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DIAGONALIZATION METHODS FOR THE GENERAL BILINEAR HAMILTONIAN OF AN ASSEMBLY OF BOSONS

by

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Abastract

The problem of the exact diagonalization of the Hamiltonian of an assembly of N bilinearly interacting bosons is discussed in what concerns the eigenvalues as well as for the expression of the new boson operators in terms of the old ones. The method is equivalent to the standard equation-of-motion approach, nevertheless sensibly more concise. Three sets of operational rules are indicated, and their use is exhibited in some examples. In some particular cases of practical importance (for example, when all the coefficients of the Hamiltonian are real), the research of the eigenvalues has been compacted as much as possible.

Résumé

Le problème de la diagonalisation exacte de l'Hamiltonien d' une assemblée de N bosons en interaction bilinéaire, est traité aussi bien en ce qui concerne les valeurs propres que pour l'expression des nouveaux opérateurs de bosons en fonction des anciens. La méthode est équivalente à l'approche standard avec l'équation de mouvement, cependant elle est sensiblement plus concise. Trois ensembles de règles opérationnelles et leur usage sur quelques exemples sont indiqués. Dans quelques cas particuliers d'importance pratique (par exemple, lorsque tous les coefficients de l'Hamiltonien sont réels), la recherche des valeurs propres a été compactée autant que possible.

1 - Introduction

It is well known from long date (at least from the date of Bogolyubov's paper[1] on superfluidity in 1947), that the Hamiltonian of an assembly of N bilinearly interacting bosons (or fermions) is susceptible of exact diagonalization, in terms of new non interacting bosons (or fermions). The standard method used to perform such a diagonalization is the so called "equation-of-motion approach", proposed by Bogolyubov and Tyablikov[2,3,4] in the years 1947-49 and by Bohm and Pines[5] in 1953. This approach is formally presented (see for example Refs. [6] and [7]) and discussed[8,9] by several authors. It is equally useful for fermion problems[10-13] (see Refs. [14] and [15] for supercondutivity) and boson problems[1,16-18] (phonon-phonon[16], photon-optical phonon[16], magnon-magnon[16,17], phonon-pseudomagnon[18] interactions, etc.) Because of the wideness of the applications of this diagonalization problem, we thought it was worth while trying to put it in compact operational rules, and this is the purpose of the present work. However, only the boson case is extensively examined, as in the fermion case, the canonical transformation between old and new fermions is given by an unitary matrix with no further complications. This is not so for the boson case, where the canonical transformation is governed by a matrix related to not necessarily positive metric, a fact which introduces a certain amount of "pathology" in the case.

In Sect. 2 the Hamiltonian we are going to deal with is presented; in Sect. 3 appear the basic ideas of the diagonalization, which lead to the three sets of operational rules of Sect. 6; in Sections 4 and 5 appear a particular canonical transformation and the treatment of particular Hamiltonians respectively; we conclude in Sect. 7 by a practical comparison between the three diagonalizing methods exposed in this paper; finally in Appendix are treated the cases N = 1, 2, 3 (N = 1 corresponds to the historical form of Bogolyubov's transformation).

2 - Hamiltonian

Let us consider an assembly of N bilinearly interacting bosons, which might be particles or quasi-particles The most general* quadratic Hamiltonian (which needs not to conserve the number of bosons) might be written as follows:

$$\mathcal{H} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{ 2\,\omega_{ij}\, b_i^+ b_j + \nu_{ij}^1\, b_i^+ b_j^+ + \nu_{ij}^2\, b_i b_j \right\} \tag{1}$$

where factor 2 has been introduced for future commodity; ω_{ij}, ν_{ij}^1 and ν_{ij}^2 are complex number, and the creation and annihilation operators satisfy

$$[b_i, b_j] = [b_i^+, b_j^+] = 0 \qquad \forall (i,j) \tag{2a}$$

$$[b_i, b_j^+] = \delta_{ij} \equiv \text{Kroenecker's delta} \qquad \forall (i,j) \tag{2b}$$

Our final purpose is of course to present this Hamiltonian in the form

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* Eventual terms linear in boson operators can be easily removed by defining new boson operators related to the old ones by $a = b + \mu$ and $a^+ = b^+ + \mu^*$, where $\mu \in \mathbb{C}$. Additive constants in Hamiltonians are not going to be explicitely written is this paper, because of their simpleness and quite frequent irrelevance.

$$\mathcal{H} = \sum_{j=1}^{N} 2\,\Omega_j\, B_j^+ B_j \tag{3}$$

where the Ω_j should be known real positive functions of the previous parameters, and the new boson operators known linear combinations of the old ones.

Let us use the notation ω, ν^1 and ν^2 for denoting the matrix $\{\omega_{ij}\}$, $\{\nu^1_{ij}\}$ and $\{\nu^2_{ij}\}$ respectively. Because of commutation rules (2a) we may always consider ν^1 and ν^2 as symmetric matrix. Furthermore, hermiticity of $\mathcal{H}$ implies hermiticity of ω as well as $\nu^{2*} = \nu^1 \equiv \nu$, where (*) denotes the complex conjugate. Hence (1) may be re-written as follows:

$$\mathcal{H} = \sum_{i,j}\left\{\omega_{ij}\, b_i^+ b_j + \omega_{ij}^*\, b_i b_j^+ + \nu_{ij}\, b_i^+ b_j^+ + \nu_{ij}^*\, b_i b_j\right\} \tag{1'}$$

where $\omega = \omega^+$ and $\nu = \nu^T$ ((+) and (T) denote the adjoint and the transposed matrix respectively). Let us now introduce the nomenclature

$$|b\rangle \equiv \begin{pmatrix} b_1 \\ \vdots \\ b_N \\ b_1^+ \\ \vdots \\ b_N^+ \end{pmatrix} \quad ; \quad \langle b| \equiv |b\rangle^+ = \left(b_1^+, \ldots, b_N^+, b_1, \ldots, b_N\right)$$

$$H \equiv \left(\begin{array}{c|c} \omega & \nu \\ \hline \nu^* & \omega^* \end{array}\right) \qquad \left(H^+ = H\right)$$

Remark that if $\mathcal{H}$ conserves the number of b-bosons*, then $\nu = 0$. The Hamiltonian (1') and the commutation rules (2a, 2b) may be written as follows:

$$\mathcal{H} = \langle b | H | b \rangle \tag{1''}$$

$$|b\rangle\langle b| - \left(|b^+\rangle\langle b^+|\right)^T = J \equiv \left(\begin{array}{c|c} 1_N & 0_N \\ \hline 0_N & -1_N \end{array}\right) \tag{2''}$$

where $|\ldots\rangle\langle\ldots|$ means the matrix direct product, and 1_N and 0_N denote the N x N unity and zero matrix respectively. Remark that $|b^+\rangle \neq |b\rangle^+$.

3 - Diagonalizing method

Let us first of all state a basic property: the Hamiltonian given by (1') will be diagonal in b's operators (this is to say $\nu = 0$ and ω diagonal) if and only if

$$[\mathcal{H}, b_i] = -2\,\omega_{ii}\, b_i \qquad \forall i .$$

The proof is straightforward once we have remarked that in general

$$[\mathcal{H}, b_i] = -2 \sum_j \left\{ \omega_{ij} b_j + \nu_{ij} b_j^+ \right\} \qquad \forall i .$$

* In any case $\mathcal{H}$ is going to conserve the number of B-bosons, which are to be introduced.

