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# PHASE SPACE CALCULATIONS

by

G. E. A. Fialho

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

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# PHASE SPACE CALCULATIONS*†

G. E. A. Fialho‡
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro

A new approach is used for the calculation of volumes in phase space, using the saddle point approximation to solve the integrals. Thermodynamic concepts of temperature, free energy, and entropy are used. Results are compared with the nonrelativistic, extreme relativistic, and Fermi approximations. Applications are made to several systems expected (by the Gell-Mann and Pais scheme) to result from collisions usually produced in laboratory. A rough empirical method of calculation is also given.

## I. INTRODUCTION

In processes involving elementary-particle reactions, where the matrix elements are unknown, some information can nevertheless be obtained by using the statistical model introduced by Fermi [1]. For this model, the values of the phase space volume are needed. Up to now, several approaches have been used for these calculations. Fermi[1] gives the volume in the non-relativistic approximation without or with momentum conservation (this last case is referred to here as the

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as the N.R. approximation); and, for processes in which nucleons and pions result, he treats the pions as extreme relativistic and the nucleons as nonrelativistic, and considers that only the nucleons satisfy the condition of momentum conservation (this approach is referred to here as F approximation). Lepore and Stuart [2] do the calculations for the extreme relativistic case with momentum conservation [3] (referred to here as E.R. approximation). Christian and Yang[4] make calculations of momentum distribution of pions for multiple meson production, where the phase space volume is calculated by numerical integration.

In this paper a different approach is used. The particles are treated relativistically and the momentum conservation is shared by all particles. The approximation method used is to solve the integrals by the saddle point method. This corresponds to Fowler's statistical mechanics approach and is roughly equivalent to Stirling's approximation. In this way, thermodynamical quantities such as temperature, entropy and free energy will be defined for the system. The results improve as the number of particles increases. Comparison of this calculation with the two extremes of energy approximation (N.R. and E.R.) and with all values of energies when only two particles come out (this case can be solved exactly), permits the introduction of a semi-empirical correction for small number of particles resulting.

This paper is designed to facilitate these phase space calculations, and, therefore, the formulas are prepared for numerical evaluations. Thus, decimal logarithms (log) are used throughout, except in the derivations of the formulas in Sec. II where natural logarithms ($\ell n$) are used. Several tables and numerical examples are given.

In Sec. VII the phase-space volumes are given as a function of the total kinetic energy of several systems of particles. Sections V through X are self-sufficient, in order that readers, not interested in mathematical derivations, can go directly to these sections. Everywhere the pion mass is taken as the unit of mass and $c = 1$.

The related problem of momentum distribution for one of the particles coming out is treated in Sec. VI.

## II. DERIVATION OF FORMULAS - A

If angular momentum conservation is disregarded the momentum space volume per unit energy range, in the c.m. system, is

$$dQ_N(W,0)/dW = (2\pi)^{-4} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d\lambda e^{i\lambda W} \int_{-\infty}^{+\infty} d\sigma \, \Pi_j \int_{-\infty}^{+\infty} dp_j \, \exp\left\{-i\sigma \cdot p_j - i\lambda W_j\right\} \tag{1}$$

where $\epsilon$ is a small positive quantity.

The last integral can be evaluated,[2] giving:

$$I_j = \frac{2\pi^2 m_j^2 \lambda}{\lambda^2 - \sigma^2} \left\{ H_o^{(1)}\left[-m_j(\lambda^2-\sigma^2)^{\frac{1}{2}}\right] + \frac{2H_1^{(1)}\left[m_j(\lambda^2-\sigma^2)^{\frac{1}{2}}\right]}{m_j(\lambda^2-\sigma^2)^{\frac{1}{2}}} \right\} \tag{2}$$

where $c = 1$ and the H's are Hankel functions.

The $\sigma$ integration in (1),

$$\int_{-\infty}^{+\infty} d\sigma \, \Pi_j I_j = 2\pi \int_{-\infty}^{+\infty} \sigma^2 d\sigma \exp\left\{ \left( \sum_j \ln I_j \right) \right\}$$

can then be performed by the saddle point method, which is given by the condition:

$$\sum_j \partial \ln I_j / \partial \sigma = 0.$$

The logarithmic derivative of $I_j$ is proportional to $\sigma$, so this condition is satisfied by $\sigma = 0$. The $\sigma$ integration becomes, after evaluation:

$$(2\pi)^{3/2} \exp\left\{ \sum_j \ln I_j - \tfrac{3}{2} \ln \sum_j \left( -\partial^2 \ln I_j / \partial \sigma^2 \right) \right\}_{\sigma = 0}$$

If this last expression is substituted in the $\lambda$ integral in (1), it finally becomes, after the substitution $i\lambda = \beta$:

$$dQ_N(W,0)/dW = -(2\pi)^{-5/2} \, i \int d\beta \, \exp\left\{ \beta W - \beta F \right\}, \tag{3}$$

where the function F is defined by

$$-\beta F = \left\{ \sum_j \ln I_j - \tfrac{3}{2} \ln \sum_j \left( -\partial^2 \ln I_j / \partial \sigma^2 \right) \right\}_{\substack{\sigma = 0 \\ i\lambda = \beta}} \tag{4}$$

The $\beta$ integration in (3) can likewise be evaluated by saddle point, given by the condition $\partial(\beta W - \beta F)/\partial \beta = 0$, or

$$W = \partial(\beta F)/\partial \beta = F + \beta \partial F/\partial \beta. \tag{5}$$

This relation gives the total energy W as a function of the variables F and $\beta$ and permits the identification of $\beta$ as the inverse temperature 1/kT, with k the Boltzmann constant and F the free energy. The quantity $k\beta(W - F) = S$ is then the entropy.

The evaluation of (3) gives:

$$dQ_N(W,0)/dW = (2\pi)^{-2} e^{S/k} \left( \partial^2 S / k \partial \beta^2 \right)^{-\frac{1}{2}}. \tag{6}$$

To evaluate this expression the following functions are defined:

$$\alpha'(m_j\beta) = m_j\beta + 3\,\ell n\,\beta + [\ell n I_j]_{\substack{\sigma=0\\ i\lambda=\beta}} - \frac{3}{2}\,\ell n(1 + m_j\beta), \qquad (7)$$

$$b'(m_j\beta) = -\beta^2[\partial^2\,\ell n I_j/\partial\,\sigma^2]_{\substack{\sigma=0\\ i\lambda=\beta}} - (1 + m_j\beta), \qquad (8)$$

where the prime is here used to denote the represented quantity multiplied by the natural logarithm of 10: $\alpha' = \alpha\,\ell n 10$, for example, and $D'' = D(\ell\,n10)^2$. This device facilitates the use of natural logarithms in this section, but expresses the final results with decimal logarithms and exponentials of 10, which are more adequate for numerical computation. All the tabulated quantities are those without primes, being simply related to the quantities used in this section.

Substituting (2) in (7) and (8), these become:

$$\alpha'(m_j\beta) = \ell n\left\{2\pi^2(m_j\beta)^2[iH_0^{(1)}(im_j\beta) - 2H_1^{(1)}(im_j\beta)/m_j\beta\right\} + m_j\beta - \frac{3}{2}\,\ell n(1+m_j\beta), \qquad (9)$$

$$b'(m_j\beta) = 3 - m_j\beta + m_j\beta / \left\{\frac{-H_0^{(1)}(im_j\beta)}{H_1^{(1)}(im_j\beta)} + \frac{2}{m_j\beta}\right\}, \qquad (10)$$

whose numerical values are given in column 1 of Table I and the $\pi$ column of Table II. These functions, as they are bounded and have little variation, are very adequate for tabulation purposes. Results for $\alpha(m_j\beta)$ and $b(m_j\beta)$ with other values of $m_j$ can then be obtained by interpolation. Those of $b(m_j\beta)$, corresponding to the particle masses given in Table V, are tabulated in the other columns of Table II.

Then the new functions:

$$A'(m_j\beta) = \alpha'(m_j\beta) + \frac{3}{2}\ln(1+m_j\beta), \qquad (11)$$

$$B'(m_j\beta) = b'(m_j\beta) + 1 + m_j\beta, \qquad (12)$$

are defined and evaluated, the results being given in Tables IV and VI for the related functions A and B.

Also the bounded function

$$d''(X) = \frac{3}{2}\left\{\left[b'(X) + 1\right]^2 + X\left[2b'(X) - 3\right]\right\} \qquad (13)$$

is tabulated in Table I, column 2 to facilitate the calculation of the function

$$D''(m_j\beta) = d''(m_j\beta) + 15\, m_j\beta/2, \qquad (14)$$

which is tabulated in Table III for the set of masses given in Table V.

