^2)$ has a denumerable contribution, and an analytic regularization procedure can be used to control the divergences of the theory.

In order to impose boundary conditions the functional integral must be taken over functions restricted to the geometric configurations. The generating function can be rewritten as

$$\ln Z(\beta,\Omega,h) = \frac{1}{Q_0(\sigma,h)}\frac{\partial^2}{\partial h^2}\,Q_0(\sigma,h)\left(-\frac{\alpha}{2}+\frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{s=0}\right), \tag{9}$$

where α is a infinite constant and $\zeta_D(s)$ is the spectral zeta-function associated with the operator D.

Let us consider now the situation in which the system is finite along each one of the spatial dimensions, i.e., $x_i \in [0,L]$, $i=1,2,...,d-1$. For the Euclidean time we assume periodic boundary conditions (Kubo-Martin-Schwinger KMS [61] [62] conditions) and for the Euclidean spatial dimensions we assume Dirichlet-Dirichlet boundary conditions. We call this latter situation "hard" boundaries. See for example the Ref. [63]. For different kinds of confining boundaries see [64] [65]. The choice of the hard boundary provides an easy solution to the eigenvalue problem, so that explicit and complete calculation using the spectral-zeta function can be performed without difficulty.

It follows that the operator D has the spectrum given by $\lambda_{n_1,\ldots,n_d}$ where

$$\lambda_{n_1,\ldots,n_d} = \left[\left(\frac{n_1\pi}{L}\right)^2+...+\left(\frac{n_{d-1}\pi}{L}\right)^2+\left(\frac{2n_d\,\pi}{\beta}\right)^2+(1-\sigma)m_0^2\right], \tag{10}$$

$n_1,n_2,\ldots,n_{d-1}$ are natural numbers and n_d are integer numbers. The spectral zeta-function associated with the operator D in this situation reads

$$\zeta_D(s) = \sum_{n_1,\ldots,n_d}^{\infty}\lambda_{n_1,\ldots,n_d}^{-s}\,, \tag{11}$$

where s is a complex parameter. The series above converges for $\mathrm{Re}\,s > \frac{d}{2}$ and its analytic continuation defines a meromorphic function of s, analytic at $s=0$. To take into account the scaling properties we should have to introduce an arbitrary parameter μ with dimension of a mass to define all the dimensionless physical quantities and in particular make the change

$$\frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{s=0} \to \frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{s=0} - \frac{1}{2}\ln\left(\frac{1}{4\pi\mu^2}\right)\zeta_D(s)|_{s=0}. \tag{12}$$

Before continue, it is possible to show that there is no scaling in the situation that we are interested in. Let us first show that there is no scaling when the length of the size of the hypercube is large compared with β. The spectral zeta function in this situation is given by

$$\zeta_D(s) \equiv \frac{V}{(2\pi)^{d-1}\beta}\sum_{n=-\infty}^{\infty}\int d^{d-1}k\frac{1}{\left(\vec{k}^2+(\frac{2\pi n}{\beta})^2+(1-\sigma)\,m_0^2\right)^s}. \tag{13}$$

Defining the quantity $\nu^{\,2} = \left(\frac{\beta}{2\pi}\right)^2 \left(\vec{k}^2 + (1-\sigma)\, m_0^2\right)$, we have that the spectral zeta function can be written as

$$\zeta_D(s) = \frac{V}{(2\pi)^{d-1}\beta}\left(\frac{\beta}{2\pi}\right)^{2s}\int d^{d-1}k \sum_{n=-\infty}^{\infty}\frac{1}{(\nu^{\,2}+n^{\,2})^s}. \tag{14}$$

Here, it is useful to define a modified Epstein-Hurwitz zeta-function in the complex plane s, $\zeta(s,\nu)$ by:

$$\zeta(s,\nu) = \sum_{n=-\infty}^{\infty}(n^2+\nu^2)^{-s}, \qquad \nu^2 \; > 0. \tag{15}$$

Note that we wrote the spectral zeta function in terms of the modified Epstein-Hurwitz zeta-function. The series defined by Eq.(15) converges absolutely and defines in the complex s plane an analytic function for Re(s) $> \frac{1}{2}$. It is possible to analytically extend the modified Epstein-Hurwitz zeta-function where the integral representation is valid for Re(s) < 1, [66] [67]. For a different representation for the analytic extension of the modified Epstein-Hurwitz zeta function in terms of the modified Bessel function $K_\alpha(z)$ or the Macdonald's function, see Ref. [68]. As we discussed, the series representation for $\zeta(s,\nu)$ converges for Re(s) $> \frac{1}{2}$ and its analytic continuation defines a meromorphic function of s which is analytic at $s=0$. The modified Epstein-Hurwitz zeta-function has poles at $s=\frac{1}{2}, -\frac{1}{2}$, etc. It is not difficult to show that the values for the modified Epstein-Hurwitz zeta function $\zeta(s,\nu)$, at $s=0$ and $\frac{\partial}{\partial s}\,\zeta(s,\nu)|_{s=0}$ are given respectively by

$$\zeta(s,\nu)|_{s=0} = 0, \tag{16}$$

and also

$$\frac{\partial}{\partial s}\,\zeta(s,\nu)|_{s=0} = -2\ln\left(2\,\sinh\pi\nu\right). \tag{17}$$

Therefore, there is no scaling in this situation. The same result was obtained by Hawking in Ref. [55]. For the case of $d=4$ and two sumations, using the Refs. [66] [67] and [68], is easy to prove that in the presence of boundaries the $\zeta_D(s)|_{\,s=0}$ vanishes, therefore there is no need for scaling in this situation also. This result can be generalized. The spectral zeta-function is related to the heat-kernel or diffusion operator via a Mellin transform. The trace of the diffusion operator is the integral of the diagonal part of the heat-kernel over the manifold. It is possible to perform an asymptotic expansion for the heat-kernel and this asymptotic expansion shows that the spectral zeta-function is a meromorphic function of the complex variable s possessing simple poles where the residues of the poles depends on the B_n coefficients which depends on the Seeley-DeWitt coefficients, the second fundamental form on the boundary and the induced geometry on the boundary. See for example the Ref. [56] and [69]. It is possible to show that the polar structure of the analytic extension of the spectral zeta function in a compact manifold with boundary is given by

$$\zeta_D(s) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Gamma(s)}\left[\sum_{n=0}^{\infty}\frac{B_n}{n-\frac{d}{2}+s} + g_2(s)\right], \tag{18}$$

for n integer or odd-half integer, where $g_2(s)$ is an analytic function in $\mathbf{C}$. As was stressed by Blau et al [70], in a four dimensional flat spacetime with massless particles and thin boundaries the geometric coefficient B_2 vanishes. This result can be generalized for the hypercube with Dirichlet boundary conditions (see the appendix A). Therefore in the case that we are interested there is no scaling.