This is the property we shall use to find the new boson operators B's which put $\mathcal{H}$ into diagonal form, this is to say

$$\mathcal{H} = \langle B | H_D | B \rangle \tag{3'}$$

where
$$H_D \equiv \left(\begin{array}{c|c} \Omega & 0_N \\ \hline 0_N & \Omega \end{array}\right)$$

and
$$\Omega \equiv \begin{pmatrix} \Omega_1 & 0 & \cdots & 0 \\ 0 & \Omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_N \end{pmatrix}$$

To perform the diagonalization let us propose

$$\langle B | = \langle b | T \qquad \text{and} \qquad | B \rangle = T^+ | b \rangle$$

where T is a 2N x 2N matrix to be found. In order to have that B_j^+ be the adjoint of B_j, T must have a particular form:

$$T = \left(\begin{array}{c|c} T_1 & T_2 \\ \hline T_2^* & T_1^* \end{array}\right) \tag{4}$$

As we want the B's to be boson operators, they must also satisfy the commutation rules

$$|B\rangle\langle B| - \left(|B^+\rangle\langle B^+|\right)^T = J \tag{5}$$

which implies (once we have remarked that $|B^+\rangle = T^* |b^+\rangle$) that

$$T^+ J T J = 1_{2N} \quad \text{hence} \quad T^{-1} = J T^+ J \tag{5'}$$

We see as a corollary that the modulus of the determinant of T equals one. It is also easily verified that the ensemble of matrix T satisfying (4) and (5') constitutes a Lie group (in general non abelian). Relation (5') may be rewritten

$$T_1^+ T_1 - T_2^T T_2^* = 1_N \tag{5"a}$$

$$T_2^+ T_1 - T_1^T T_2^* = 0_N \tag{5"b}$$

To be sure that $\mathcal{H}$ is diagonal we impose

$$[\mathcal{H}, B_j] = -2\,\Omega_j\, B_j \qquad \forall j$$

$$[\mathcal{H}, B_j^+] = 2\,\Omega_j\, B_j^+ \qquad \forall j$$

or more compactly

$$[\mathcal{H}, |B\rangle] = -2\, J\, H_D |B\rangle \tag{6}$$

Taking into account that

$$[\mathcal{H}, |B\rangle] = [\mathcal{H}, T^+ |b\rangle] = T^+ [\mathcal{H}, |b\rangle] = -2\, T^+ J H |b\rangle \,,$$

relation (6) immediately implies that

$$T^{+} J H = J H_D T^{+} \tag{6'}$$

hence
$$T^{-1} H J T = H_D J \tag{6''}$$

where we have used relation (5'). And taking into account the particular form of T, (6") may be rewritten as follows

$$\left(\begin{array}{c|c} P_H & R_H \\ \hline -R_H^{+} & -P_H^{T} \end{array}\right) = \left(\begin{array}{c|c} \Omega & O_N \\ \hline O_N & -\Omega \end{array}\right)$$

where

$$P_H \equiv T_1^{+}\omega T_1 + T_2^{T}\omega^{*} T_2^{*} - T_1^{+}\nu T_2^{*} - T_2^{T}\nu^{*} T_1 = P_H^{+} \tag{6'''a}$$

$$R_H \equiv T_1^{+}\omega T_2 + T_2^{T}\omega^{*} T_1^{*} - T_1^{+}\nu T_1^{*} - T_2^{T}\nu^{*} T_2 = R_H^{T} \tag{6'''b}$$

Let us formulate in another way what we are doing:

$$\mathcal{H} = \langle b|H|b\rangle = \langle b|(TT^{-1})H(J(T(JJ)T^{-1})J)|b\rangle$$

$$= (\langle b|T)(T^{-1}HJTJ)(JT^{-1}J|b\rangle) = \langle B|H_D|B\rangle$$

where we have used relation (5') in the last step.

Before going on, a few words about a frequent particular case, namely when $\nu = 0$. In this (and only this) case the solution is given by $T_2 = O_N$, and we have to deal with a standard N x N diagonalization problem:

$$T_1^{+}\omega T_1 = \Omega \qquad \text{with} \qquad T_1^{+} T_1 = 1_N$$

Let us now turn back to the general situation. The secular equation of our diagonalization problem is given by

$$\det\,(HJ - \Omega_j \, 1_{2N}) \equiv N\text{-th degree polynome in } \Omega_j^2 = 0 \quad \forall j \tag{7}$$

where the fact that only even powers of Ω_j appear, will soon become clear. So our problem will be practically solved if we find a matrix T which simultaneously diagonalizes the matrix HJ and satisfies restrictions (5"a , 5"b). The discussion of the existence and unicity of such a matrix T is beyond the scope of this paper. However let us point out a very suggestive fact: the number of unknown real quantities is exactly the same as the number of real relations between them*. We have indeed $(4N^2 + N)$ real unknown quantities: $2\,N^2$ for the complex matrix T_1, $2\,N^2$ for the complex matrix T_2 and N for the real diagonal matrix Ω . On the other hand, we have $(4\,N^2 + N)$ real equations to solve: N^2 for (5"a) (notice that the concerned matrix is hermitic), $N(N-1)$ for (5"b) (notice that the concerned matrix is antisymmetric), N^2 for (6"'a) (notice that the concerned matrix is hermitic), $N(N + 1)$ for (6"'b) (notice that the concerned matrix is symmetric) and finally N for (7).

Let us now prove that in the secular equation (7), only even powers of Ω_j appear. Relation (6') may be rewritten as follows:

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* This is not a sufficient condition for the existence of the solution, therefore strictly speaking it guarantees nothing beyond a strong suspicion.

$$HJT = T H_D J$$

or, more explicitely,

$$\left(\begin{array}{c|c} \omega & -\nu \\ \hline \nu^* & -\omega^* \end{array}\right)\left(\begin{array}{c|c} T_1 & T_2 \\ \hline T_2^* & T_1^* \end{array}\right) = \left(\begin{array}{c|c} T_1 & T_2 \\ \hline T_2^* & T_1^* \end{array}\right)\left(\begin{array}{c|c} \Omega & O_N \\ \hline O_N & -\Omega \end{array}\right)$$

in other words, the j-th column of the left-half of T is nothing but the eigenvector associated to the j-th eigenvalue of Ω (namely Ω_j), while the j- th column of the right half of T constitutes the eigenvector associ ated to ($-\Omega_j$). Then we see that the eigenvectors associated to Ω_j and to ($-\Omega_j$) are intimately related, and that the segular equation (7) contains only powers of Ω_j^2.