When (9) and (12) are substituted in (4), the latter becomes:

$$-\beta F = \sum_j \left\{\alpha'(m_j\beta) - 3\ln(m_j\beta) - m_j\beta + \frac{3}{2}\ln(1 + m_j\beta)\right\}$$

$$+ 3\ln\beta - \frac{3}{2}\ln\left\{\sum_j B'(m_j\beta)\right\} + 3\sum_j \ln m_j,$$

which, substituted in (5), gives for the kinetic energy:

$$\beta E = \beta(W - \sum_j m_j) = \sum_j b'(m_j\beta) - 9 +$$

$$+ \sum_j D''(m_j\beta) / \sum_j B'(m_j\beta) \quad ,$$

and the expression (24) follows.

The entropy will then be given by:

$$S/k = \beta E + \sum_j A'(m_j\beta) - \frac{3}{2}\ln\sum_j B'(m_j\beta) - 3(N-1)\ln\beta,$$

and with the definition (25) of the function $\chi$, (5) becomes:

$dQ_N(W,0)/dW =$

$$= (2\pi)^{-2}(\log e)^{\frac{3}{2}} \beta^{-3(N-1)} 10^{\chi} (\partial^2 S/k\, \partial\beta^2)^{-\frac{1}{2}} \quad (15)$$

The last term of this expression can be calculated by observing, from the definition of S, that

$$\partial^2 S/k\,\partial\beta^2 = -\partial E/\partial\beta$$

The exact calculation of $\partial E/\partial\beta$ from (1) will give quite an elaborate expression for numerical evaluation, but this can be approximated if the quantity $\xi$ is defined by

$$\tilde{\beta} = \xi\,\beta, \quad \text{and} \quad \beta E = (N-1)b'(\tilde{\beta}). \quad (16)$$

The functions $\tilde{\beta}E$ and $(N-1)b'(\tilde{\beta})$ have curves of similar shape and the above relation (16) adjusts the parameter $\xi$ so that the curves intersect at the point of interest. If as an approximation, we consider that they have the same tangent at this point, or, what is the same, that the parameter $\xi$ has a slow variation, then

$$\beta^2(\partial/\partial\beta)E \approx (N-1)\tilde{\beta}^2(\partial/\partial\tilde{\beta})\left[b'(\tilde{\beta})/\tilde{\beta}\right], \quad (17)$$

(a function of $\tilde{\beta}$ ) is, on using (16), the inverse of the function $b(x)$ of the argument $\beta E \log e/(N-1)$.

If we define the functions

$$\phi(N,\beta) = \log\left\{(\log e)^{3/2}\beta^{-3N+4}(N-1)^{-\frac{1}{2}}\right\},$$

(given in Table VII) and

$\psi[\beta E \log e/(N-1)]$

$$= \frac{1}{2}\log\left\{-\tilde{\beta}^2(\partial/\partial\tilde{\beta})\left[b'(\tilde{\beta})/\tilde{\beta}\right]\right\} + \log 4\pi^2$$

(given in Table VIII), then the logarithm of expression (15) becomes the expression (26) without the last term.

The small variation of the function $\psi$ gives us confidence in the above approximation, as both sides of (17) have the same limits for $\beta \to 0$ or $\beta \to \infty$.

## III. CORRECTION OF THE SADDLE POINT APPROXIMATION

The saddle point approximation method gives good results for large N. For small N some discrepancy is to be expected. To check the error, it is possible to compare the saddle point approximation for all N with the N.R. and E.R. approximations and, in the critical case N = 2, to compare the saddle point approximation with the exact calculation.

N.R. case - To get this approximation, the Hankel functions must be replaced by their asymptotic expansions [5] when $\beta \to \infty$, and the formula worked out. The relation between temperature and energy becomes $\beta E \sim \frac{3}{2}(N-1)$ as expected, and the volume in momentum space is

$$\frac{dQ}{dW} = (2\pi)^{3(N-1)/2} \left[\frac{\prod_1^N m_i}{\sum_1^N m_i}\right]^{3/2} \frac{E^{3N/2-5/2}}{(3N/2-5/2)!} C_{NR}$$

which agrees with the direct calculation on the N.R. limit except for the correction factor:

$$C_{NR} = \left(\frac{3N-3}{3N-5}\right)^{-3N/2+2} \frac{\Gamma(3N/2-3/2)}{\Gamma_s(3N/2-3/2)} e ,$$

where $\Gamma_s(N+1)$ is Stirling's approximation $(n/e)^n(2\pi n)^{\frac{1}{2}}$ for the gamma function $\Gamma(n+1)$. The correction $C_{N.R.}$ approaches one for large N and for small N has the values:

| N | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $C_{N.R.}$ | 0.957 | 1.014 | 1.013 | 1.011 | 1.009 | 1.008 |

These are quite good approximations, even for small N.

E.R. case - In this approximation ($\beta \to 0$), the Hankel functions are substituted for by means of the expressions [6]:

$$i\pi H_0^{(1)}(2Xi) = \sum_{\nu=0}^{\infty} \frac{2\psi(\nu) - 2\ln X}{(\nu!)^2} X^{2\nu},$$

and

$$-\pi H_1^{(1)}(2Xi) = \frac{1}{X} - X\sum_{\nu=0}^{\infty} \frac{\psi(\nu) + \psi(\nu+1) - 2\ln X}{\nu!(\nu+1)!} X^{2\nu},$$

where

$$\psi(0) = -C,$$
$$\psi(\nu) = -C + 1 + \frac{1}{2} \dots + \frac{1}{\nu},$$

and C = 0.577 is Euler's or Mascheroni's constant.

The temperature-energy relation becomes $\beta E \approx 3(N-1)$, and the volume in momentum space is

$$\frac{dQ}{dW} = \left(\frac{\pi}{2}\right)^{N-1} \frac{(4N-4)!E^{3N-4}}{(2N-1)!(2N-2)!(3N-4)} C_{E.R.}, \qquad (19)$$

which agrees with the direct calculation on the E.R. limit, except for the correction factor:

$$C_{E.R.} = \left(1 - \frac{1}{N}\right)^{3/2} \left(\frac{3N-4}{3N-3}\right)^{3N-7/2} \left(\frac{2N-1}{2N-2}\right)^{2N-\frac{1}{2}}$$
$$\times \frac{\Gamma_s(4N-3)}{\Gamma(4N-3)} \frac{\Gamma(2N)}{\Gamma_s(2N)} \frac{\Gamma(2N-1)}{\Gamma_s(2N-1)} \frac{\Gamma(3N-3)}{\Gamma_s(3N-3)} .$$

This correction approaches 1 for large N, but for small N it departs too much from 1 to be disregarded, being 0.580 for N = 2.

The function

$$\Xi(N) = \log C_{E.R.}(N)/\log C_{E.R.}(N=2)$$

is given in Table IX.

N = 2 case - In this case the formula can be analytically integrated, giving:

$$dQ/dW = (\pi/2)\left[(W^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2\right]^{\frac{1}{2}} \left\{1 - \left[(m_1^2 - m_2^2)/W^2\right]^2\right\} . \tag{20}$$

The exact calculation has been made for processes in which the particles $2\pi$, $N\pi$, $\Lambda K$, $\Sigma K$ and 2N come out. The differences between $\log(dQ/dW) = L$ calculated by this process and L calculated by the saddle point method are plotted in Fig. 7 as a function of the relation between the kinetic energy and the sum of the masses $E/\sum m$. These differences fall pretty well on the same curve $\eta(E/\sum m)$. If we assume that for other values of N the corrections have the same functional dependence on $E/\sum m$, then with (20) these corrections are given by $\eta(E/\sum m)\Xi(N)$, which is the last term of expression (26).