Let us study in Eq. (9) the contribution arising from the spectral zeta-function which takes into account the geometric constrains upon the scalar field. Using the spectrum of the D operator give by Eq. (10) and the definition of the spectral zeta-function given by Eq. (11), we get that the derivative of the spectral zeta-function in $s = 0$ yields

$$\frac{d}{ds}\,\zeta_D(s)|_{\,s=0} = -\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\biggl(\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) + \ln\Bigl(1+\frac{a^2\beta^2}{4n_d^2L^2+q^2\beta^2}\Bigr)\biggr), \tag{19}$$

where $\vec{n}_{d-1} = (n_1, n_2, ..., n_{d-1})$, $q^2 = n_1^2 + n_2^2 + ... + n_{d-1}^2$ and $a^2 = \Bigl(\frac{(1-\sigma)m_0^2L^2}{\pi^2}\Bigr)$. Note that in Eq. (19) we are using that $\zeta(s,\nu)|_{s=0} = 0$. Using the following identity [71]

$$\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) = \int_1^{(\frac{\pi\,\beta\,q}{L})^2}\frac{d\theta^2}{\theta^2+(2\pi n_d)^2} + \ln\Bigl(1+(2\pi n_d)^2\Bigr)\,, \tag{20}$$

we can see that the first term in the right hand side of Eq. (19) gives a divergent contribution. To proceed we use another useful identity given by

$$\sum_{n_d=-\infty}^{\infty}\frac{1}{\theta^2+(2\pi n_d)^2} = \frac{1}{2\theta}\Bigl(1+\frac{2}{e^{\theta}-1}\Bigr). \tag{21}$$

Using both identities given by Eq. (20) and Eq. (21), it is possible to express the double summation that appears in Eq. (19) by a single summation given by

$$\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) = 2\sum_{\vec{n}_{d-1}=1}^{\infty}\int_1^{(\frac{\pi\,\beta\,q}{L})} d\theta\Bigl(\frac{1}{2}+\frac{1}{e^{\theta}-1}\Bigr) + \alpha_1, \tag{22}$$

where $\alpha_1 = \sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\Bigl(1+(2\pi n_d)^2\Bigr)$. Carrying out the θ integration, we finally arrive that Eq. (22) can be written as

$$\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) = 2\sum_{\vec{n}_{d-1}=1}^{\infty}\biggl(\frac{\pi\,\beta\,q}{2L} + \ln\Bigl(1-e^{-\frac{\pi\,\beta\,q}{L}}\Bigr)\biggr) + \alpha_2\,, \tag{23}$$

where $\alpha_2 = \alpha_1 - \sum_{\vec{n}_{d-1}=1}^{\infty}\Bigl(1+2\ln(1-e^{-1})\Bigr)$. Since this divergent contribution α_2 is β-independent we will see that can be eliminated using the third law of thermodynamics. The first term on the right side of Eq. (23) is a divergent contribution, corresponding to the zero-point energy term. Using the following mathematical result [72] [73] given by

$$\prod_{n=-\infty}^{\infty}\left(1+\frac{a^2}{n^2+b^2}\right)=\frac{\sinh^2(\pi\sqrt{a^2+b^2})}{\sinh^2(\pi\, b)}\,, \tag{24}$$

we can write the last term of Eq. (19) in a more manageable way. Using the Eq. (23) and Eq. (24), the derivative of the spectral zeta-function in $s=0$ can be rewritten as

$$\frac{d}{ds}\,\zeta_D(s)|_{s=0}=-2\sum_{\vec{n}_{d-1}=1}^{\infty}\left[\ln\left(\frac{\sinh\left(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\right)}{\sinh\left(\frac{\pi\beta q}{2L}\right)}\right)+\ln\left(1-e^{-\frac{\pi\,\beta\,q}{L}}\right)+\frac{\pi\,\beta\,q}{2L}\right]-\alpha_2\,. \tag{25}$$

It is possible to show that in the finite temperature case the independent-value generating function $Q_0(\sigma,h)$ satisfies $Q_0(\sigma,h)|_{h=\sigma=0}=1$, and

$$\frac{\partial^2}{\partial\,h^2}\,Q_0(\sigma,h)|_{\,h=\sigma=0}=\frac{\Gamma(\frac{2}{p})}{2p\,g_0^{\frac{2}{p}}\,(p\,!)^{\frac{p}{2}}}. \tag{26}$$

See the appendix A for the derivation. In the next section we show that it is possible to obtain a quantum bound in the spatially bounded system defined by a self-interacting scalar field in the strong-coupling regime, in high temperatures. As we will see, for the cases of intermediate or low temperatures, the sign of the renormalized zero-point energy is crucial for the validity of a quantum bound in the specific entropy.

4 The specific entropy for strongly coupled $(g_0\,\varphi^p)_d$ theory

In this section we compute the specific entropy $\frac{S}{E}$ of the system. For simplicity, let us define $\ln Z(\beta,\Omega,h)|_{h=0}=\ln Z(\beta,\Omega)$. From Eq. (6) and Eq. (7), and using for simplicity that the mean energy $E(\beta,\Omega)=E$ and the specific entropy $S(\beta,\Omega)=S$, is given by

$$\frac{S}{E}=\beta-\ln Z(\beta,\Omega)\left(\frac{d}{d\beta}\ln Z(\beta,\Omega)\right)^{-1}. \tag{27}$$

Substituting Eq. (25) and Eq. (26) in Eq. (9) we have that $\ln Z(\beta,\Omega)$ is given by

$$\ln Z(\beta,\Omega)=-\frac{\Gamma(\frac{2}{p})}{2p\,(p\,!)^{\frac{p}{2}}g_0^{\frac{2}{p}}}\left(\frac{\alpha\,'}{2}+I_2(\beta)\right)\,, \tag{28}$$

where $\alpha' = \alpha + \alpha_2$ and the quantity $I_2(\beta)$ is given by

$$I_2(\beta) = \sum_{\vec{n}_{d-1}=1}^{\infty} \left[\ln\left(\frac{\sinh\left(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\right)}{\sinh\left(\frac{\pi\beta q}{2L}\right)} \right) + \ln\left(1 - e^{-\frac{\pi\,\beta\,q}{L}}\right) + \frac{\pi\,\beta\,q}{2L} \right] . \tag{29}$$

Defining C_1 and $C_2 = -\frac{2C_1}{\alpha'}$ that depend only of p and g_0 and do not depend on β as

$$C_1 = -\frac{\alpha'\,\Gamma(\frac{2}{p})}{4p\,(p\,!)^{\frac{p}{2}}\, g_0^{\frac{2}{p}}} , \tag{30}$$

the quantity $\ln Z(\beta,\Omega)$ can be written in a general form as

$$\ln Z(\beta,\Omega) = C_1 - C_2\, I_2(\beta). \tag{31}$$

It is worth to mention that the quantity C_1 corresponds to a divergent expression, C_2 is finite and the summation term in the right-hand side of Eq. (25) is proportional to the zero-point energy. In order to renormalize $\ln Z(\beta,\Omega)$ we first can use the third law of thermodynamics. The derivative of $\ln Z(\beta,\Omega)$ with respect of β yields

$$\frac{d}{d\beta}\ln Z(\beta,\Omega) = -C_2\,\frac{d}{d\beta} I_2(\beta), \tag{32}$$

where the derivative of $I_2(\beta)$ with respect to β is given by

$$\frac{d}{d\beta} I_2(\beta) = \frac{\pi}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} \left(\left(\sqrt{q^2+a^2}\,\coth\left(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\right) - q\,\coth\left(\frac{\pi\beta\,q}{2L}\right)\right) + \frac{2\,q}{e^{\frac{\pi\,\beta\,q}{L}} - 1} + q \right) . \tag{33}$$