Let us assume we found a particular solution* (noted $\bar{\bar{T}}$) of equation (6'"b). If we write now

$$T \equiv \bar{\bar{T}} S \qquad \text{with} \qquad S^{-1} = J S^+ J$$

relation (6'') may be rewritten as follows

$$S^{-1}\left(\bar{\bar{T}}^{-1} HJ \bar{\bar{T}}\right) S = S^{-1}\left(\begin{array}{c|c} Q_H & O_N \\ \hline O_N & -Q_H^T \end{array}\right) S = \left(\begin{array}{c|c} \Omega & O_N \\ \hline O_N & -\Omega \end{array}\right)$$

where

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* Equation (6'"b) admits of course a great number of solutions, from which only one (independent) satisfies simultaneously (6'"a).

$$Q_H \equiv \bar{T}_1^+ \omega \bar{T}_1 + \bar{T}_2^T \omega^* \bar{T}_2^* - \bar{T}_1^+ \nu \bar{T}_2^* - \bar{T}_2^T \nu^* \bar{T}_1 = Q_H^+$$

The solution S may be written as follows

$$S = \left(\begin{array}{c|c} S_1 & 0_N \\ \hline 0_N & S_1^* \end{array}\right) \tag{8}$$

therefore $\qquad S_1^{-1} Q_H S_1 = \Omega$

with $\qquad S_1^{-1} = S_1^+$

In this way our problem, as in the case $\nu = 0$ has been reduced to a standard diagonalization problem of the N x X hermitic matrix Q_H. The matrix T will be given by

$$T_1 = \bar{T}_1 S_1 \qquad \text{and} \qquad T_2 = \bar{T}_2 S_1^*$$

In all usual* physical Hamiltonians, we want the $\{\Omega_j\}$ to be real (and positive) numbers, therefore

$$\det(H) = \prod_{j=1}^{N} \Omega_j^2 > 0 \tag{9}$$

We can also see that

$$F \equiv (HJ)^2 = \left(\begin{array}{c|c} F_1 & F_2 \\ \hline F_2^* & F_1^* \end{array}\right)$$

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* Hamiltonians adapted to describe displacive phase transitions might constitute an exception.

where $$F_1 \equiv \omega^2 - \nu\nu^* = F_1^+$$
$$F_2 \equiv \nu\omega^* - \omega\nu = -F_2^T$$

If a matrix T diagnonalizes HJ necessarily it diagonalizes also F (<u>the opposite is not true</u>*), therefore

$$T^{-1} F T = (H_D J)^2 = \left(\begin{array}{c|c} \Omega^2 & O_N \\ \hline O_N & \Omega^2 \end{array}\right) \tag{10}$$

It follows then that all diagonal elements of F_1 are positive, this is to say

$$\sum_{j=1}^{N} \left\{ |\omega_{ij}|^2 - |\nu_{ij}|^2 \right\} > 0 \qquad \forall i \tag{11}$$

It is clear that conditions (9) and (11) are necessary but in general not sufficient.

The general form (4) for T leads to

$$T^{-1} F T = \left(\begin{array}{c|c} P_F & R_F \\ \hline -R_F^+ & P_F^T \end{array}\right)$$

with $$P_F \equiv T_1^+ F_1 T_1 - T_2^T F_1^* T_2^* + T_1^+ F_2 T_2^* - T_2^T F_2^* T_1 = P_F^+$$
$$R_F \equiv T_1^+ F_2 T_1^* - T_2^T F_2^* T_2 + T_1^+ F_1 T_2 - T_2^T F_1^* T_1^* = -R_F^T$$

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* The reason is that the eigenvalues $|\Omega_j|$ and $-|\Omega_j|$ lead, in F, to a degenerate bidimensional subspace related to the eigenvalue Ω_j^2. It is clear that $\nu \neq O_N$ and $\nu\omega^* - \omega\nu = O_N$ are compatible (an example is given in Appendix II).

and relation (10) may be rewritten as follows:

$$P_F = \Omega^2 \tag{10'a}$$

$$R_F = O_N \tag{10'b}$$

If we assume we found a particular solution* (noted $\bar{\bar{\bar{T}}}$) of equation (10'b), the matrix T may be written as

$$T = \bar{\bar{\bar{T}}}\, S$$

with S given by (8) and satisfying

$$S_1^{-1} Q_F S_1 = \Omega^2$$

$$S_1^{-1} = S_1^{+}$$

where

$$Q_F \equiv \bar{\bar{\bar{T}}}_1^{+} F_1 \bar{\bar{\bar{T}}}_1 - \bar{\bar{\bar{T}}}_2^{T} F_1^{*} \bar{\bar{\bar{T}}}_2^{*} + \bar{\bar{\bar{T}}}_1^{+} F_2 \bar{\bar{\bar{T}}}_2^{*} - \bar{\bar{\bar{T}}}_2^{T} F_2^{*} \bar{\bar{\bar{T}}}_1 = Q_F^{+}$$

Two immediate corollaries are

$$Q_F = Q_H^2$$

and $\det(F) = \left[\det(Q_F)\right]^2 = \left[\det(\Omega)\right]^4 = \prod_{j=1}^{N} \Omega_j^4$

The preliminar reserch of $\bar{\bar{\bar{T}}}$ might be of practical importance: $\bar{\bar{\bar{T}}}^{-1} HJ\, \bar{\bar{\bar{T}}}$ might be not diagonal, but is expected to be much easier to diagonalize than HJ.

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* Every $\bar{\bar{T}}$ is also a $\bar{\bar{\bar{T}}}$, but the opposite is not true.

4 - Particular solution $\overset{\equiv}{T}$:

Let us discuss in this Section a general way to construct a particular solution $\overset{\equiv}{T}$ of equation (10'b). We shall first of all treat the general case N = 2. Let us use the notation

$$F_1 \equiv \begin{pmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{pmatrix} \quad \text{and} \quad F_2 \equiv \begin{pmatrix} 0 & \tilde{f}_{12} e^{i\varphi_{12}} \\ -\tilde{f}_{12} e^{i\varphi_{12}} & 0 \end{pmatrix}$$

where $f_{11}, f_{22}, \tilde{f}_{12}$ and φ_{12} are real numbers and let us propose the following form for $\overset{\equiv}{T}$:

$$\overset{\equiv}{T}_1 = \operatorname{ch} \psi_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overset{\equiv}{T}_2 = e^{i\chi_{12}} \operatorname{sh} \psi_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with ψ_{12} and χ_{12} being also real numbers. We verify that restriction (5') is automatically satisfied, therefore it is enough to impose (10'b), which leads to the solution

$$\chi_{12} = \varphi_{12}$$

$$\text{and} \quad \operatorname{th} 2\psi_{12} = \frac{2\tilde{f}_{12}}{f_{22} - f_{11}}$$

We see however that our proposal for $\overset{\equiv}{T}$ is not satisfactory if $f_{11} = f_{22}$, so let us make another one for this particular situation:

$$\overset{\equiv}{T}_1 = \operatorname{ch} \psi_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overset{\equiv}{T}_2 = e^{i\chi_{12}} \operatorname{sh} \psi_{12} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where ψ_{12} and χ_{12} are again real numbers. Restriction (5') is again satisfied and relation (10'b) leads to the solution

$$\chi_{12} = \varphi_{12}$$

and
$$\operatorname{th} 2\psi_{12} = \frac{2\tilde{f}_{12}}{f_{12} + f_{12}^{*}}$$

We see that we have again troubles in the particular case of f_{12} being a pure imaginary number (let us note $f_{12} = i\, f_{12}''$) simultaneously with $f_{11} = f_{22}$. In order to be complete let us treat this case by making a new proposal

$$\overline{\overline{T}}_1 = \operatorname{ch} \psi_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{\overline{T}}_2 = i e^{i\chi_{12}} \operatorname{sh} \psi_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where ψ_{12} and χ_{12} are once more real numbers. Once more restriction (5') is satisfied and (10'b) leads to the solution

$$\chi_{12} = \varphi_{12}$$

and
$$\operatorname{th} 2\psi_{12} = \frac{\tilde{f}_{12}}{f_{12}''}$$

Let us finally say that if f_{12}'' vanishes also, then $\tilde{f}_{12}$ must vanish, otherwise Ω_j^2 should become a complex number*. So we may choose $\overline{\overline{T}} = 1_4$ as F will be diagonal by hypothesis.