## IV. DERIVATION OF FORMULAS - B

If the total energy-momentum is $W'$, $\vec{P}' \neq 0$, the volume in momentum space per unit energy range is

$$dQ_N(W',\vec{P}')/dW' = \int \delta(\vec{P}' - \textstyle\sum_j \vec{p}_j{}')\, \delta(\sum_j W_j' - W') \prod_{j=1}^{N} d\vec{p}_j{}' . \tag{21}$$

Under a Lorentz transormation in the X-direction:

$$\alpha_{11} = \alpha_{44} = \gamma; \quad \alpha_{14} = -\alpha_{41} = iv\gamma ;$$
$$\gamma^{-2} = 1 - v^2 = W^2/W'^2 = (W'^2 - p'^2)/W'^2 ,$$

the product $\delta(\vec{P}' - \sum_j \vec{p}_j{}')\delta(\sum_j W_j' - W')$ and the ratio $d\vec{p}_j/W_j$ are invariants, so (21) becomes

$$dQ_N(W',\vec{P}')dW' = \int \delta(\textstyle\sum_j \vec{p}_j)\delta(\sum_j W_j - W) \prod_{j=1}^{N} \frac{W_j'}{W_j} d\vec{p}_j \,.$$

This, with the replacement of the δ functions by their Fourier transforms, and with

$$W_j' = \gamma(W - vp_{jx}) \,,$$

becomes

$$dQ_N(W',\vec{P}')/dW' = (2\pi)^{-4}\int d\lambda\, e^{i\lambda W}\int d\vec{\sigma} \times \prod_{j=1}^{N}\int \gamma\left(1 - \frac{vp_{jx}}{W_j}\right) d\vec{p}_j \exp\left\{-i\vec{\sigma}\vec{p} - i\lambda W_j\right\}, \tag{22}$$

or

$$dQ_N(W',\vec{P}')/dW' = \gamma^N(2\pi)^{-4}\int d\lambda\, e^{i\lambda W}\int d\vec{\sigma} \times \left(1 - v\sum_i \frac{J_i}{I_i} + v^2\sum_{i,k}\frac{J_iJ_k}{I_iI_k} + \dots\right)\prod_{j=1}^{N} I_j, \tag{23}$$

where $I_i$ is defined in (2) and

$$J_i = \int \frac{p_{ix}}{W_i}\exp\left\{-\vec{\sigma}\cdot\vec{p}_i - i\lambda W_i\right\} d\vec{p}_i \,.$$

In the $\vec{\sigma}$ integration in (23), the terms depending on $J_i$ can be taken out of the integral and evaluated at the saddle point, giving zero results. The only remaining term is one independent of $J_i$, and expression (31) results.

To obtain the correction to be applied to expression (31), the exact integrals of the terms depending on $J_i$ must be evaluated. It is then readily seen that the terms with an odd number of J's are zero, and the result is a sequence in even powers of v.

In the worst case of this saddle point approximation, when N = 2, the correction for (31) can be analytically integrated, giving [7]

$$dQ_2(W',\vec{P}')/dW' = C(W'/W)^2 dQ_2(W,0)/dW,$$

where

$$C = 1 - \frac{1}{3}\left(\frac{v}{c}\right)^2 \frac{1-2(m_1^2+m_2^2)/W^2+(m_1^2-m_2^2)/W^4}{1-(m_1^2-m_2^2)^2/W^4}$$

This result shows that (31) still is a good approximation.

## V. THERMODYNAMIC DISTRIBUTION

In a system of N particles of masses $m_1 \ldots m_j \ldots m_N$, with a total kinetic energy E, it is possible to define an equilibrium temperature $\beta = 1/kT$ (k = Boltzmann constant) given by:

$$\beta E \log e = \sum_j b(m,\beta) + \sum_j D(m,\beta) / \sum_j B(m_j\beta) - 3.909, \tag{24}$$

where $b(m_j\beta)$, $D(m_j\beta)$ and $B(m_j\beta)$ are functions given in Tables II, III and IV as functions of $\beta$, with masses corresponding to several particles as given in Table V.[8] Then the auxiliary function

$$X = \beta E \log e + \sum_j A(m_j\beta) - (3/2) \log \sum_j B(m_j\beta) \tag{25}$$

can be calculated, with $A(m_j\beta)$ as given in Table VI.

The volume of the momentum space per unit energy range, when the total momentum is zero and the total energy is W, is given by:

$$L = \log\left[dQ_N(W,0)/dW\right] = \phi(N,\beta) - \psi\left[\beta E \log e/(N-1)\right] + \\ + X + \eta(E/\textstyle\sum_j m_j)\,\Xi(N), \tag{26}$$

where the functions $\phi$, $\psi$, $\Xi$, and $\eta$ are given respectively in Tables VII, VIII, IX and X.

## VI. NUMERICAL EXAMPLE

Consider the case where two nucleons and one pion come out. Relation (24) gives $\beta$ as an implicit function of E. It is then preferable to start with a value of $\beta$ and get the corresponding E. The choice of a convenient $\beta$ can be facilitated by inspection of Figs. 1, 2, and 3 where graphs of $\beta E/(N-1)$ as a function of $E/(N-1)$ are given for several sets of outgoing particles.

For $\beta = 0.2$ we have:

| Table | Function | Nucleon | Pion | |
|---|---|---|---|---|
| II | $b =$ | 0.973 | 1.224 | $\sum b = 3.170$ |
| III | $D =$ | 5.460 | 4.577 | $\sum D = 15.477$ |
| IV | $B =$ | 1.992 | 1.745 | $\sum B = 5.729$ |
| VI | $A =$ | 1.832 | 1.483 | $\sum A = 5.147$ |
| V | $m/m_x =$ | 6.73 | 1 | $\sum m/m_\pi = 14.46$ |

then formulas (24) and (25) yield $\beta E \log e = 1.963$ and $\chi = 5.973$. From Tables VII to X we find:

| Arguments | Table | |
|---|---|---|
| $N = 3,\ \beta = 0.2$ | VII | $\phi = 2.801$ |
| $\beta E \log e/(N-1) = 0.981$ | VIII | $\psi = 1.808$ |
| $E/\sum m = 1.56$ | X | $\eta = 0.233$ |
| $N = 3$ | IX | $\Xi = 0.624$ |

Therefore, finally, with formula (26), we obtain $L = 7.111$ or $dQ/dW = 1.29 \times 10^7$.

## VII. APPLICATIONS

In the processes of collision of two particles, the $\pi^-p$, $\pi^-n$, $\pi^+p$, $\pi^+p$, pp, and pn collisions are easily obtained in the laboratory. Momentum space calculations for all of the sets of two or three particles resulting from such collisions and compatible with the Gell-Mann and Pais [9] scheme of selection rules are presented in this paper. Calculations were extended for all compatible sets of four particles resulting from the pn collision, as these are useful in reference to the antiproton production on the Berkeley Bevatron.[10] Also calculations were made on the 2, 3, 4, and 5-pion systems because of their interest as reference curves (since the pion mass is here taken as unity) or for eventual calculations on annihilation processes.

The relations between temperature and energy as calculated by (24) are given in Figs. 1, 2, and 3 because they are useful as an orientation for the choice of a temperature corresponding to a convenient energy range in any new process.

The logarithms (decimal) L of the momentum space volumes are given in Table XI, XII as a correction to the N.R. and E.R. approximations, because this gives a better tabulation than the volume itself, which is an increasing monotonic function of the energy. The N.R. and E.R. approximations are given by formulas (27) and (28), and the quantities s, t, q, and r for the different systems are given in Tables XI and XII. The results for two-particle systems are included, although these are calculated with the exact formula (19), because they are useful as references. Figure 4 gives L for the system of two nucleons and one pion; the N.R. E.R., and F approximations are there represented by straight lines.

## VIII. ESTIMATION OF MOMENTUM SPACE VOLUMES

The logarithms of momentum space volumes in the N.R. and E.R. approximations are:

$$\text{N.R.} \quad L_{N.R.} = s + t \log E \tag{27}$$

$$\text{E.R.} \quad L_{E.R.} = q + r \log E, \tag{28}$$

where $s = s' + \frac{3}{2}M$,

$$M = \Sigma_j \log m_j - \log(\Sigma_j m_j), \tag{29}$$

and $s'$, $t$, $q$, and $r$ are functions of the number of particles N. These functions of N are (when $m_\pi = c = 1$):

| | | | | |
|---|---|---|---|---|
| N = | 2 | 3 | 4 | 5 |
| s' = | 1.250 | 2.094 | 2.526 | 2.710 |
| t = | 1/2 | 2 | 7/2 | 5 |
| q = | 0.196 | -0.541 | -1.986 | -3.661 |
| r = | 2 | 5 | 8 | 11 |

For larger N, see formulas (18) and (19) with $C_{E.R.}=1$.

The energy of the crossing point between the two approximations is then readily found from the equations:

$$\begin{aligned} N = 2, \quad \log E_1 &= M + 0.703, \\ N = 3, \quad \log E_1 &= \tfrac{1}{2} M + 0.878, \\ N = 4, \quad \log E_1 &= \tfrac{1}{3} M + 0.983 \\ N = 5, \quad \log E_1 &= \tfrac{1}{4} M + 1.062 \end{aligned} \tag{30}$$

If, as in Tables XI and XII, $\Delta L$ denotes the difference between L calculated by saddle-point and by the N.R. and E.R. approximations, then the curves of $\Delta L/(N-1)$ vs $E/E_1$ do not much differ.

Figure 5 was obtained by taking a mean of all calculations in this paper. If this curve is used for any other calculation, then a value of $\Delta L$ and so of $L$ is readily obtained. This method of using a mean curve (Fig. 5) is inexact and the resulting calculation will be relatively crude. Nevertheless, this procedure represents an improvement over the N.R. and E.R. approximations.