Substituting Eq. (31) and Eq. (32) in the definition of the entropy given by Eq. (7), we have that the entropy of the system can be written as

$$S = C_1 - \beta C_2 \left(\frac{I_2(\beta)}{\beta} - \frac{d}{d\beta} I_2(\beta) \right) . \tag{34}$$

This expression of the entropy must satisfy the third law of thermodynamics, i.e., the entropy of a system has a limiting property that $\lim_{\beta\to\infty} S = 0$. To proceed, lets analyze the limit given by

$$\lim_{\beta\to\infty} \frac{I_2(\beta)}{\beta} = \lim_{\beta\to\infty} \frac{d}{d\beta} I_2(\beta) = \frac{\pi a^2}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} \frac{1}{\sqrt{q^2+a^2}+q} + \frac{\pi}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} q\,. \tag{35}$$

Substituting Eq. (35) in Eq. (34), and using the third law of thermodynamics, we get

$$\lim_{\beta\to\infty} S = C_1 = 0\,. \tag{36}$$

Therefore the first step to find a finite result for $\ln Z(\beta,\Omega)$, was achieved, since we were able to renormalize C_1 to zero using the third law of thermodynamics. After this step we have

$$\ln Z(\beta,\Omega) = -C_2\, I_2(\beta)\,. \tag{37}$$

Note that in the $\ln Z(\beta,\Omega)$ expression (see Eq. (29)) we still have the contribution coming from the zero-point energy, which is given by

$$E_0 = \frac{\pi}{2L}\sum_{\vec{n}_{d-1}=1}^{\infty}(n_1^2+n_2^2+...+n_{d-1}^2)^{\frac{1}{2}}\,. \tag{38}$$

After an analytic continuation we obtain the renormalized zero-point energy defined by $E_0^{(r)}$, and, consequently, a finite result for $\ln Z(\beta,\Omega)$.

Substituting Eq. (37) in Eq. (27) we can see that for the case $a=0$, i.e., the massless case, the quotient $\frac{S}{E}$ yields

$$\frac{S}{E} = 2\pi R\, T_d(\xi)\,, \tag{39}$$

where we are defining the dimensionless variable ξ given by $\xi=\beta/L$. Since the field is confined in a hypercube, the radius of the smallest $(d-1)$-dimensional sphere that circumscribe this system should be given by $R=\frac{1}{2}\sqrt{(d-1)}\,L$. The function $T_d(\xi)$ defined in Eq. (39) is given by

$$T_d(\xi) = \frac{1}{\pi\sqrt{d-1}}\,\frac{\xi\, P_d(\xi)+R_d(\xi)}{\varepsilon_d^{(r)}+P_d(\xi)}\,, \tag{40}$$

where $\varepsilon_d^{(r)} = LE_0^{(r)}$ and the positive functions $P_d(\xi)$ and $R_d(\xi)$ are defined respectively by

$$P_d(\xi) = \sum_{\vec{n}_{d-1}=1}^{\infty}\pi\, q\left(e^{\pi\,\xi\, q}-1\right)^{-1} \tag{41}$$

and

$$R_d(\xi) = -\sum_{\vec{n}_{d-1}=1}^{\infty}\ln\left(1-e^{-\pi\,\xi\, q}\right)\,. \tag{42}$$

Now let us study the function $T_d(\xi)$ given by Eq. (39). The quantum bound holds whenever $T_d(\xi)\leq 1$ for all values of ξ. From the definition of the function $T_d(\xi)$, given by Eq. (40), we have that $T_d(\xi)$ has a divergent value only if the renormalized zero-point energy is negative. For the point $\xi=\xi_0$ which satisfy $\varepsilon_d^{(r)}+P_d(\xi_0)=0$, the quantum bound is invalidated.

Numerical calculations can help us understand the quantum bound. In the Fig. (1) we present the plot of the function $T_d(\xi)$ in the case of $d=3$ over the interval $0<\xi<2$. Since the renormalized zero-point energy is positive [74], the function $T_d(\xi)$ also is positive for all values of

ξ. There is a maximum for some value of ξ that we are calling ξ_{max} which is near one. For this case there is a quantum bound. In Fig.(2) we present the function $T_d(\xi)$ in the case of $d = 4$ over the interval $0 < \xi < 2$. Since in this case the renormalized zero-point energy is negative we have that for some value of $\xi = \xi_0$ the function $T_d(\xi)$ diverges. There exists a critical value ξ_c where for $\xi > \xi_c$, the specific entropy is unbounded above.

Let us analyze two cases. The first one is where the renormalized zero-point energy is positive (see Fig. 1) when a maximum value for $T_d(\xi)$ appears. The second case, with a negative renormalized zero-point energy, invalidate the quantum bound. For even space-time dimensions, the renormalized zero-point energy is always negative. For the odd space-time dimensional case, it is known that for $d \leq 29$ this quantity is positive and for $d > 29$ it changes the sign [10].

For the cases of positive renormalized zero-point energy, an equation for the maximum value of $T_d(\xi)$ can be found. The equation for the maximum is given by $R_d(\xi_{max}) = \varepsilon_d^{(r)}\,\xi_{max}$. Substituting this ξ_{max} in Eq. (40) we can find that $T_d(\xi_{max}) = \frac{\xi_{max}}{\pi\sqrt{d-1}}$. Using the same procedure in Eq. (39) we get $\frac{S}{E} = \beta_{max}$, where $\beta_{max} = L\,\xi_{max}$. Therefore we can conclude that for odd space-time dimensions $d \leq 29$ there exists a maximum value for the function $T_d(\xi)$.