We are able now to make a porposal for $\overline{\overline{T}}$ for any value of N.

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* If $f_{11} = f_{22}$ and $f_{12} = 0$ the secular equation for F leads to the roots $\Omega_{1,2}^2 = f_{11} \pm i\, \tilde{f}_{12}$.

Let us use the notation

$$F_1 \equiv \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1N} \\ f_{12}^* & f_{22} & \cdots & f_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1N}^* & f_{2N}^* & \cdots & f_{NN} \end{pmatrix}$$

and

$$F_2 \equiv \begin{pmatrix} 0 & \tilde{f}_{12} e^{i\varphi_{12}} & \cdots & \tilde{f}_{1N} e^{i\varphi_{1N}} \\ -\tilde{f}_{12} e^{i\varphi_{12}} & 0 & \cdots & \tilde{f}_{2N} e^{i\varphi_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{f}_{1N} e^{i\varphi_{1N}} & -\tilde{f}_{2N} e^{i\varphi_{2N}} & \cdots & 0 \end{pmatrix}$$

where $\{f_{ij}\}$, $\{\tilde{f}_{ij}\}$ and $\{\varphi_{ij}\}$ are real numbers. Our proposal will be

$$\overline{\overline{T}} = \overline{\overline{T}}^{12}\, \overline{\overline{T}}^{13} \cdots \overline{\overline{T}}^{1N}\, \overline{\overline{T}}^{23}\, \overline{\overline{T}}^{24} \cdots \overline{\overline{T}}^{2N} \cdots \overline{\overline{T}}^{N-1,N} \tag{12}$$

where we have $\frac{1}{2} N(N-1)$ factors and each $\overline{\overline{T}}^{ij}$ is a 2N x 2N matrix depending on two real numbers (namely ψ_{ij} and χ_{ij}) and its form is given by

$$\bar{\bar{\bar{T}}}_1^{ij} = \begin{pmatrix} 1 & 0 & \cdots & & & & & \\ 0 & 1 & \cdots & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ & & & \mathrm{ch}\,\psi_{ij} & \cdots & 0 & & \\ & & & \vdots & \ddots & \vdots & & \\ & & & 0 & \cdots & \mathrm{ch}\,\psi_{ij} & & \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & \cdots & 1 & 0 \\ & & & & & & \cdots & 0 & 1 \end{pmatrix} \tag{13}$$

and i-th column j-th column

$$\bar{\bar{\bar{T}}}_2^{ij} = e^{i\chi_{ij}}\,\mathrm{sh}\,\psi_{ij} \begin{pmatrix} 0 & 0 & \cdots & & & & & \\ 0 & 0 & \cdots & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ & & & \tau_{11} & \cdots & \tau_{12} & & \\ & & & \vdots & \ddots & \vdots & & \\ & & & \tau_{12} & \cdots & \tau_{22} & & \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & \cdots & 0 & 0 \\ & & & & & & \cdots & 0 & 0 \end{pmatrix} \tag{14}$$

where the central matrix τ is given by

$$\tau_{11} = \tau_{22} = 0 \quad \text{and} \quad \tau_{12} = e^{i\chi_{ij}} \, sh\, \psi_{ij} \tag{14a}$$

if $f_{ii} \neq f_{jj}$, hence $\sum_{k=1}^{N} \left\{ |\omega_{ik}|^2 - |\omega_{jk}|^2 - |\nu_{ik}|^2 + |\nu_{jk}|^2 \right\} \neq 0$;

$$\tau_{12} = 0 \quad \text{and} \quad \tau_{11} = -\tau_{22} = e^{i\chi_{ij}} \, sh\, \psi_{ij} \tag{14b}$$

if $f_{ii} = f_{jj}$ and f_{ij} is not a pure imaginary number, hence
$\sum_{k=1}^{N} \left\{ \omega_{ik}\omega_{kj} + \omega_{ki}\omega_{jk} - \nu_{ik}\nu^*_{kj} - \nu_{jk}\nu^*_{ki} \right\} \neq 0$;

$$\tau_{12} = 0 \quad \text{and} \quad \tau_{11} = \tau_{22} = i\, e^{i\chi_{ij}} \, sh\, \psi_{ij} \tag{14c}$$

if $f_{ii} = f_{jj}$ and f_{ij} is a pure imaginary number.

It is easy to see that in all cases, $\overset{\equiv}{T}$ satisfies automatically restriction (5'). On the other hand relation (10'b) leads to N(N-1) real equations which in principle enable us to find the N(N-1) unknown quantities $\{\psi_{ij}\}$ and $\{\chi_{ij}\}$.

5 - Particular cases

We entend to expose here a few particular situations which deserve attention because of their practical importance. Let us express the matrix T which diagonalizes HJ in the form

$$T = L\, X$$

with $$L \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} 1_N & -1_N \\ \hline 1_N & 1_N \end{array} \right) ,$$

therefore $$T^{-1} H J T = X^{-1} M_L X$$

where $$M_L \equiv \frac{1}{2} \left(\begin{array}{c|c} (\omega-\omega^*)+(\nu-\nu^*) & (\omega+\omega^*)-(\nu+\nu^*) \\ \hline (\omega+\omega^*)+(\nu+\nu^*) & (\omega-\omega^*)-(\nu-\nu^*) \end{array} \right)$$

It follows that in what concerns the eigenvalues our diagonalization is equivalent to a standard diagonalization of M_L, whose secular equation is given by

$$\det \left(M_L - \lambda \, 1_{2N} \right) = 0 \tag{15}$$

Now, if $\omega + \omega^* = \pm (\nu + \nu^*)$ this leads to

$$\det \left((\omega - \omega^* + \nu - \nu^*)^2 - 4 \lambda^2 1_N \right) = 0$$

where we have used that the determinant of matrix is invariant through transposition. Our work is now simplified as we have to deal with a N x N matrix.

Another relatively simple situation occurs when $\omega = \omega^*$ and $\nu = \nu^*$, as in this case equation (15) leads to

$$\left| \begin{array}{c|c} \lambda 1_N & -\omega+\nu \\ \hline -\omega-\nu & \lambda 1_N \end{array} \right| = 0$$

A simplified method for calculating such a determinant is given in Appendix IV.