As an illustration, let us examine again the example of Sec. VI. Now we start with the kinetic energy $E = 22.6$. For two nucleons and one pion, relation (29) gives $M = 0.496$, which, substituted in (30), gives $\log E_1 = 1.126$. Then in Fig. 5, with $\log(E/E_1) = 0.228$, we get $\Delta L = 0.842$, this difference being referred to the E.R. case. This approximation is $L_{E.R.} = 6.230$, giving (with the above value of $\Delta L$) 7.072 or $dQ/dW = 11.8\times10^6$. This value is to be compared with the saddle point result $12.9\times10^6$, the E.R. result $1.70\times10^6$, and the N.R. result $0.352\times10^6$. For the same energy the Fermi approximation gives $36.3\times10^6$.

## IX. MOMENTUM DISTRIBUTION

If the total energy-momentum is $W', \vec{P}' \neq 0$, the volume in momentum space $dQ_N(W',\vec{P}')/dW'$ is related to the volume with zero total momentum by a Lorentz transformation, which within the saddle point approximation, gives:

$$dQ_N(W',\vec{P}')/dW' = (W'/W)^N dQ_N(W,0)/dW, \qquad (31)$$

where

$$W = (W'^2 - \vec{P}'^2)^{\frac{1}{2}} . \qquad (32)$$

In the statistical model, Fermi [11] considers that the square of the matrix element is simply proportional to $(\Omega/V)^N$, where

V is the normalizing volume and $\Omega$ is the volume resulting from the contraction of the sphere of interaction $\Omega_o = 4\pi R^3/3$. The Lorentz contraction factor is $(2/W^t)$ times the mass of the nucleon, with $W^t$ the total energy and R the Compton wavelength of the pion ($\hbar/\mu c = 1.4 \times 10^{-13}$ cm). With this assumption, the cross section for one particle coming out with an energy $W^*$ and momentum $p^*$ in the range $dp^*$ will be proportional to [11]

$$S_N(W^t, p^*) = 4\pi(\Omega/h^3)p^{*2}(W^t - W^*)^{N-1}dQ_{N-1}(W,0)/dW, \tag{33}$$

where

$$W = \left[(W^t - W^*)^2 - (p^t - p^*)^2\right]^{\frac{1}{2}} \tag{34}$$

and $W^t$, $p^t$ are the total energy and momentum of the system. The total cross section is proportional to

$$S_N{}^t(W^t) = (\Omega/h^3)(W^t/W)^N dQ_N(W,0)/dW, \tag{35}$$

where W is given by (32) with $W^t$ substituted for W'.

## X. NUMERICAL EXAMPLE

As an example, let us compute the cross section in the center-of-mass (c.m.), system for a nucleon-nucleon collision at the c.m. energy $W^t = 19.835$ (2.2 Bev in the lab System), in which two nucleons and one pion of momentum $p^* = 2m_\pi c$ result. With $m_\pi = c = 1$, we have: $W^* = (p^{*2} + m_\pi{}^2c^2)^{\frac{1}{2}} = 2.236$, $W^t - W^* = 17.599$. Since $p^t = 0$, $p^t - p^* = -2$. Then, also, from (34), $W = 17.49$,

$$\log(\Omega/h^3) = \log(2m_N\Omega_0/h^3W^t) = 0.05918 - 2,$$

and, from Table XI, with $E = W - \Sigma_j m_j = 17.49 - 13.46 = 4.05$ for two nucleons: $\log E_1 = 1.229$, $\log(E/E_1) = -0.622$, $t = \frac{1}{2}$, $s = 2.040$, one gets $\Delta L = 0.173$, and $L = s + \frac{1}{2}\log E + sL = 2.157$. Finally, with $N = 3$, $\log \Omega/h^3 = 0.5918 - 2$, $p^* = 2$, $W = 17.49$, $W^t - W^* = 17.599$, and $L = \log[dQ_N(W,0)/dW] = 2.517$, formula (33) gives $\log S = 0.401 - 2$.

In this way the curves S vs $p^*/m_\pi C$ for the production of two nucleons and 1,2, or 3 pions were obtained. They are represented in Fig. 6, where each curve is normalized to unit area.

These curves agree, within the precision of graphing, with the ones obtained by Christian and Yang's [4] numerical calculations and serve as a check on the present approximation method.

The author wishes to thank his sponsor, Professor Robert Seber, for suggesting this problem and for his guidance and encouragement during the course of the work.

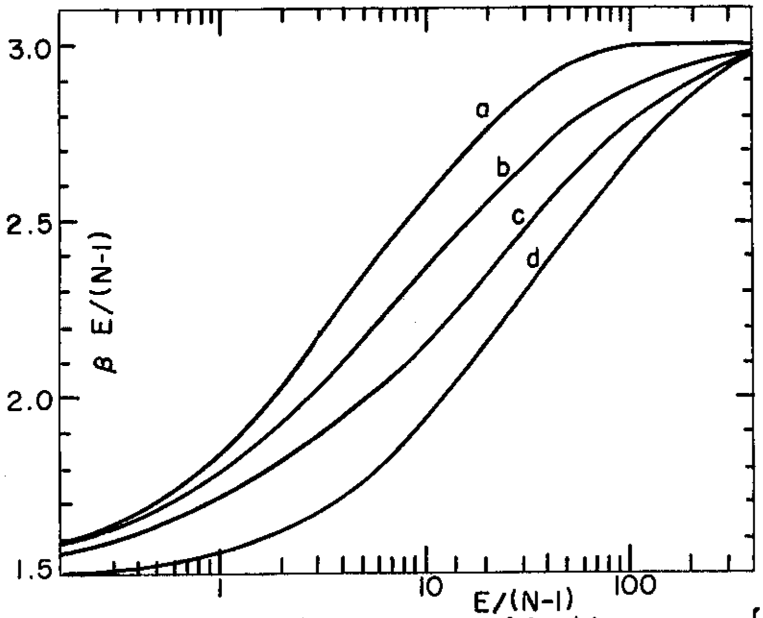


Fig. 1. Relation between mean kinetic energy [E/(N-1)] in $m_\pi c^2$ units and temperature (=1/kβ; k = Boltzmann constant) as a function of the mean kinetic energy, for systems consisting of: (a) 3 pions; (b) 1 nucleon and 2 pions; (c) 2 nucleons and 1 pion; (d) 3 nucleons.

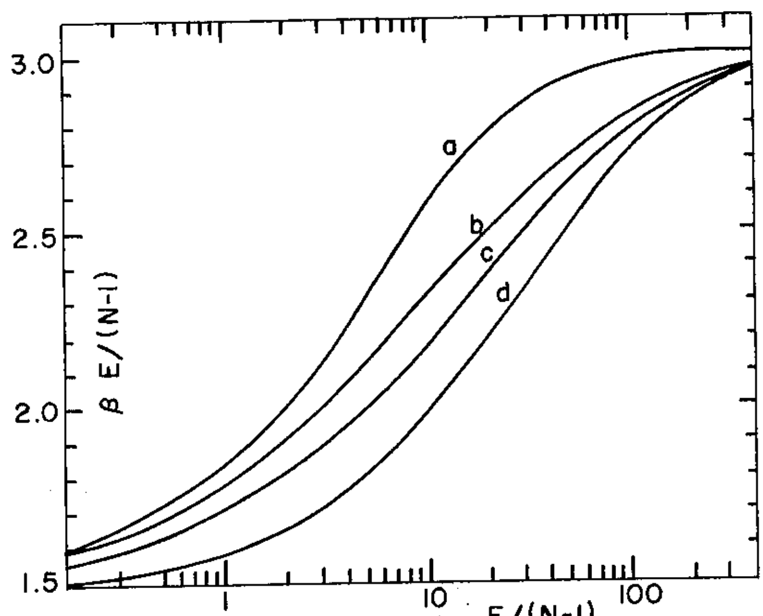


Fig. 2. Relation between mean kinetic energy [E/(N-1)] in $m_\pi c^2$ units and temperature (=1/kβ; k = Boltzmann constant) as a function of the mean kinetic energy, for systems consisting of: (a) 4 pions; (b) 2 nucleons and 2 pions; (c) 1 nucleon, 1 hyperon (Σ), 1 heavy meson (K), 1 pion; (d) 4 nucleons.