We can see that the maximum value of $T_d(\xi)$ depends on the renormalized zero-point energy, where for the case $d = 3$ is less than one. To prove that for odd $d \leq 29$, we have that $T_d(\xi)$ satisfies the inequality $T_d(\xi) < 1$, let us define an auxiliary function $R'_d(\xi)$ that satisfies $R_d(\xi) < R'_d(\xi)$. This function is given by

$$R'_d(\xi) = -\int_{\Omega_R} d\Omega_{d-1}\int_0^\infty dr\, r^{d-2}\ln\left(1 - e^{-\pi\,\xi\, r}\right)\,, \tag{43}$$

where the angular domain of integration Ω_R correspond to the region where $r_i > 0$. Performing this integral [73] we have that

$$R'_d(\xi) = S_{d-1}\,\Gamma(d-1)\,\zeta(d)\left(\frac{1}{\pi\xi}\right)^{d-1}. \tag{44}$$

where the angular term is $S_{d-1} = \frac{(\sqrt{\pi})^{d-1}}{2^{d-2}\Gamma(\frac{d-1}{2})}$. Using the Eq. (44) in the equation for the maximum, i.e., $R_d(\xi_{max}) = \varepsilon_d^{(r)}\,\xi_{max}$, we can find that $\xi_{max} < \xi'_{max}$, where

$$\xi'_{max} = \left(\frac{2}{(2\sqrt{\pi})^{d-1}}\frac{\Gamma(d-1)\,\zeta(d)}{\Gamma(\frac{d-1}{2})\,\varepsilon_d^{(r)}}\right)^{\frac{1}{d}}\,, \tag{45}$$

and we have that $T_d(\xi_{max}) < \frac{\xi'_{max}}{\pi\sqrt{d-1}}$. In the table 1 we present the maximum values for $d = 3$ until $d = 29$ for odd d's.

d	**3**	**5**	**7**	**9**	**11**	**13**
$\varepsilon_d^{(r)}$	4.1×10^{-2}	6.2×10^{-3}	1.1×10^{-3}	2.2×10^{-4}	4.4×10^{-5}	9.4×10^{-6}
$T_d(\xi_{max}) <$	0.3763	0.2645	0.2303	0.2130	0.2025	0.1953

d	**15**	**17**	**19**	**21**	**23**	**25**
$\varepsilon_d^{(r)}$	2.0×10^{-6}	4.5×10^{-7}	1.0×10^{-8}	2.2×10^{-8}	5.0×10^{-9}	1.1×10^{-9}
$T_d(\xi_{max}) <$	0.1901	0.1861	0.1829	0.1804	0.1784	0.1769

d	**27**	**29**	**31**
$\varepsilon_d^{(r)}$	2.3×10^{-10}	3.0×10^{-11}	-1.1×10^{-11}
$T_d(\xi_{max}) <$	0.1761	0.1781	no maximum

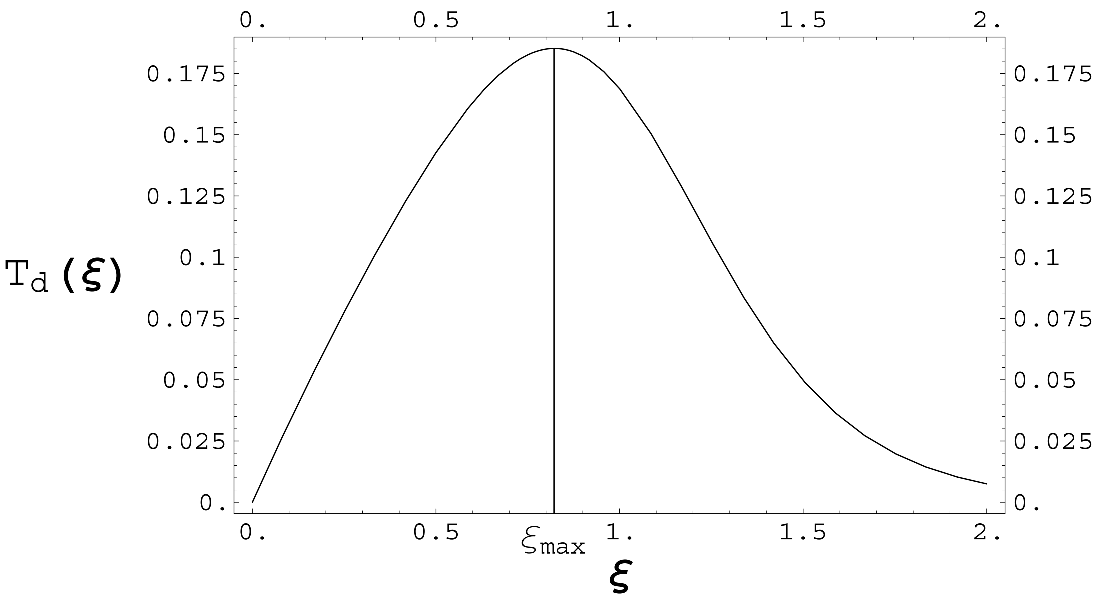

Figure 1: $T_d(\xi)$ as a function of ξ for the case of positive renormalized zero-point energy for $d = 3$.

Until now we studied the quantum bound for general dimensions based on the summations given by Eq. (41) and Eq. (42). Nevertheless we can find an upper bound function $T'_d(\xi)$ of the function $T_d(\xi)$ which is more manageable. For this purpose, in a similar way as we have defined the function $R'_d(\xi)$, let us define also the auxiliary functions $P'_d(\xi)$ and $P''_d(\xi)$, that satisfy

$$P_d(\xi) < P'_d(\xi)\,, \tag{46}$$

and

$$P_d(\xi) > P''_d(\xi)\,, \tag{47}$$

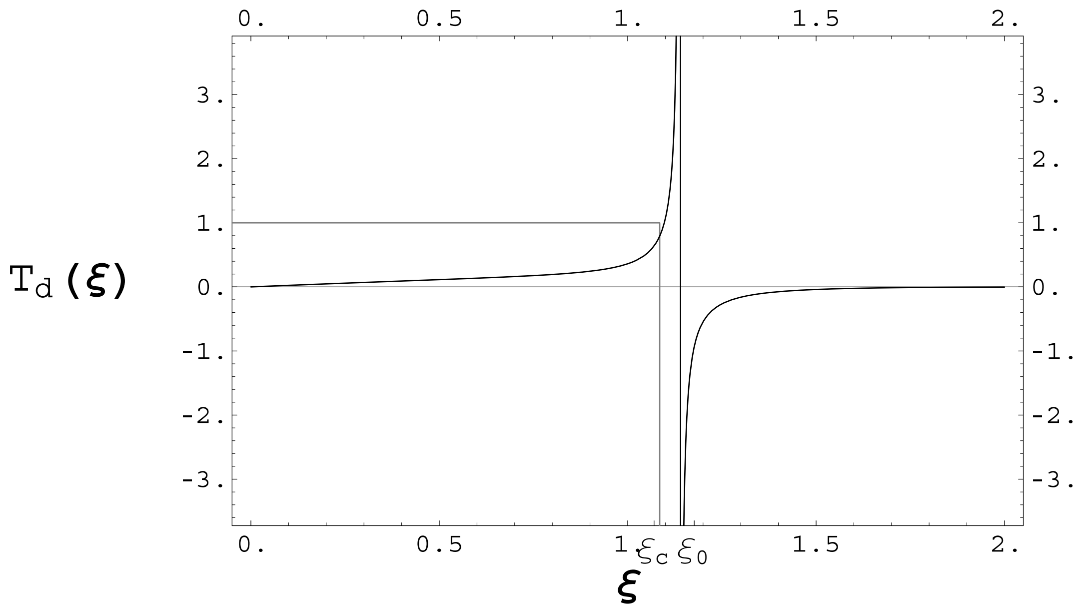

Figure 2: $T_d(\xi)$ as a function of ξ for the case of negative renormalized zero-point energy for $d = 4$.