Let us now turn to another situation. We shall now express T as follows

$$T = K\,Y$$

with

$$K \equiv \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} 1_N & -i\,1_N \\ \hline -i\,1_N & 1_N \end{array}\right),$$

therefore

$$T^{-1} H J T = Y^{-1} M_K Y$$

with

$$M_K \equiv \frac{1}{2}\left(\begin{array}{c|c} (\omega-\omega^*)-i(\nu+\nu^*) & (\nu^*-\nu)+i(\omega+\omega^*) \\ \hline (\nu^*-\nu)-i(\omega+\omega^*) & (\omega-\omega^*)+i(\nu+\nu^*) \end{array}\right)$$

As before, and <u>only</u> for the research of the eigenvalues, we may use the secular equation

$$\det\left(M_K - \lambda\, 1_{2N}\right) = 0 \tag{16}$$

In the case $\nu^*-\nu = \pm i\left(\omega+\omega^*\right)$ this leads to

$$\det\left(\left((\omega-\omega^*)-i(\nu+\nu^*)\right)^2 - 4\lambda^2\, 1_N\right) = 0$$

which again is a simpler problem.

Finally, if $\omega-\omega^* = \nu+\nu^* = 0_N$, equation (16) leads to

$$\left|\begin{array}{c|c} \lambda 1_N & \nu - i\omega \\ \hline \nu + i\omega & \lambda 1_N \end{array}\right| = 0$$

This determinant can be calculated with the method indicated in Appendix IV. Let us recall that the cases $\omega - \omega^* = \nu \pm \nu^* = 0_N$ are very frequent in Physics.

6 - Methods

From the ideas developed in previous Sections, operational methods emerge, which are exposed here. In all of them we must, first of all, present the Hamiltonian to be diagonalized, into the form

$$\mathcal{H} = \langle b | H | b \rangle$$

which defines the 2N x 2N matrix

$$H = \left(\begin{array}{c|c} \omega & \nu \\ \hline \nu^* & \omega^* \end{array}\right)$$

The problem will be considered completely solved if we attain the knowledge (as functions of ω and ν) of the N real eigenvalues $\{\Omega_j\}$ (which define the diagonalized Hamiltonian H_D), and of the 2N x 2N complex matrix T (which defines the new boson operators $\langle B| = \langle b| T$ in terms of the old ones). We recall that T has the form

$$T = \left(\begin{array}{c|c} T_1 & T_2 \\ \hline T_2^* & T_1^* \end{array}\right)$$

which gives also the N first eigenvalues $\{\vec{T}_j\}$ by the N x 2N matricial relation

$$\begin{pmatrix} T_1 \\ \hline T_2^* \end{pmatrix} \equiv \begin{pmatrix} \vec{T}_1 \end{pmatrix} \begin{pmatrix} \vec{T}_2 \end{pmatrix} \cdots \begin{pmatrix} \vec{T}_N \end{pmatrix}$$

where

$$\vec{T}_j \equiv \begin{pmatrix} t_j^1 \\ \vdots \\ t_j^N \\ \hline t_j^{N+1} \\ \vdots \\ t_j^{2N} \end{pmatrix}$$

So the knowledge of T implies in the knowledge of $4N^2$ real numbers (only $2N^2$ if T is real). We recall that

$$J \equiv \left(\begin{array}{c|c} 1_N & 0_N \\ \hline 0_N & -1_N \end{array}\right)$$

<u>Method I</u>:

1) Find the roots of the secular equation

det $\left(HJ - \lambda 1_{2N} \right) =$ N-th degree polynome in $\lambda^2 = 0$

then $\qquad \Omega_j = |\lambda_j| \qquad\qquad (j = 1, 2, \ldots, N)$

2) Write, for each value of j, the set of 4N real equations (only 2N if $\mathcal{T}$ is real)

$$\left[\left(HJ - \Omega_j 1_{2N} \right) \vec{T}_j \right]_k = 0 \qquad (k = 1, 2, \ldots, 2N)$$

where by $[\ldots]_k$ we are noting the k-th component of the vector. Then eliminate an arbitrary one between them and replace it by the real one

$$\sum_{k=1}^{N} |t_j^k|^2 - \sum_{k=N+1}^{2N} |t_j^k|^2 = 1 \tag{17}$$

We have in this way a set of 4N independent real equations (only 2N if $\mathcal{T}$ is real) which in principle leads to the knowledge of the 2N complex numbers $\{t_j^k\}$ associated to the chosen value of j. An example of use of this method is given in Appendix I.

2') Alternative possibility for step (2): Find a particular solution $\overline{\overline{T}}$ of the equations

$$\left(\overline{\overline{T}}_1^{+} \omega \overline{\overline{T}}_2 + \overline{\overline{T}}_2^{T} \omega^* \overline{\overline{T}}_1^{*} - \overline{\overline{T}}_1^{+} \nu \overline{\overline{T}}_1^{*} - \overline{\overline{T}}_2^{T} \nu^* \overline{\overline{T}}_2 \right)_{ij} = 0 \qquad (i \geq j)$$

where by $(\ldots)_{ij}$ we note the ij-th element of the matrix, and where the norm relation (17) must also be satisfied.

3') Calculate the matrix

$$Q_H = \overline{\overline{T}}_1^{+} \omega \overline{\overline{T}}_1 + \overline{\overline{T}}_2^{T} \omega^* \overline{\overline{T}}_2^{*} - \overline{\overline{T}}_1^{+} \nu \overline{\overline{T}}_2^{*} - \overline{\overline{T}}_2^{T} \nu^* \overline{\overline{T}}_1$$

and solve the standard diagonalization problem

$$\left(S_1^{-1} Q_H S_1 \right)_{ij} = \Omega_j \delta_{ij}$$

with $S_1^{-1} = S_1^{+}$ and $\delta_{ij} \equiv$ Kroenecker's delta.

4') T is given by

$$T_1 = \overline{\overline{T}}_1 S_1 \qquad \text{and} \qquad T_2 = \overline{\overline{T}}_2 S_1^{*}$$

Method II

1) The same as step (1) of Method I

2) Calculate the matrix

$$F_1 \equiv \begin{pmatrix} f_{11} & \cdots & f_{1N} \\ \vdots & \ddots & \vdots \\ f_{1N}^{*} & \cdots & f_{NN} \end{pmatrix} = \omega^2 - \nu\nu^{*}$$

$$F_2 \equiv \begin{pmatrix} 0 & \tilde{f}_{12} e^{i\varphi_{12}} & \cdots & \tilde{f}_{1N} e^{i\varphi_{1N}} \\ -\tilde{f}_{12} e^{i\varphi_{12}} & 0 & \cdots & \tilde{f}_{2N} e^{i\varphi_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{f}_{1N} e^{i\varphi_{1N}} & -\tilde{f}_{2N} e^{i\varphi_{2N}} & \cdots & 0 \end{pmatrix}$$

$$= \nu\omega^{*} - \omega\nu$$

3) Write, for each one of the $\frac{1}{2}N(N-1)$ values of (ij)(ij = 12,13,...,1N, 23, 24 ..., 2N, ...,(N-1)N), the 2N x 2N matrix

$$\overline{\overline{\overline{T}}}^{ij} \equiv \left(\begin{array}{c|c} \overline{\overline{\overline{T}}}_1^{ij} & \overline{\overline{\overline{T}}}_2^{ij} \\ \hline \left(\overline{\overline{\overline{T}}}_2^{ij}\right)^* & \left(\overline{\overline{\overline{T}}}_1^{ij}\right)^* \end{array}\right)$$

where

$\overline{\overline{\overline{T}}}_1^{ij}$ is given by expression (13), and

$\overline{\overline{\overline{T}}}_2^{ij}$ is given

by expressions (14) and (14a) if $f_{ii} \neq f_{jj}$;

by expressions (14) and (14b) if $f_{ii} = f_{jj}$ and $\mathcal{R}(f_{ij}) \neq 0$;

by expressions (14) and (14c) if $f_{ii} = f_{jj}$ and $\mathcal{R}(f_{ij}) = 0$.