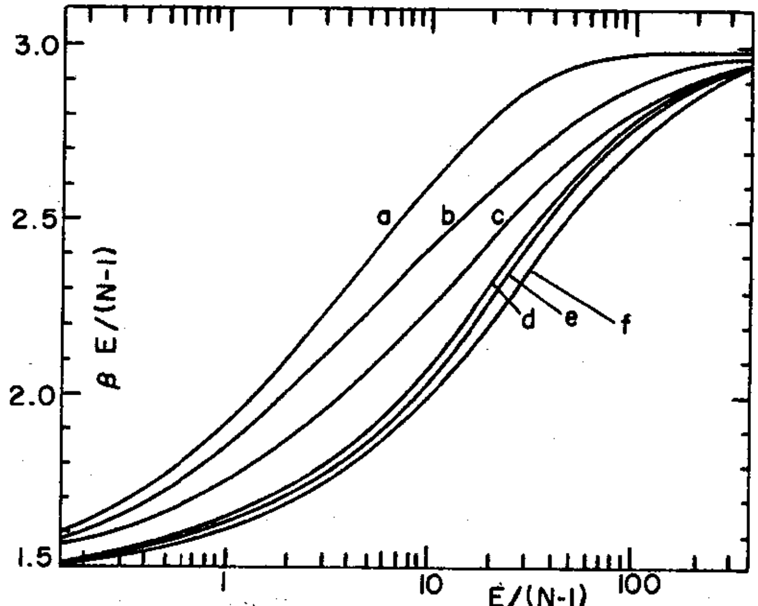


Fig. 3. Relation between mean kinetic energy [E/(N-1)] in $m_\pi c^2$ units and temperature (=1/kβ; k = Boltzmann constant) as a function of the mean kinetic energy, for systems consisting of: (a) 5 pions; (b) 2 nucleons and 3 pions; (c) 1 hyperon (Λ or Σ), 1 heavy meson (K), and one pion; (d) 1 nucleon and 2 heavy mesons (K); (e) 1 hyperon (Ω) and 2 heavy mesons (K); (f) 1 nucleon, 1 hyperon (Σ or Λ) and 1 heavy meson (K).

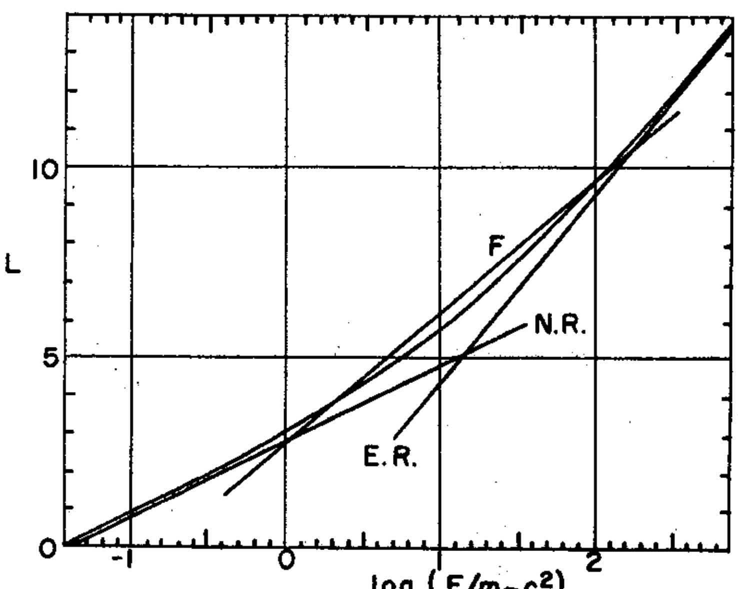


Fig. 4. Variation of L = log dQ/dW with log E for a system consisting of 2 nucleons and 1 pion. The straight lines represent: N.R., the nonrelativistic approximation; E.R., the extreme relativistic approximation; F, the Fermi approximation, where the nucleons and pions are treated respectively as N.R. and E.R. particles and only the nucleons share the momentum conservation.

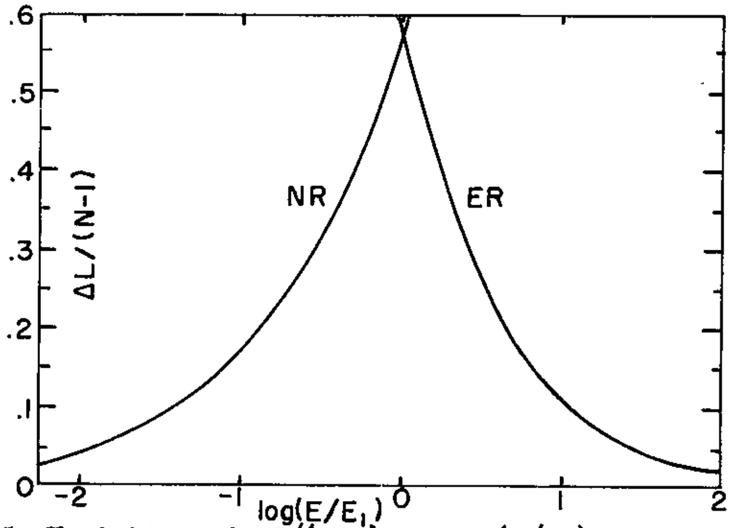


Fig. 5. Variation of $\Delta L/(N-1)$ vs $\log(E/E_i)$, where $\Delta L = L - L_{R.N.}$ for the curve marked N.R. and $\Delta L = L - L_{E.R.}$ for the curve marked E.R; $E_i$ is the total kinetic energy of the system for which $L_{N.R.} = L_{E.R.}$ $L = \log dQ/dW$; $L_{N.R.}$ and $L_{E.R.}$ are the N.R. and E.R. approximations of L. The values of $\Delta L$ in this figure are the average of all the values calculated in the present paper and summarized in Tables XI and XII.

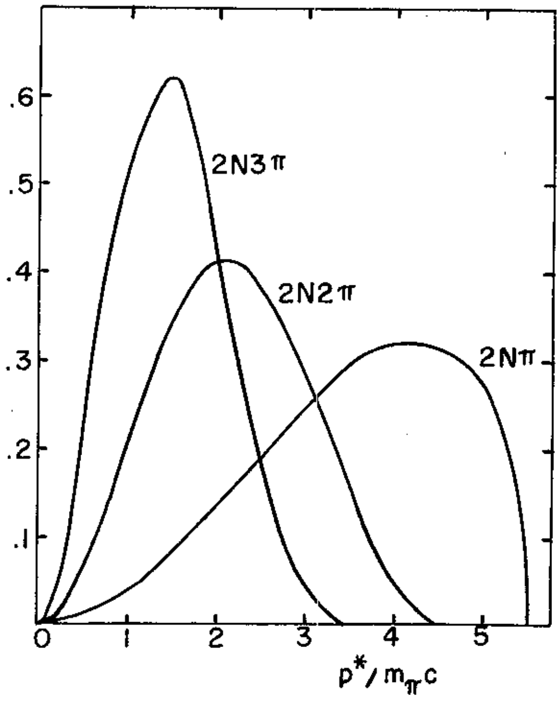


Fig. 6. Momentum distribution of mesons for nucleon-nucleon collision at 19.835 $m_\pi c^2$ in the c.m. system (2.2 Bev in Laboratory System). The curves have been normalized to unit area.

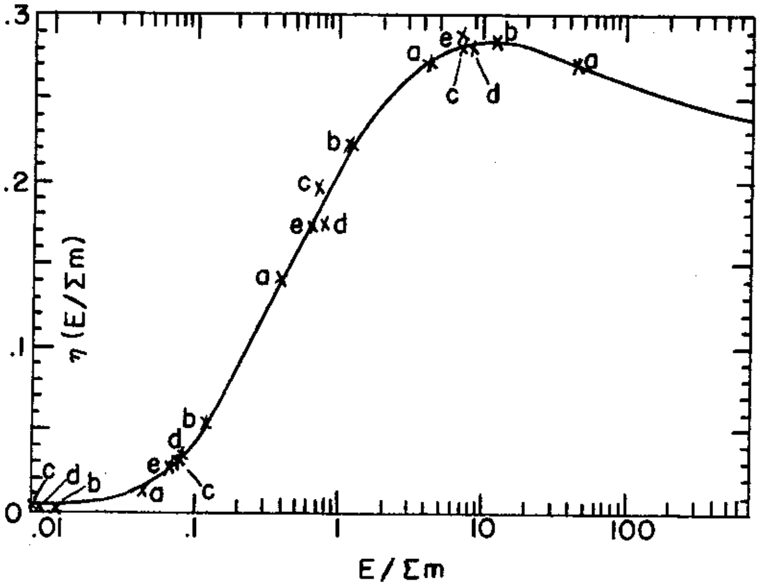


Fig. 7. Difference $\eta$ between $L = \log dQ/dW$ calculated by the exact formulas and calculated by saddle point approximation. E = kinetic energy of the system, $\Sigma m$ sum of particle masses ($m_\pi = c = 1$). The points are for the systems: (a) 2 pions; (b) 1 nucleon and 1 pion; (c) 1 hyperon ($\Sigma$) and 1 heavy meson (K); (d) 1 hyperon ($\Lambda$) and 1 heavy meson (K); (e) 2 nucleons.