so that the specific entropy satisfies the following inequality

$$\frac{S}{E} < 2\pi R \, T'_d(\xi)\,, \tag{48}$$

where

$$T'_d(\xi) = \frac{1}{\pi\sqrt{d-1}} \, \frac{\xi \, P'_d(\xi) + R'_d(\xi)}{\varepsilon_d^{(r)} + P''_d(\xi)}\,. \tag{49}$$

Without loss of generality we can choose as the auxiliary functions $P'_d(\xi)$ and $P''_d(\xi)$ the integrals

$$P'_d(\xi) = \pi \int_{\Omega_R} d\Omega_{d-1} \int_0^\infty dr \, r^{d-1} \left(e^{\pi\,\xi\, r} - 1\right)^{-1} \tag{50}$$

and

$$P''_d(\xi) = \pi \int_{\Omega_R} d\Omega_{d-1} \int_1^\infty dr \, r^{d-1} \left(e^{\pi\,\xi\, r} - 1\right)^{-1}\,. \tag{51}$$

Performing these integrals [73], we obtain that $P'_d(\xi)$ and $P''_d(\xi)$ are given by

$$P'_d(\xi) = \pi S_{d-1} \Gamma(d) \zeta(d) \left(\frac{1}{\pi\xi}\right)^d \tag{52}$$

and

$$P''_d(\xi) = \pi S_{d-1}\Big(\Gamma(d)\zeta(d) - f(d)\Big)\left(\frac{1}{\pi\xi}\right)^d, \tag{53}$$

where the series $f(d)$ is given by

$$f(d) = \sum_{l=0}^{\infty} \frac{B_l}{(d+l-1)l!}. \tag{54}$$

To obtain an upper bound for the specific entropy in a generic Euclidean d-dimensional spacetime we have only to substitute Eq. (44), Eq. (52) and Eq. (53) into Eq. (49). We have that

$$T'_d(\xi) = \frac{h_1(d)}{\varepsilon_d^{(r)}\,\xi^{d-1} + h_2(d)\,\xi^{-1}}, \tag{55}$$

where

$$h_1(d) = \frac{S_{d-1}}{\pi^d\sqrt{d-1}}\zeta(d)\Big(\Gamma(d) + \Gamma(d-1)\Big), \tag{56}$$

and

$$h_2(d) = \frac{S_{d-1}}{\pi^{d-1}}\Big(\Gamma(d)\,\zeta(d) - f(d)\Big). \tag{57}$$

It is interesting to study the behavior of the specific entropy for low and high temperatures. For the case of high temperatures, we get

$$\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{h_2(d)}\,\xi\,. \tag{58}$$

This behavior of the specific entropy increasing with β in the high-temperature limit was obtained by Deutsch in Ref. [24]. Bekenstein using the condition $\beta \ll R$ (high temperature limit) also obtained the same behavior in Ref. [21]. Since the thermal energy can compensate the negative renormalized zero-point energy, the quantum bound holds.

When considering the low temperature behavior of the specific entropy, we can see that the problem of the sign of the renormalized zero-point energy can invalidate the quantum bound. In this limit we have

$$\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{\varepsilon_d^{(r)}}\xi^{1-d}\,. \tag{59}$$

It is well known that the renormalized zero-point energy for massless scalar fields in a cube, assuming Dirichlet boundary conditions change sign with the dimension, i.e., for $d = 2$ we have $E_0^{(r)} < 0$, for $d = 3$ we have $E_0^{(r)} > 0$ and for the important case $d = 4$ we have $E_0^{(r)} < 0$. Although some authors claim that the energy of the boundaries of such systems can compensate the negative renormalized-zero point energy yielding a net positive energy, this is still an open question in the literature.

5 Conclusions and discussions

In this paper we study self-interacting scalar fields in the strong-coupling regime in equilibrium with a thermal bath, also in the presence of macroscopic boundaries. In the strong-coupling perturbative expansion we may split the problem of defining the generating functional into two parts: how to define precisely the independent-value generating functional and how to go beyond the independent-value approximation, taking into account the perturbation part. The presence of the spectral zeta-function allow us to introduce the boundary conditions in the problem. Using the Klauder representation for the independent-value generating functional, and up to the order $(g_0)^{-\frac{2}{p}}$, we show that it is possible to obtain a quantum bound in the system defined by a self-interacting scalar field in the strong-coupling regime. We established a bound on information storage capacity of the strong-coupled system in a framework independent of gravitational physics.

We have shown, in the strong-coupling regime, at low and intermediate temperatures ($\beta \approx L$), the quantum bound depends on the sign of the renormalized zero-point energy given by $E_0^{(r)}$. For even spacetime dimensions d and also for odd values satisfying the inequality $d > 29$, $E_0^{(r)}$ is a negative quantity. Therefore the quantum bound is invalidated. For odd values of d, satisfying the inequality $d \leq 29$, $E_0^{(r)}$ is a positive quantity. In this situation the specific entropy satisfies a quantum bound. Defining $\varepsilon_d^{(r)}$ as the renormalized zero-point energy for the free theory per unit length, we get the following functional dependencies. For low temperatures we get $\frac{S}{E} < 2\pi R \,\frac{h_1(d)}{\varepsilon_d^{(r)}\xi^{\,d-1}}$, where R is the radius of the smallest sphere circumscribing the system. For the case of high temperature, we get that the specific entropy always satisfies a quantum bound, given by $\frac{S}{E} < 2\pi R \,\frac{h_1(d)}{h_2(d)}\,\xi$.

Although our results are based in a quite particular choice of the shape of the macroscopic boundaries that confine the field in the volume Ω, it is extremely interesting to point out that the quantum bound that we are obtaining is independent of the shape of the boundaries. In other words, even though we choose the hypercube to confine the fields in a finite volume, an arbitrary boundary should give the same results. If we have a domain G and if we consider the eigenvalue problem for a self-adjoint second-order partial differential operator acting on scalar functions, an important property of monoticity of the eigenvalues associated with the Dirichlet boundary condition lead us to the result that under the Dirichlet boundary condition, the n^{th} eigenvalue of the domain G never exceeds the n^{th} eigenvalue of the sub-domain G^*. Also, the asymptotic behavior of the eigenvalues does not depend on the shape, but only on the size of the fundamental domain. These two results can be used to show that for any shape we will get the same results. This argument also was presented by Schiffer and Bekenstein.