4) Calculate the 2N x 2N matrix

$$\overline{\overline{\overline{T}}} = \overline{\overline{\overline{T}}}^{12}\, \overline{\overline{\overline{T}}}^{13} \ldots \overline{\overline{\overline{T}}}^{1N}\, \overline{\overline{\overline{T}}}^{23}\, \overline{\overline{\overline{T}}}^{24} \ldots \overline{\overline{\overline{T}}}^{2N} \ldots \overline{\overline{\overline{T}}}^{N-1,N}$$

which will be now expressed in terms of the N(N-1) real numbers $\{\varphi_{ij}\}$ and $\{\chi_{ij}\}$. Then present $\overline{\overline{\overline{T}}}$ in the form

$$\overline{\overline{\overline{T}}} \equiv \left(\begin{array}{c|c} \overline{\overline{\overline{T}}}_1 & \overline{\overline{\overline{T}}}_2 \\ \hline \overline{\overline{\overline{T}}}_2^{\,*} & \overline{\overline{\overline{T}}}_1^{\,*} \end{array}\right)$$

which leads to knowledge of $\overline{\overline{\overline{T}}}_1$ and $\overline{\overline{\overline{T}}}_2$ separately.

5) Determine $\{\psi_{ij}\}$ and $\{\chi_{ij}\}$ by solving the N(N-1) real equations given by the matricial relation

$$\overline{\overline{T}}_1{}^{+}F_2\,\overline{\overline{T}}_1{}^{*}-\overline{\overline{T}}_2{}^{T}F_2{}^{*}\overline{\overline{T}}_2+\overline{\overline{T}}_1{}^{+}F_1\,\overline{\overline{T}}_2-\overline{\overline{T}}_2{}^{T}F_1{}^{*}\overline{\overline{T}}_1{}^{*}=O_N$$

Substitute the solutions in the expression of $\overline{\overline{T}}$ obtained in step (4), which will now be a function of $\{f_{ij}\}$, $\{\tilde{f}_{ij}\}$ and $\{\varphi_{ij}\}$.

6) Calculate the N x N matrix

$$Q_F=\overline{\overline{T}}_1{}^{+}F_1\,\overline{\overline{T}}_1-\overline{\overline{T}}_2{}^{T}F_1{}^{*}\overline{\overline{T}}_2{}^{*}+\overline{\overline{T}}_1{}^{+}F_2\,\overline{\overline{T}}_2{}^{*}-\overline{\overline{T}}_2{}^{T}F_2{}^{*}\overline{\overline{T}}_1$$

7) Proceed to a standard diagonalization of the hermitic matrix Q_F by a unitary matrix S_1 presented in the following form:

$$S_1\equiv\begin{pmatrix}\vec{S}_1\end{pmatrix}\begin{pmatrix}\vec{S}_2\end{pmatrix}\cdots\begin{pmatrix}\vec{S}_N\end{pmatrix}$$

To perform this, write, for each value of j, the set of 2N real equations

$$\left[(Q_F-\Omega_j^2\,1_N)\vec{S}_j\right]_k=0 \qquad (k = 1, 2, \ldots, N)$$

then eliminate an arbitrary one between them and replace it by the real equation

$$\|\vec{S}_j\|=1$$

The solution of this set of 2N real equations (only N if S_1 is real) gives the vector $\vec{S}_j$.

8) Calculate the matrix

$$T_1' = \overline{\overline{\overline{T}}}_1 S_1 \qquad \text{and} \qquad T_2' = \overline{\overline{\overline{T}}}_2 S_1^*$$

and then

$$T' = \left(\begin{array}{c|c} T_1' & T_2' \\ \hline (T_2')^* & (T_1')^* \end{array}\right)$$

9) Calculate the matrix

$$H' = (T')^{-1} H J T' J$$

and enter in step (2) or step (2') of Method I.

An example of use of this Method is given in Appendix II.

Method III: This Method is applicable only for the research of the eigenvalues $\{\Omega_j\}$ and only for some particular cases:

1st case: $\omega + \omega^* = \pm(\nu + \nu^*)$

Find the roots of the secular equations

$$\det\left((\omega - \omega^* + \nu - \nu^*)^2 - \mu 1_N\right) = 0$$

then $\Omega_j = \frac{1}{2}\sqrt{\mu_j}$ (j = 1, 2, ..., N)

2nd case: $\omega + \omega^* = \pm i(\nu - \nu^*)$

Find the roots of the secular equation

$$\det\left(\left((\omega - \omega^*) - i(\nu + \nu^*)\right)^2 - \mu 1_N\right) = 0$$

then $\Omega_j = \frac{1}{2}\sqrt{\mu_j}$ (j = 1, 2, ..., N)

3rd case: $\omega - \omega^* = \nu - \nu^* = 0_N$

Find the roots of the equation

$$\sum_{m=0}^{N} (-1)^m C_m \mu^{N-m} = 0$$

where C_n is given in Appendix IV with

$$A \equiv -\omega + \nu \qquad \text{and} \qquad B \equiv -\omega - \nu$$

and then $\Omega_j = \sqrt{\mu_j}$ (j = 1, 2, ..., N)

4th case: $\omega - \omega^* = \nu + \nu^* = 0_N$

Find the roots of the equation

$$\sum_{m=0}^{N} (-1)^m C_m \mu^{N-m} = 0$$

where C_n is given in Appendix IV with

$$A \equiv \nu - i\omega \qquad \text{and} \qquad B \equiv \nu + i\omega$$

and then $\Omega_j = \sqrt{\mu_j}$ (j = 1, 2 ..., N)

Examples of the use of this Method are given in Appendix III.

7 - Conclusion

Let us conclude by saying that the exposed method for diagonalizing any Hamiltonian of N bilinearly interacting bosons is absolutely equivalent to the so called "equation-of-motion approach". However systematic exploitation of the peculiar boson properties had led to a concise mathematic formulation which allows for the establishment of operational rules. We have talked all the time of N bosons; nevertheless the method is equally aplicable to N families (or branches) of bosons, by simple identification of the boson operators $(b_1 \equiv b_q \, , b_2 \equiv b_{-q} \, , \text{etc})$ as it was done, for example, in Refs. [1], [16], [17] and [18].

Finally let us compare the different methods presented in this paper. Method I (steps (1) and (2)) should be considered the most standard way of performing the diagonalization, however if the matrix H is rather complicate (low symmetry, no zeros) the more delayed procedure indicated in Method I (steps (1), (2'), (3') and (4')) could be preferable. Furthermore, if H is very complicate, the highly delayed procedure indicated in Method II could be worth while. If we are interested only in the eigenvalues (as it is frequently the case in Statistical Mechanics), there is no doubt that Method III should be adopted if we are in face of one of its four cases; if not, the problem will be solved by Method I (step (1)).