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3 The author is indebted to Professor T. D. Lee for calling his attention to an error in formula (20) of reference 2. The factorial in the numerator is (4n-4)! instead of (4n-3)!.

4 R. Christian and C. N. Yang, unpublished Brookhaven Report, 1953.

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6 Jahnke-Ende, Tafeln Hoherer Funktionen (B. G. Teubner, Leipzig, 1952), p. 136, p. 19.

7 M. M. Block (unpublished).

8 H. A. Bethe and F. Hoffmann, Mesons and Fields (Row, Peterson and Corp., New York - 1955) 2nd Vol. p. 374.

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11 The statistical factors for the multiplicity of particles are not included here.

TABLE I

Values of the functions a(x) and d(x).

| x | a | d | x | a | d |
|---|---|---|---|---|---|
| 0 | 1.400 | 4.523 | 4 | 1.233 | 3.163 |
| .01 | 1.397 | 4.526 | 10 | 1.213 | 2.981 |
| .02 | 1.396 | 4.531 | 20 | 1.205 | 2.907 |
| .04 | 1.392 | 4.537 | 40 | 1.201 | 2.865 |
| .1 | 1.380 | 4.453 | 100 | 1.199 | 2.846 |
| .2 | 1.364 | 4.274 | 200 | 1.198 | 2.835 |
| .4 | 1.341 | 4.035 | 400 | 1.197 | 2.831 |
| 1 | 1.298 | 3.675 | 1000 | 1.197 | 2.829 |
| 2 | 1.259 | 3.401 | $\infty$ | 1.197 | 2.829 |

TABLE II

Values of the function b(β m) for masses of several particles

| β | Ω | Σ | Λ | N | K | π | μ |
|---|---|---|---|---|---|---|---|
| | particles | | | | | | |
| 0 | 1.303 | 1.303 | 1.303 | 1.303 | 1.303 | 1.303 | 1.303 |
| .01 | 1.274 | 1.278 | 1.281 | 1.288 | 1.300 | 1.302 | 1.303 |
| .02 | 1.231 | 1.238 | 1.244 | 1.256 | 1.286 | 1.302 | 1.302 |
| .04 | 1.163 | 1.174 | 1.181 | 1.199 | 1.253 | 1.302 | 1.302 |
| .1 | 1.033 | 1.050 | 1.061 | 1.090 | 1.172 | 1.273 | 1.284 |
| .2 | .919 | .934 | .947 | .973 | 1.081 | 1.224 | 1.249 |
| .4 | .814 | .828 | .838 | .864 | .965 | 1.161 | 1.182 |
| 1 | .726 | .733 | .738 | .752 | .825 | 1.029 | 1.072 |
| 2 | .691 | .695 | .698 | .707 | .748 | .913 | .955 |
| 4 | .671 | .674 | .677 | .681 | .705 | .812 | .847 |
| 10 | .658 | .659 | .661 | .662 | .673 | .725 | .743 |
| 20 | .656 | .656 | .656 | .657 | .662 | .690 | .701 |
| 40 | .654 | .654 | .654 | .654 | .657 | .671 | .678 |
| 100 | .652 | .653 | .653 | .653 | .655 | .660 | .661 |
| 200 | .651 | .651 | .651 | .652 | .653 | .655 | .656 |
| 400 | .651 | .651 | .651 | .651 | .652 | .654 | .654 |
| 1000 | .651 | .651 | .651 | .651 | .651 | .652 | .653 |
| ∞ | .651 | .651 | .651 | .651 | .651 | .651 | .651 |

TABLE III

Values of the function D(β m) for masses of several particles

| β | particles Ω | Σ | Λ | N | K | π | μ |
|---|---|---|---|---|---|---|---|
| 0 | 4.523 | 4.523 | 4.523 | 4.523 | 4.523 | 4.523 | 4.523 |
| .01 | 4.593 | 4.596 | 4.597 | 4.607 | 4.590 | 4.540 | 4.534 |
| .02 | 4.554 | 4.559 | 4.568 | 4.574 | 4.604 | 4.559 | 4.548 |
| .04 | 4.579 | 4.562 | 4.557 | 4.554 | 4.573 | 4.594 | 4.586 |
| .1 | 5.039 | 4.947 | 4.876 | 4.774 | 4.714 | 4.594 | 4.606 |
| .2 | 6.113 | 5.889 | 5.752 | 5.460 | 4.810 | 4.557 | 4.571 |
| .4 | 8.564 | 8.059 | 7.754 | 7.101 | 5.539 | 4.601 | 4.571 |
| 1 | 16.46 | 15.12 | 14.15 | 12.57 | 8.208 | 5.090 | 4.858 |
| 2 | 29.85 | 27.16 | 25.51 | 21.99 | 13.06 | 6.230 | 5.664 |
| 4 | 56.74 | 51.35 | 48.03 | 40.97 | 22.97 | 8.821 | 7.554 |
| 10 | 137.5 | 124.0 | 115.7 | 98.06 | 52.96 | 17.13 | 13.78 |
| 20 | 272.2 | 245.2 | 228.6 | 193.2 | 103.0 | 31.20 | 24.44 |
| 40 | 541.5 | 487.6 | 454.4 | 383.6 | 203.1 | 59.45 | 45.89 |
| 100 | 1349 | 1215 | 1132 | 954.8 | 503.6 | 144.3 | 110.4 |
| 200 | 2696 | 2427 | 2261 | 1907 | 1004 | 285.8 | 217.9 |
| 400 | 5390 | 4850 | 4518 | 3811 | 2006 | 568.7 | 432.9 |
| 1000 | 13470 | 12122 | 11291 | 9523 | 5010 | 1417 | 1078 |

TABLE IV

Values of the function B(β m) for masses of several particles

| β | Particles | | | | | | |
|---|---|---|---|---|---|---|---|
| | Ω | Σ | Λ | N | K | π | μ |
| 0 | 1.737 | 1.737 | 1.737 | 1.737 | 1.737 | 1.737 | 1.737 |
| .01 | 1.749 | 1.749 | 1.750 | 1.751 | 1.749 | 1.740 | 1.740 |
| .02 | 1.748 | 1.746 | 1.747 | 1.748 | 1.751 | 1.745 | 1.743 |
| .04 | 1.762 | 1.757 | 1.754 | 1.750 | 1.748 | 1.753 | 1.749 |
| .1 | 1.880 | 1.856 | 1.842 | 1.816 | 1.760 | 1.750 | 1.751 |
| .2 | 2.180 | 2.112 | 2.074 | 1.992 | 1.822 | 1.745 | 1.749 |
| .4 | 2.902 | 2.751 | 2.658 | 2.467 | 2.014 | 1.769 | 1.748 |
| 1 | 5.294 | 4.889 | 4.638 | 4.109 | 2.796 | 1.897 | 1.836 |
| 2 | 9.394 | 8.573 | 8.063 | 6.987 | 4.257 | 2.216 | 2.049 |
| 4 | 17.64 | 16.00 | 14.97 | 12.81 | 7.289 | 2.983 | 2.601 |
| 10 | 42.44 | 38.31 | 35.75 | 30.32 | 16.48 | 5.502 | 4.478 |
| 20 | 83.78 | 75.53 | 70.40 | 59.55 | 31.84 | 9.810 | 7.737 |
| 40 | 166.5 | 150.0 | 139.7 | 118.0 | 62.59 | 18.48 | 14.42 |
| 100 | 414.5 | 373.3 | 347.7 | 293.4 | 154.8 | 44.52 | 34.11 |
| 200 | 828.0 | 745.5 | 694.2 | 585.6 | 308.6 | 87.95 | 67.11 |
| 400 | 1655 | 1490 | 1387 | 1170 | 616.0 | 174.8 | 133.1 |
| 1000 | 4136 | 3723 | 3467 | 2924 | 1538 | 435.4 | 331.2 |