A Appendix: The Klauder representation for the independent value generating functional

To give meaning to the independent value generating functional $Q_0(\sigma,h)$, we may either discretize the space or use the Klauder's result, as the formal definition of the independent-value generating functional derived for scalar fields in a d-dimensional Euclidean space. This generating functional is a mean zero Gaussian functional integral and using the fact that the fields defined in each point of the Euclidean time are statistically independent we are able to write

$$Q_0(\sigma,h) = \exp\left(-\int d^dx\, L(\sigma,h(x))\right), \tag{A.1}$$

for some function $L(\sigma,h(x))$. The formula above is fundamental for our study. Let us see how it is possible to extract some information from it. Before studying the interacting case, let us analyze a simple example, i.e., $g_0=0$. In this case we have

$$Z(\beta,\Omega,h)_{\,g_0=0} = \exp\left(-\frac{1}{2}\int d^dx\int d^dx'\,\frac{\delta}{\delta h(x)}K(m_0,\sigma\,;x-x')\frac{\delta}{\delta h(x')}\right)\,Q_0(\sigma,h)|_{\,g_0=0}, \tag{A.2}$$

where $Q_0(\sigma,h)$, the independent-value generating functional, is given by

$$Q_0(\sigma,h)|_{\,g_0=0} = \mathcal{N}\int[d\varphi(x)]\,\exp\left(\int d^dx\,\left(-\frac{1}{2}\,\sigma\,m_0^2\,\varphi^2(x)+h(x)\varphi(x)\right)\right), \tag{A.3}$$

where once more the modified kernel $K(m_0,\sigma;x-x')$ was defined by Eq.(5).

The free independent-value functional must satisfies $Q_0(\sigma,h)|_{h=g_0=0}=1$. We would like to point out that in Klauder's derivation for the free independent-value model a result was obtained which is well defined for all functions which are square integrable in R^n i.e., $h(x)\,\varepsilon\,L^2(R^n)$. Since we are assuming that $h=cte$, we have to normalize our expressions. Therefore we have

$$Q_0(\sigma,h)|_{\,g_0=0} = \exp\left(-\frac{1}{2V\sigma\,m_0^2}\int\,d^dx\,h^2(x)\right). \tag{A.4}$$

The generalization for the self-interaction scalar field with the $\frac{g_0}{p!}\,\varphi^p(x)$ self-interacting contribution is straightforward. It is possible to show that the independent-value generating function can be written as

$$Q_0(\sigma,h) = \exp\left(-\frac{1}{2V}\int\,d^dx\int_{-\infty}^{\infty}\frac{du}{|u|}\,(1-\cos(hu))\exp\left(-\frac{1}{2}\,\sigma\,m_0^2\,u^2-\frac{g_0}{p\,!}\,u^p\right)\right). \tag{A.5}$$

There is no need to go into details of this derivation. The reader can find it in Ref. [49]. In order to study $Q_0(\sigma,h)$ let us define $E(m_0,\sigma,g_0,h)$ given by

$$E(m_0,\sigma,g_0,h) = \int_{-\infty}^{\infty}\frac{du}{|u|}\,(1-\cos(hu))\exp\left(-\frac{1}{2}\,\sigma\,m_0^2\,u^2-\frac{g_0}{p\,!}\,u^p\right). \tag{A.6}$$

Using a series representation for $\cos x$ and using the fact that the series obtained $(\sum_{k=1}^{\infty} c_k\, f_k(u))$ not only converges on the interval $[0,\infty)$, but also converges uniformly there, the series can be integrated term by term. It is not difficult to show that

$$E(m_0,\sigma,g_0,h) = 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}h^{2k}\int_0^{\infty} du\, u^{2k-1}\exp\left(-\frac{1}{2}\,\sigma\, m_0^2\, u^2 - \frac{g_0}{p\,!}\, u^p\right). \tag{A.7}$$

Now let use the fact that the σ parameter can be choosen in such a way that the calculations becomes tractable. This is the main difference from Klauder's result. Analysing only the independent-value generating functional it is not possible to write $Q_0(\sigma,h)$ in a closed form even in the case of constant external source. Let us choose $\sigma = 0$. Therefore we have

$$E(m_0,\sigma,g_0,h)|_{\sigma=0} = 2\sum_{k=1}^{\infty}\frac{(-1)^k}{2k!}h^{2k}\int_0^{\infty} du\, u^{2k-1}\exp(-\frac{g_0}{p\,!}\, u^p). \tag{A.8}$$

At this point let us use the following integral representation for the Gamma function [72]

$$\int_0^{\infty} dx\, x^{\nu-1}\exp(-\mu\, x^p) = \frac{1}{p}\,\mu^{-\frac{\nu}{p}}\,\Gamma\left(\frac{\nu}{p}\right), \quad \mathrm{Re}(\mu)>0 \quad \mathrm{Re}(\nu)>0 \;\; p>0. \tag{A.9}$$

At this point it is clear that the $(g_0\,\varphi^p)$ theory, for even $p>4$, can also easily handle applying our method. Using the result given by Eq.(A.9) in Eq.(A.8) we have

$$E(m_0,\sigma,g_0,h)|_{\sigma=0} = \sum_{k=1}^{\infty} g(p,k)\frac{h^{2k}}{g_0^{\frac{2k}{p}}}, \tag{A.10}$$

where the coefficients $g(p,k)$ are given by

$$g(p,k) = \frac{2}{p}\frac{(-1)^k}{(2k)!}(p\,!)^{\frac{2k}{p}}\Gamma(\frac{2k}{p}). \tag{A.11}$$

Substituting the Eq.(A.10) and Eq.(A.11) in Eq.(A.5) we obtain that the independent-value generating function $Q_0(\sigma,h)|_{\sigma=0}$ can be written as

$$Q_0(\sigma,h)|_{\sigma=0} = \exp\left[-\frac{1}{2\Omega\beta}\int_0^{\beta} d\tau\int d^{d-1}x\sum_{k=1}^{\infty} g(p,k)\frac{h^{2k}}{g_0^{\frac{2k}{p}}}\right]. \tag{A.12}$$

It is easy to calculate the second derivative for the independent-value generating function with respect to h. Note that $Q_0(\sigma,h)|_{h=\sigma=0} = 1$. Thus we have

$$\frac{\partial^2}{\partial h^2}Q_0(\sigma,h)|_{\sigma=0} = \left(-\frac{1}{2}\sum_{k=1}^{\infty} g(p,k)(2k)(2k-1)\frac{h^{2k-2}}{g_0^{\frac{2k}{p}}}\right)\exp\left(-\frac{1}{2}\sum_{k=1}^{\infty} g(p,k)\frac{h^{2k}}{g_0^{\frac{2k}{p}}}\right) + G(g_0,p,h), \tag{A.13}$$

where $G(g_0,p,h)$ is given by

$$G(g_0,p,h) = \left(\sum_{k,\,q=1}^{\infty} g(p,k,q)\frac{h^{2k+2q-2}}{g_0^{\frac{2(k+q)}{p}}}\right)\exp\left(-\frac{1}{2}\sum_{k=1}^{\infty} g(p,k)\frac{h^{2k}}{g_0^{\frac{2k}{p}}}\right), \tag{A.14}$$

and $g(k,q) = k\,q\,g(k)g(q)$. We are interested in the case $h=0$, therefore the double series does not contributes to the Eq.(A.13), since $lim_{h\to 0}G(h) = 0$. Using the fact that we are interested in the case $h=0$, we have the simple result that in the Eq.(A.13) only the term $k=1$ contributes. We get

$$\frac{\partial^2}{\partial h^2}\,Q_0(\sigma,h)|_{\,h=\sigma=0} = \frac{1}{2p\,g_0^{\frac{2}{p}}}\frac{\Gamma(\frac{2}{p})}{(p\,!)^{\frac{p}{2}}}. \tag{A.15}$$