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APPENDIX I

Let us treat, by the Method I, the cases N = 1 and N = 2. For N = 1 we have $\omega \in \mathbb{R}$ and $\nu = |\nu|e^{i\varphi} \in \mathbb{C}$. The secular equation is

$$\begin{vmatrix} \omega - \lambda & -|\nu|\, e^{i\varphi} \\ |\nu|\, e^{-i\varphi} & -\omega - \lambda \end{vmatrix} = 0 \quad \text{hence} \quad \lambda = \pm\left(\omega^2 - |\nu|^2\right)^{1/2} \quad \text{hence}$$

$\Omega = \left(\omega^2 - |\nu|^2\right)^{1/2}$. We see that it must be $\omega > |\nu|$. Let us propose

$$\overline{T}_1 = \operatorname{ch}\psi \qquad \text{and} \qquad \overline{T}_2 = e^{i\chi}\operatorname{sh}\psi$$

therefore (performing step (2') of Method I)

$$\chi = \varphi \qquad \text{and} \qquad \operatorname{th} 2\psi = |\nu| / \omega$$

hence

$$\overline{T}_1 = \frac{1}{\sqrt{2}}\, \frac{\left(\omega + \sqrt{\omega^2 - |\nu|^2}\right)^{1/2}}{\left(\omega^2 - |\nu|^2\right)^{1/4}}$$

and

$$\overline{T}_2 = \frac{e^{i\varphi}}{\sqrt{2}}\, \frac{\left(\omega - \sqrt{\omega^2 - |\nu|^2}\right)^{1/2}}{\left(\omega^2 - |\nu|^2\right)^{1/4}}$$

We may then verify that $Q_H = \sqrt{\omega^2 - |\nu|^2}$, as it is natural. In this case, obviously $T = \overline{T}$.

For N = 2 we shall only find the eigenvalues. The most general situation is given by

$$\omega \equiv \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega^*_{12} & \omega_{22} \end{pmatrix} \quad \text{and} \quad \nu \equiv \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{12} & \nu_{22} \end{pmatrix}$$

with ω_{11} , ω_{22} being real numbers and the rest being complex. The secular equation is given by

$$\begin{vmatrix} \omega_{11}-\lambda & \omega_{12} & -\nu_{11} & -\nu_{12} \\ \omega^*_{12} & \omega_{22}-\lambda & -\nu_{12} & -\nu_{22} \\ \nu^*_{11} & \nu^*_{12} & -\omega_{11}-\lambda & -\omega^*_{12} \\ \nu^*_{12} & \nu^*_{22} & -\omega_{12} & -\omega_{22}-\lambda \end{vmatrix} = \lambda^4 - c_1\lambda^2 + c_2 = 0$$

where

$$c_1 \equiv \omega_{11}^2 + \omega_{22}^2 + 2|\omega_{12}|^2 - |\nu_{11}|^2 - |\nu_{22}|^2 - 2|\nu_{12}|^2$$

and

$$\begin{aligned} c_2 \equiv\ & \omega_{11}^2\omega_{22}^2 + |\omega_{12}|^4 - 2\omega_{11}\omega_{22}|\omega_{12}|^2 - \omega_{11}^2|\nu_{22}|^2 - \omega_{22}^2|\nu_{11}|^2 \\ & - 2\left(\omega_{11}\omega_{22} + |\omega_{12}|^2\right)|\nu_{12}|^2 + |\nu_{11}|^2|\nu_{22}|^2 + |\nu_{12}|^4 \\ & - 2\,\mathcal{R}\left(\omega_{12}^2\nu_{11}^*\nu_{22}\right) - 2\,\mathcal{R}\left(\nu_{11}\nu_{22}\,\nu_{12}^{*\,2}\right) \\ & + 4\,\omega_{11}\,\mathcal{R}\left(\omega_{12}\nu_{12}^*\nu_{22}\right) + 4\,\omega_{22}\,\mathcal{R}\left(\omega_{12}\nu_{12}\nu_{11}^*\right) \end{aligned}$$

therefore $$\Omega_{1,2} = +\left(\frac{C_1}{2} \pm \sqrt{\frac{C_1^2}{4} - C_2} \right)^{1/2}$$

We see that it must be

$$C_1 \geqslant 0 \qquad \text{and} \qquad C_1^2 \geqslant 4C_2 \geqslant 0 .$$

The particular case $\nu_{11} = \nu_{22} = 0$ and $\omega_{12} = \nu_{12} \in \mathbb{R}$ appears in Ref. [18]. On the other hand, if we assume that $\omega_{22} = \omega_{12} = \nu_{22} = \nu_{12} = 0$, we reobtain the case $N = 1$.

APPENDIX II

We shall treat here the case N = 2 in the particular case $\omega_{11} = \omega_{22} = 1$, $\nu_{11} = \nu_{22} = 0$ and $\omega_{12} = \nu_{12} \in \mathbb{R}$. The eigenvalues have already been obtained in Appendix I:

$$\Omega_{1,2} = \sqrt{1 \pm 2\,\omega_{12}}$$

therefore it must be $|\omega_{12}| < \frac{1}{2}$. We verify immediately that $F_2 = 0_2$, hence $\overline{\overline{T}} = 1_4$, therefore

$$Q_F = F_1 = \begin{pmatrix} 1 & 2\omega_{12} \\ 2\omega_{12} & 1 \end{pmatrix} \quad \text{and} \quad H' = H .$$

Now we enter into step (2) of Method I. The equations to determine T are

$$\left[\left(HJ - \sqrt{1+2\omega_{12}}\ 1_4\right)\vec{T}_1\right]_k = 0 \qquad (k = 1,2,3)$$

$$(t_1^1)^2 + (t_1^2)^2 - (t_1^3)^2 - (t_1^4)^2 = 1$$

$$\left[\left(HJ - \sqrt{1-2\omega_{12}}\ 1_4\right)\vec{T}_2\right]_k = 0 \qquad (k = 1,2,3)$$

$$(t_2^1)^2 + (t_2^2)^2 - (t_2^3)^2 - (t_2^4)^2 = 1$$

The solution (attained through very boring calculations!) is given by:

$$T_1 = \begin{pmatrix} t_1^1 & t_2^1 \\ t_1^1 & -t_2^1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} t_1^3 & t_2^3 \\ t_1^3 & -t_2^3 \end{pmatrix}$$

with

$$t_1^1 = \frac{|\omega_{12}|}{D_1} \quad ; \quad t_1^3 = \frac{\omega_{12}}{|\omega_{12}|} \, \frac{1 + \omega_{12} - \sqrt{1 + 2\omega_{12}}}{D_1}$$

$$t_2^1 = \frac{|\omega_{12}|}{D_2} \quad ; \quad t_2^3 = \frac{\omega_{12}}{|\omega_{12}|} \, \frac{-1 + \omega_{12} + \sqrt{1 - 2\omega_{12}}}{D_2}$$

$$D_1 \equiv \sqrt{2} \left[\omega_{12}^2 - \left(1 + \omega_{12} - \sqrt{1 + 2\omega_{12}} \right)^2 \right]^{1/2}$$

$$D_2 \equiv \sqrt{2} \left[\omega_{12}^2 - \left(1 - \omega_{12} - \sqrt{1 - 2\omega_{12}} \right)^2 \right]^{1/2}$$