TABLE VI

Values of the function $A(\beta m)$ for masses of several particles

| $\beta$ | particles $\Omega$ | $\Sigma$ | $\Lambda$ | N | K | $\pi$ | $\mu$ |
|---|---|---|---|---|---|---|---|
| 0 | 1.400 | 1.400 | 1.400 | 1.400 | 1.400 | 1.400 | 1.400 |
| .01 | 1.440 | 1.436 | 1.434 | 1.427 | 1.416 | 1.403 | 1.405 |
| .02 | 1.478 | 1.470 | 1.465 | 1.455 | 1.430 | 1.409 | 1.408 |
| .04 | 1.552 | 1.537 | 1.528 | 1.508 | 1.459 | 1.418 | 1.414 |
| .1 | 1.732 | 1.704 | 1.488 | 1.639 | 1.541 | 1.442 | 1.433 |
| .2 | 1.957 | 1.918 | 1.893 | 1.832 | 1.661 | 1.483 | 1.463 |
| .4 | 2.257 | 2.207 | 2.175 | 2.100 | 1.586 | 1.560 | 1.523 |
| 1 | 2.746 | 2.686 | 2.646 | 2.551 | 2.221 | 1.750 | 1.676 |
| 2 | 3.158 | 3.094 | 3.051 | 2.949 | 2.579 | 1.975 | 1.875 |
| 4 | 3.590 | 3.524 | 3.479 | 3.372 | 2.979 | 2.281 | 2.152 |
| 10 | 4.174 | 4.107 | 4.061 | 3.952 | 3.544 | 2.775 | 2.619 |
| 20 | 4.621 | 4.554 | 4.509 | 4.397 | 3.984 | 3.188 | 3.021 |
| 40 | 5.070 | 5.002 | 4.956 | 4.846 | 4.430 | 3.620 | 3.447 |
| 100 | 5.666 | 5.597 | 5.551 | 5.441 | 5.023 | 4.205 | 4.029 |
| 200 | 6.117 | 6.048 | 6.002 | 5.891 | 5.473 | 4.653 | 4.476 |
| 400 | 6.568 | 6.500 | 6.453 | 6.342 | 5.924 | 5.102 | 4.924 |
| 1000 | 7.165 | 7.097 | 7.050 | 6.939 | 6.521 | 5.698 | 5.519 |

TABLE VII

Values of the function $\phi(N, \beta)$ for several multiplicities of particles

| $\beta$ | N = 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| .01 | 3.457 | 9.306 | 15.218 | 21.156 | 27.107 | 34.068 |
| .02 | 2.855 | 7.801 | 12.810 | 17.844 | 22.893 | 27.950 |
| .04 | 2.252 | 6.296 | 10.402 | 14.533 | 18.678 | 22.833 |
| .1 | 1.457 | 4.306 | 7.218 | 10.156 | 13.107 | 16.068 |
| .2 | .855 | 2.801 | 4.810 | 6.844 | 8.893 | 10.950 |
| .4 | .252 | 1.296 | 2.402 | 3.533 | 4.678 | 5.833 |
| 1 | .471-1 | .306-1 | .218-1 | .156-1 | .107-1 | .068-1 |
| 2 | .855-2 | .801-3 | .810-4 | .844-5 | .893-6 | .950-7 |
| 4 | .252-2 | .296-4 | .402-6 | .533-8 | .678-10 | .833-12 |
| 10 | .457-3 | .306-6 | .218-9 | .156-12 | .107-15 | .068-18 |
| 20 | .855-4 | .801-8 | .810-11 | .844-16 | .893-20 | .950-24 |
| 40 | .252-4 | .296-9 | .402-14 | .533-19 | .678.24 | .833-29 |
| 100 | .457-5 | .306-11 | .219-17 | .156-23 | .107-29 | .068-35 |
| 200 | .855-6 | .801-13 | .810-20 | .844-27 | .893-34 | .950-41 |
| 400 | .252-6 | .296-14 | .402-22 | .533-30 | .678-38 | .833-46 |
| 1000 | .457-7 | .306-16 | .218-25 | .156-34 | .107-43 | .068-52 |

TABLE VIII

Values of the function $\psi(x)$

| x | $\Psi$ | x | $\Psi$ | x | $\Psi$ | x | $\Psi$ |
|---|---|---|---|---|---|---|---|
| .651 | 1.694 | .710 | 1.720 | .820 | 1.767 | .960 | 1.804 |
| .655 | 1.687 | .720 | 1.725 | .830 | 1.770 | .980 | 1.808 |
| .660 | 1.691 | .730 | 1.729 | .840 | 1.773 | 1.000 | 1.811 |
| .665 | 1.694 | .740 | 1.734 | .850 | 1.776 | 1.025 | 1.816 |
| .670 | 1.698 | .750 | 1.739 | .860 | 1.779 | 1.050 | 1.819 |
| .675 | 1.701 | .760 | 1.744 | .870 | 1.782 | 1.075 | 1.823 |
| .680 | 1.704 | .770 | 1.749 | .880 | 1.784 | 1.100 | 1.825 |
| .685 | 1.707 | .780 | 1.753 | .890 | 1.787 | 1.150 | 1.829 |
| .690 | 1.710 | .790 | 1.757 | .900 | 1.790 | 1.200 | 1.833 |
| .695 | 1.713 | .800 | 1.761 | .920 | 1.795 | 1.250 | 1.834 |
| .700 | 1.715 | .810 | 1.764 | .940 | 1.800 | 1.303 | 1.834 |

TABLE V

Masses of the particles as used in this paper

| Particles | $m/m_\pi$ | m in Mev. | Particles | $m/m_\pi$ | m in Mev. |
|---|---|---|---|---|---|
| $\Omega$ | 9.52 | 1329 | K | 3.54 | 495 |
| $\Sigma$ | 8.57 | 1195 | $\pi$ | 1 | 139.5 |
| $\Lambda$ | 7.98 | 1115 | $\mu$ | .76 | 106 |
| N | 6.73 | 938 | | | |

TABLE IX

Function $\Xi$ (N) of the number of particles that multiply the correction for the saddle point calculated value of L( = log dQ/dW)

| N | $\Xi$ | N | $\Xi$ | N | $\Xi$ | N | $\Xi$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 8 | .219 | 14 | .122 | 20 | .082 |
| 3 | .624 | 9 | .193 | 15 | .116 | 25 | .066 |
| 4 | .462 | 10 | .173 | 16 | .106 | 30 | .058 |
| 5 | .361 | 11 | .157 | 17 | .101 | 100 | .031 |
| 6 | .298 | 12 | .143 | 18 | .094 | 1000 | .002 |
| 7 | .251 | 13 | .130 | 19 | .087 | | |

TABLE X

Difference $\eta$ between the exact value of L(= log dQ/dW) and L calculated by saddle point in case of two outgoing particles, as a function of the relation between the kinetic energy E and the total mass $\Sigma$ m of the system. For N $\neq$ 2 this correction must be multiplied by $\Xi$ (N), as given in Table IX.

| E/$\Sigma$m | $\eta$ | E/$\Sigma$m | $\eta$ | E/$\Sigma$m | $\eta$ | E/$\Sigma$m | $\eta$ |
|---|---|---|---|---|---|---|---|
| 0 | .000 | .26 | .100 | .85 | .195 | 5.0 | .275 |
| .02 | .004 | .28 | .104 | .90 | .200 | 5.5 | .277 |
| .03 | .007 | .30 | .109 | .95 | .204 | 6 | .279 |
| .04 | .012 | .32 | .113 | 1.0 | .208 | 7 | .282 |
| .05 | .016 | .34 | .118 | 1.1 | .213 | 8 | .284 |
| .06 | .020 | .36 | .122 | 1.2 | .218 | 10 | .285 |
| .07 | .024 | .38 | .126 | 1.3 | .223 | 15 | .284 |
| .08 | .028 | .40 | .130 | 1.4 | .228 | 20 | .283 |
| .09 | .034 | .42 | .135 | 1.5 | .232 | 30 | .278 |
| .10 | .040 | .44 | .140 | 1.6 | .234 | 40 | .274 |
| .11 | .044 | .46 | .143 | 1.7 | .236 | 50 | .271 |
| .12 | .050 | .48 | .146 | 1.8 | .239 | 60 | .268 |
| .13 | .055 | .50 | .149 | 1.9 | .242 | 80 | .264 |
| .14 | .059 | .52 | .153 | 2.0 | .244 | 100 | .260 |
| .15 | .064 | .54 | .156 | 2.2 | .247 | 150 | .256 |
| .16 | .068 | .56 | .160 | 2.4 | .250 | 200 | .252 |
| .17 | .072 | .58 | .163 | 2.6 | .253 | 300 | .248 |
| .18 | .076 | .60 | .166 | 2.8 | .256 | 400 | .245 |
| .19 | .080 | .65 | .173 | 3.0 | .259 | 600 | .242 |
| .20 | .084 | .70 | .178 | 3.5 | .263 | 800 | .240 |
| .22 | .090 | .75 | .184 | 4.0 | .268 | 1000 | .238 |
| .24 | .095 | .80 | .189 | 4.5 | .272 | $\infty$ | .232 |

## TABLE XI

Summary of momentum space volumes per unit of energy range. The logarithm L of these volumes is given by the difference $\Delta L = L - L_{NR}$ (where $L_{NR} = s + t \log E$ is the N.R. approximation), as a function of the relation between the kinetic energy E of the system and the kinetic energy $E_i$, for which the N.R. and the $E_i$, for which the N.R. and the E.R. approximations are the same.