B Appendix: Proof that the value of the spectral zeta-function in the origin vanishes, i.e., $\zeta_D(s)|_{s=0} = 0$

As we discussed before, to take into account the scaling properties we should have to introduce an arbitrary parameter μ with dimension of a mass to define all the dimensionless physical quantities and in particular make the change

$$\frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{\,s=0} \to \frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{\,s=0} - \frac{1}{2}\ln\left(\frac{1}{4\pi\mu^2}\right)\zeta_D(s)|_{\,s=0}. \tag{B.1}$$

In this appendix we have a proof that the spectral zeta-function in $s=0$ is zero, consequently there is no scaling in the theory. The Epstein zeta-function is defined by

$$Z_p\,(a_1,...,a_p\,;2s) = \sum_{n_1,...,\,n_p=-\infty}^{\infty\;\;\prime}\Big((a_1\,n_1)^2+...+(a_p\,n_p)^2\Big)^{-s}, \tag{B.2}$$

where the prime indicates that the term for which all $n_i = 0$ is to be omitted. This summation is convergent only for $2s > p$. Nevertheless, we can find an integral representation which gives an analytic continuation for the Epstein zeta-function except for a pole at $p = 2s$ [9]. This representation is given by

$$\begin{aligned}&(\pi\,\eta)^{-s}\,\Gamma(s)\,Z_p\,(a_1,...,a_p\,;2s) = \\ &-\frac{1}{s}+\frac{2}{p-2s}+\eta^{-s}\int_{\eta}^{\infty}dx\,x^{s-1}\left(\vartheta(0,...,0;a_1^2\,x,...,a_p^2\,x)-1\right)\\ &+\eta^{(2s-p)/2}\int_{1/\eta}^{\infty}dx\,x^{(p-2s)/2-1}\left(\vartheta(0,...,0;x/a_1^2,...,x/a_p^2)-1\right),\end{aligned} \tag{B.3}$$

where $\eta^{p/2}$ is the product of the $p's$ parameters a_i given by $\eta^{p/2} = a_1...a_p$, and the generalized Jacobi function $\vartheta(z_1, ..., z_p; x_1, ..., x_p)$, is defined by

$$\vartheta(z_1, ..., z_p\,; x_1, ..., x_p) = \prod_{i=1}^{p} \vartheta(z_i; x_i)\,, \tag{B.4}$$

with $\vartheta(z; x)$ being the Jacobi function, i.e.,

$$\vartheta(z; x) = \sum_{n=-\infty}^{\infty} e^{\pi(2nz-n^2x)}\,. \tag{B.5}$$

Using this integral expression for the Epstein zeta-function, given by Eq. (B.3), we can find that

$$Z_p(a_1, ..., a_p\,; 2s)|_{s=0} = -1\,, \tag{B.6}$$

for any $p \geq 1$. To proceed, let us define the function $Z_p^{(q)}(a_1, ..., a_p\,; 2s)$, given by

$$Z_p^{(q)}(a_1, ..., a_p\,; 2s) = \sum_{n_1,...,n_q=1}^{\infty}\ \sum_{n_{q+1},...,n_p=-\infty}^{\infty} \Big((a_1\,n_1)^2 + ... + (a_p\,n_p)^2\Big)^{-s}\,. \tag{B.7}$$

Using the result given in Eq. (B.6) we can show that, after performing the analytic continuation of the function $Z_p^{(q)}(a_1, ..., a_p\,; 2s)$, the following property holds

$$Z_p^{(q)}(a_1, ..., a_p; 2s)|_{s=0} = 0\,, \tag{B.8}$$

where this result is valid only for $0 < q < p$. We can prove this property by induction. First, let us verify that for $q = 1$ the above property hold. Therefore, assuming that is valid for q, we have only to show that is true for $q + 1$. For $q = 1$ we have that

$$Z_p(a_1, ..., a_p\,; 2s)|_{\,s=0} = Z_p(a_2, ..., a_p\,; 2s)|_{\,s=0} + 2Z_p^{(1)}(a_1, ..., a_p\,; 2s)|_{\,s=0}\,. \tag{B.9}$$

Since $p > 1$ we can use the property given by Eq. (B.6), for the two first terms of Eq. (B.9) and verify that $Z_p^{(1)}(a_1, ..., a_p\,; 2s)|_{s=0} = 0$. The next step in the proof by induction is to assume the validity of this property for some q, i.e., $Z_p^{(q)}(a_1, ..., a_p\,; 2s)|_{\,s=0} = 0$, with p being arbitrary, but satisfying the condition $0 < q < p$, then we must to verify the validity of this property for $q+1$, i.e., $Z_{p'}^{(q+1)}(a_1, ..., a_{p'}\,; 2s)|_{\,s=0} = 0$ with p' also being arbitrary but satisfying the condition $0 < q + 1 < p'$. ¿From the following property

$$Z_{p'}^{(q)}(a_1, ..., a_{p'}\,; 2s)|_{s=0} = Z_{p'-1}^{(q)}(a_1, ..., a_q, a_{q+2}, ..., a_{p'}; 2s)|_{s=0} + 2Z_{p'}^{(q+1)}(a_1, ..., a_{p'}; 2s)|_{\,s=0}\,, \tag{B.10}$$

since $0 < q < p' - 1$ and using the assumption of the validity of this property for arbitrary q, given by Eq. (B.8), we can see that the two first terms in Eq. (B.10) vanish. Therefore we finally proved that $Z_{p'}^{(q+1)}(a_1, ..., a_{p'}\,; 2s)|_{\,s=0} = 0$. We are interested in a particular case of this property, given by

$$Z_p^{(p-1)}(a_1, ..., a_p; 2s)|_{s=0} = \left(\sum_{n_1,...,n_{p-1}=1}^{\infty} \sum_{n_p=-\infty}^{\infty} \Big((a_1\,n_1)^2 + ... + (a_p\,n_p)^2\Big)^{-s} \right)\Bigg|_{s=0} = 0\,. \qquad \text{(B.11)}$$

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References

[1] A. Ajdari, B. Duplantier, D. Hone, L. Peliti and J. Prost, J. Phys. II France **2**, 487 (1992).

[2] M. L. Lyra, M. Kardar and N. F. Svaiter, Phys. Rev. **E47**, 3456 (1993).

[3] M. Krech, *"The Casimir Effect in Critical Systems"*, World Scientific, Singapure (1994).

[4] J. G. Brankov, D. M. Danchev and M. S. Tonchev. *" Theory of Critical Phenomena in Finite Size Systems"*, World Scientific, Singapure (2000).

[5] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wekf. **51**, 793 (1948).

[6] G. Plunien, B. Müller and W. Greiner, Phys. Rep. **134**, 87 (1986).

[7] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).

[8] K. A. Milton, *"The Casimir Effect: Physical Manifestation of Zero-Point Energy"*, World Scientific (2001).

[9] J. Ambjorn and S. Wolfram, Ann. Phys. **147**, 1 (1983).