APPENDIX III

We shall treat here, by Method III, the case N = 3 for ω and ν real matrix. The secular equation can be written as follows

$$\mu^3 - C_1\mu^2 + C_2\mu - C_3 = 0$$

where

$$C_1 = \omega_{11}^2 + \omega_{22}^2 + \omega_{33}^2 - \nu_{11}^2 - \nu_{22}^2 - \nu_{33}^2 + 2\left(\omega_{12}^2 + \omega_{13}^2 + \omega_{23}^2 - \nu_{12}^2 - \nu_{13}^2 - \nu_{23}^2\right)$$

$$\begin{aligned}
C_2 = {} & \left(\omega_{11}\omega_{22} + \nu_{11}\nu_{22} - \omega_{12}^2 - \nu_{12}^2\right)^2 - \left(\omega_{11}\nu_{22} + \omega_{22}\nu_{11} - 2\omega_{12}\nu_{12}\right)^2 \\
& + \left(\omega_{22}\omega_{33} + \nu_{22}\nu_{33} - \omega_{23}^2 - \nu_{23}^2\right)^2 - \left(\omega_{22}\nu_{33} + \omega_{33}\nu_{22} - 2\omega_{23}\nu_{23}\right)^2 \\
& + \left(\omega_{11}\omega_{33} + \nu_{11}\nu_{33} - \omega_{13}^2 - \nu_{13}^2\right)^2 - \left(\omega_{11}\nu_{33} + \omega_{33}\nu_{11} - 2\omega_{13}\nu_{13}\right)^2 \\
& + 2\Big[\left(\omega_{11}\omega_{23} + \nu_{11}\nu_{23} - \omega_{12}\omega_{13} - \nu_{12}\nu_{13}\right)^2 - \left(\omega_{11}\nu_{23} + \nu_{11}\omega_{23} - \omega_{12}\nu_{13} - \omega_{13}\nu_{12}\right)^2 \\
& + \left(\omega_{22}\omega_{13} + \nu_{22}\nu_{13} - \omega_{12}\omega_{23} - \nu_{12}\nu_{23}\right)^2 - \left(\omega_{22}\nu_{13} + \nu_{22}\omega_{13} - \omega_{12}\nu_{23} - \omega_{23}\nu_{12}\right)^2 \\
& + \left(\omega_{33}\omega_{12} + \nu_{33}\nu_{12} - \omega_{13}\omega_{23} - \nu_{13}\nu_{23}\right)^2 - \left(\omega_{33}\nu_{12} + \nu_{33}\omega_{12} - \omega_{13}\nu_{23} - \omega_{23}\nu_{13}\right)^2\Big]
\end{aligned}$$

$$C_3 = \begin{vmatrix} \omega_{11}-\nu_{11} & \omega_{12}-\nu_{12} & \omega_{13}-\nu_{13} \\ \omega_{12}-\nu_{12} & \omega_{22}-\nu_{22} & \omega_{23}-\nu_{23} \\ \omega_{13}-\nu_{13} & \omega_{23}-\nu_{23} & \omega_{33}-\nu_{33} \end{vmatrix} \begin{vmatrix} \omega_{11}+\nu_{11} & \omega_{12}+\nu_{12} & \omega_{13}+\nu_{13} \\ \omega_{12}+\nu_{12} & \omega_{22}+\nu_{22} & \omega_{23}+\nu_{23} \\ \omega_{13}+\nu_{13} & \omega_{23}+\nu_{23} & \omega_{33}+\nu_{33} \end{vmatrix}$$

The eigenvalues are given by

$$\Omega_j = +\sqrt{\mu_j} \qquad (j = 1, 2, 3).$$

If we take in the present secular equation the particular case $\omega_{33} = \nu_{33} = \omega_{13} = \omega_{23} = \nu_{12} = \nu_{23} = 0$, we verify easily the consistence with the secular equation obtained in Appendix I for N = 2.

APPENDIX IV

We want to calculate the determinant

$$\Delta \equiv \left| \begin{array}{c|c} \lambda 1_N & A \\ \hline B & \lambda 1_N \end{array} \right|$$

where

$$A \equiv \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{1N} & \cdots & a_{NN} \end{pmatrix} = A^T$$

$$B \equiv \begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{1N} & \cdots & b_{NN} \end{pmatrix} = B^T$$

with $\{a_{ij}\}$ and $\{b_{ij}\}$ being complex numbers. A long, but not complicate, inductive process leads to

$$\Delta = \sum_{m=0}^{N} (-1)^m C_m \lambda^{2(N-m)}$$

where

$$C_0 = 1$$

$$C_1 = \sum_{i=1}^{N} a_{ii} b_{ii} + 2 \sum_{i<j} a_{ij} b_{ij}$$

$$C_m = \sum_{\{\text{all minors}\}} \left(\alpha^{(m)} \beta^{(m)} \right) \qquad \left(\left[\frac{N!}{m!\,(N-m)!} \right]^2 \text{terms} \right)$$

$$C_N = |A|\,|B|$$

$\alpha^{(n)}$ $\equiv$ determinant of a $n \times n$ minor of matrix A, constructed without touching the positions of the elements a_{ij}.

$\beta^{(n)}$ $\equiv$ determinant of a $n \times n$ minor of matrix B, which is obtained by making $a_{ij} \to b_{ij}$ in $\alpha^{(n)}$.

In order to clarify the use of this method, we present here the results for N = 2 and N = 3:

$$\begin{vmatrix} \lambda & 0 & a_{11} & a_{12} \\ 0 & \lambda & a_{12} & a_{22} \\ b_{11} & b_{12} & \lambda & 0 \\ b_{12} & b_{22} & 0 & \lambda \end{vmatrix} =$$

$$= \lambda^4 - (a_{11}b_{11} + a_{22}b_{22} + 2a_{12}b_{12})\lambda^2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & \lambda & 0 & a_{12} & a_{22} & a_{23} \\ 0 & 0 & \lambda & a_{13} & a_{23} & a_{33} \\ b_{11} & b_{12} & b_{13} & \lambda & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 & \lambda & 0 \\ b_{13} & b_{23} & b_{33} & 0 & 0 & \lambda \end{vmatrix} =$$

$$= \lambda^6 - \lambda^4(a_{11}b_{11} + a_{22}b_{22} + a_{33}b_{33} + 2a_{12}b_{12} + 2a_{13}b_{13} + 2a_{23}b_{23})$$

$$+\lambda^2\left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33}\end{vmatrix}\begin{vmatrix} b_{22} & b_{23} \\ b_{23} & b_{33}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33}\end{vmatrix}\begin{vmatrix} b_{11} & b_{13} \\ b_{13} & b_{33}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{vmatrix}\right.$$

$$\left.+2\begin{vmatrix} a_{12} & a_{23} \\ a_{13} & a_{33}\end{vmatrix}\begin{vmatrix} b_{12} & b_{23} \\ b_{13} & b_{33}\end{vmatrix} + 2\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23}\end{vmatrix}\begin{vmatrix} b_{12} & b_{22} \\ b_{13} & b_{23}\end{vmatrix} + 2\begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{23}\end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{13} & b_{23}\end{vmatrix}\right\}$$

$$-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{vmatrix}\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33}\end{vmatrix}$$

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