[10] F. Caruso, N. P. Neto, B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D43**, 1300 (1991).

[11] R. D. M. De Paola, R. B. Rodrigues and N. F. Svaiter, Mod. Phys. Lett. **A34**, 2353 (1999).

[12] L. E. Oxman. N. F. Svaiter and R. L. P. G. Amaral, Phys. Rev. **D72**, 125007 (2005).

[13] G. 't Hooft, *"Dimensional Reduction in Quantum Gravity"*, in *Salam-Festschrifft*, A. Aly, J. Ellis and S. Randbar-Daemi, Eds., World Scientific, Singapore (1993), ArXiv gr-qc/9310026.

[14] L. Susskind, J. Math. Phys. **36**. 6377 (1995).

[15] R. Bousso, Rev. Mod. Phys. **74**, 825 (2002).

[16] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973).

[17] J. D. Bekenstein, Phys. Rev. **D23**, 287 (1981).

[18] J. D. Bekenstein, Phys. Rev. **D30**, 1669 (1984).

[19] M. Schiffer and J. D. Bekenstein, Phys. Rev. **D39**, 1109 (1989).

[20] L. X. Li and l. Liu, Phys. Rev. **D46**, 3296 (1992).

[21] J. D. Bekenstein, Phys. Rev. **D49**, 1912 (1994).

[22] D. N. Page, Phys. Rev **D26**, 947 (1982).

[23] D. Unwin, Phys. Rev. **D26**, 944 (1982).

[24] D. Deutch, Phys. Rev. Lett. **48**, 286 (1982).

[25] W. G. Unruh, Phys. Rev. **D42**, 3596 (1990).

[26] D. Marolf and R. Roiban, *Note on Bound States and the Bekenstein Bound*, ArXiv hep-th/0406037 (2004).

[27] R. Bousso, J. High Energy Phys. **04**, 035 (2004).

[28] R. Bousso, *Bound States and the Bekenstein Bound* ArXiv hep-th/0310148 (2003).

[29] J. D. Bekenstein and E. I. Guendelman, Phys. Rev. **D35**, 716 (1987).

[30] J. D. Bekenstein and M. Schiffer, Int. J. Mod. Phys. **C1**, 355 (1990).

[31] K. Symanzik, Nucl. Phys. **B190**, 1 (1981).

[32] C. D. Fosco and N. F. Svaiter, J. Math. Phys. **42**, 5185, (2001).

[33] M. I. Caicedo and N. F. Svaiter, J. Math. Phys. **45**, 179 (2004).

[34] N. F. Svaiter, J. Math. Phys. **45**, 4524 (2004).

[35] M. Aparicio Alcalde, G. F. Hidalgo and N. F. Svaiter, J. Math. Phys. **47**, 052303 (2006).

[36] E. R. Caianiello, G. Scarpetta, N. Cim. **22A**, 448 (1974), ibid. Lett. N. Cim. **11**, 283 (1974).

[37] J. R. Klauder, Acta Phys. Aust. **41**, 237 (1975).

[38] J. R. Klauder, Phys. Rev. **D14**, 1952 (1976).

[39] R. Menikoff and D. H. Sharp, J. Math. Phys. **19**, 135 (1978).

[40] J. R. Klauder, Ann. Phys. **117**, 19 (1979).

[41] S. Kovesi-Domokos, Il Nuovo Cim. **33A**, 769 (1976).

[42] C. M. Bender, F. Cooper, G. S. Guralnik and D. H. Sharp, Phys. Rev. **D19**, 1865 (1979).

[43] N. Parga, D. Toussaint and J. R. Fulco, Phys. Rev. **D20**, 887 (1979).

[44] C. M. Bender, F. Cooper, G. S. Guralnik and D. H. Sharp, R. Roskies and M. L. Silverstein, Phys. Rev. **D20**, 1374 (1979).

[45] F. Cooper and R. Kenway, Phys. Rev. **D24**, 2706 (1981).

[46] C. Bender, F. Cooper, R. Kenway and L. M. Simmons, Phys. Rev. **D24**, 2693 (1981).

[47] *"The Strong Coupling Expansion and the Singularities of the Perturbative Expansion"*, N. F. Svaiter, *Proceedings of the X Brazilian School of Cosmology and Gravitation*, Rio de Janeiro, Brazil, AIP (2003), hep-th/0404070.

[48] R. J. Rivers, *"Path Integral Methods in Quantum Field Theory"*, Cambridge University Press, Cambridge (1987).

[49] J. R. Klauder, *"Beyond Conventional Quantization"*, Cambridge University Press, Cambridge (2000).

[50] J. Frohlich, Nucl. Phys. **B200**, 281 (1982).

[51] M. Aizenman and R. Graham, Nucl. Phys. **B225**, 261 (1983).

[52] D. J. E. Callaway, Phys. Rep. **167**, 241 (1981).

[53] G. Gallavotti, Rev. Mod. Phys. **57**, 471 (1985).

[54] R. T. Seeley, Am. Math. Proc. Symp. Pure Math. **10**, 288 (1967).

[55] S. W. Hawking, Comm. Math. Phys. **55**, 133 (1977).

[56] J. S. Dowker and G. Kennedy, J. Phys. **A11**, 895 (1978).

[57] A. Voros, Comm. Math. Phys. **110**, 439 (1987).

[58] W. Dittrich and M. Reuter, *"Effective Lagrangians in Quantum Electrodynamics"*, Springer-Verlag, Berlin (1986).

[59] N. F. Svaiter, Physica **A345**, 517 (2005).

[60] N. F. Svaiter, Physica **A386**, 111 (2006).

[61] R. Kubo, J. Phys. Soc. Jap. **12**, 570 (1957).

[62] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

[63] N. F. Svaiter and B. F. Svaiter, Jour. Phys. **A25**, 979 (1992).

[64] F. Caruso, R. De Paola and N. F. Svaiter, Int. Jour. Mod. Phys. **A14**, 2077 (1999).

[65] L. H. Ford and N. F. Svaiter, Phys. Rev. **58**, 065007-1 (1998).

[66] L. H. Ford, Phys. Rev. **D21**, 933 (1980).

[67] L. H. Ford and N. F. Svaiter, Phys. Rev. **51**, 6981 (1995).

[68] E. Elizalde and A. Romeo, J. Math. Phys. **30**, 1133 (1989).

[69] J. G. Moss, Class. Quant. Grav **6**, 759 (1989).

[70] S. K. Blau, M. Visser and A. Wipf, Nucl. Phys. **B310**, 163 (1988).

[71] J. J. Kapusta, *"Finite-temperature Field Theory"*, Cambridge University Press (1989).

[72] I. S. Gradshteyn and I. M. Ryzhik, *"Tables of Integrals, Series and Products"*, Academic Press Inc., New York (1980).

[73] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *"Integrals and Series"*, Vol. 1 and 2, Gordon and Breach Science Publishers (1986).

[74] J. R. Ruggiero, A. H. Zimerman and A. Villani, Rev. Bras. Fis. **7**, 663 (1977).