| System | 3N | 2Nπ | N2π | 3π | N2K | Ω 2K | NΛK | NΣK | ΛΣπ | ΣKπ |
|---|---|---|---|---|---|---|---|---|---|---|
| $-\log(E/E_i)$ | | | | | ΔL = | | | | | |
| .04 | .970 | 1.152 | 1.198 | .982 | 1.062 | 1.022 | 1.000 | 1.022 | 1.012 | 1.228 |
| .08 | .920 | 1.100 | 1.138 | .924 | 1.030 | .980 | .950 | .960 | 1.152 | 1.180 |
| .12 | .880 | 1.058 | 1.080 | .866 | .998 | .930 | .902 | .904 | 1.090 | 1.140 |
| .16 | .842 | 1.000 | 1.032 | .842 | .968 | .880 | .868 | .860 | 1.032 | 1.086 |
| .20 | .804 | .950 | .980 | .778 | .942 | .840 | .822 | .818 | .980 | 1.042 |
| .24 | .762 | .900 | .932 | .732 | .908 | .796 | .774 | .770 | .912 | .988 |
| .28 | .718 | .850 | .886 | .688 | .880 | .754 | .736 | .738 | .856 | .940 |
| .32 | .678 | .804 | .842 | .650 | .860 | .718 | .702 | .698 | .814 | .900 |
| .36 | .640 | .770 | .792 | .618 | .830 | .680 | .670 | .662 | .762 | .856 |
| .40 | .608 | .730 | .754 | .590 | .802 | .644 | .638 | .624 | .726 | .808 |
| .44 | .574 | .692 | .714 | .554 | .778 | .612 | .602 | .598 | .690 | .762 |
| .48 | .538 | .652 | .680 | .524 | .748 | .584 | .580 | .564 | .660 | .724 |
| .52 | .510 | .608 | .642 | .496 | .720 | .552 | .542 | .536 | .624 | .680 |
| .56 | .482 | .574 | .604 | .470 | .690 | .528 | .520 | .504 | .600 | .644 |
| .60 | .462 | .542 | .578 | .442 | .660 | .500 | .496 | .282 | .576 | .620 |
| .70 | .412 | .470 | .494 | .388 | .596 | .440 | .438 | .424 | .514 | .552 |
| .80 | .362 | .400 | .420 | .342 | .524 | .360 | .382 | .372 | .462 | .492 |
| .90 | .322 | .330 | .346 | .302 | .458 | .330 | .330 | .324 | .404 | .424 |
| 1.00 | .282 | .280 | .298 | .270 | .398 | .296 | .292 | .294 | .358 | .372 |
| 1.20 | .218 | .204 | .220 | .220 | .280 | .224 | .222 | .230 | .264 | .274 |
| 1.40 | .170 | .150 | .162 | .164 | .186 | .166 | .172 | .176 | .198 | .202 |
| 1.60 | .130 | .104 | .120 | .124 | .120 | .120 | .130 | .134 | .134 | .140 |
| 1.80 | .100 | .070 | .084 | .092 | .076 | .080 | .092 | .096 | .096 | .096 |
| 2.00 | .070 | .044 | .060 | .070 | .052 | .058 | .064 | .062 | .062 | .064 |
| 2.40 | .030 | .022 | .024 | .034 | .024 | .034 | .030 | .030 | .030 | .050 |
| 2.80 | .020 | .012 | .012 | .018 | .016 | .022 | .018 | .018 | .016 | .016 |
| $\log E_i$ = | 1.467 | 1.126 | .821 | .639 | 1.271 | 1.306 | 1.387 | 1.395 | 1.055 | 1.060 |
| s = | 3.862 | 2.838 | 1.924 | 1.378 | 3.372 | 3.339 | 3.620 | 3.646 | 2.624 | 2.640 |
| t = | | | | | 2 | | | | | |

| System | $2N$ | $N\pi$ | $2\pi$ | $\Sigma K$ | $\Lambda K$ | $4N$ | $2N2\pi$ | $4\pi$ | $N\Sigma K\pi$ | $2N3\pi$ | $5\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $-\log(E/E_i)$ | $\Delta L =$ | | | | | | | | | | |
| .04 | .452 | .772 | .535 | .582 | .582 | 1.392 | 1.725 | 1.428 | 1.689 | 2.232 | 1.872 |
| .08 | .438 | .751 | .515 | .564 | .560 | 1.317 | 1.650 | 1.338 | 1.608 | 2.096 | 1.740 |
| .12 | .399 | .731 | .495 | .542 | .540 | 1.254 | 1.563 | 1.263 | 1.539 | 1.980 | 1.624 |
| .16 | .373 | .709 | .481 | .529 | .521 | 1.200 | 1.497 | 1.197 | 1.464 | 1.826 | 1.536 |
| .20 | .346 | .690 | .648 | .512 | .505 | 1.116 | 1.428 | 1.134 | 1.395 | 1.796 | 1.488 |
| .24 | .321 | .670 | .451 | .494 | .489 | 1.050 | 1.350 | 1.050 | 1.326 | 1.684 | 1.364 |
| .28 | .299 | .648 | .438 | .480 | .475 | .990 | 1.275 | .990 | 1.254 | 1.596 | 1.280 |
| .32 | .280 | .624 | .422 | .468 | .458 | .918 | 1.200 | .930 | 1.197 | 1.500 | 1.204 |
| .36 | .263 | .603 | .408 | .450 | .442 | .849 | 1.128 | .867 | 1.131 | 1.404 | 1.128 |
| .40 | .250 | .580 | .393 | .438 | .430 | .810 | 1.074 | .816 | 1.080 | 1.324 | 1.072 |
| .44 | .236 | .558 | .382 | .423 | .412 | .753 | 1.017 | .762 | 1.020 | 1.248 | 1.008 |
| .48 | .222 | .538 | .370 | .411 | .400 | .717 | .960 | .720 | .963 | 1.168 | .932 |
| .52 | .202 | .521 | .356 | .398 | .386 | .663 | .897 | .675 | .912 | 1.084 | .872 |
| .56 | .190 | .500 | .346 | .385 | .376 | .624 | .737 | .627 | .855 | 1.024 | .808 |
| .60 | .180 | .483 | .332 | .372 | .368 | .594 | .780 | .588 | .810 | .960 | .760 |
| .70 | .151 | .440 | .306 | .345 | .344 | .513 | .669 | .513 | .708 | .804 | .640 |
| .80 | .129 | .398 | .282 | .321 | .318 | .448 | .567 | .444 | .612 | .680 | .540 |
| .90 | .108 | .350 | .259 | .299 | .294 | .366 | .480 | .375 | .525 | .560 | .456 |
| 1.00 | .089 | .306 | .238 | .274 | .270 | .324 | .393 | .313 | .453 | .472 | .392 |
| 1.20 | .063 | .238 | .201 | .231 | .222 | .246 | .282 | .228 | .345 | .324 | .284 |
| 1.40 | .046 | .178 | .170 | .185 | .180 | .189 | .198 | .177 | .243 | .228 | .196 |
| 1.60 | .036 | .126 | .141 | .145 | .144 | .135 | .126 | .132 | .168 | .140 | .136 |
| 1.80 | .030 | .082 | .112 | .112 | .111 | .099 | .087 | .099 | .114 | .084 | .100 |
| 2.00 | .022 | .059 | .084 | .084 | .084 | .072 | .060 | .066 | .078 | .064 | .076 |
| 2.40 | .020 | .031 | .045 | .039 | .040 | .033 | .033 | .033 | .036 | .040 | .040 |
| 2.80 | .019 | .018 | .025 | .014 | .013 | .021 | .027 | .015 | .021 | .028 | .024 |
| $\log E_i =$ | 1.229 | .642 | .401 | 1.101 | 1.092 | 1.609 | 1.138 | .781 | 1.319 | 1.172 | .887 |
| $s =$ | 2.040 | 1.160 | .798 | 1.848 | 1.834 | 5.349 | 3.227 | 1.623 | 4.045 | 3.370 | 1.662 |
| $t =$ | 1/2 | | | | | 7/2 | | | | 5 | |

## TABLE XII

Summary of momentum space volumes per unit energy range. The logarithm L of these volumes is given by the difference $\Delta L = L - L_{E.R.}$ (where $L_{E.R.} = q + r \log E$ is the N.E approximation) as a function of the relation between the kinetic energy E of the system and the kinetic energy $E_i$, for which the N.R. and the E.R. approximations are the same.

| System | $3N$ | $2N\pi$ | $N2\pi$ | $3\pi$ | $N2K$ | $\Omega 2K$ | $N\Lambda K$ | $N\Sigma K$ | $\Lambda\Sigma\pi$ | $\Sigma K\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\log(E/E_i)$ | | | | | $\Delta L =$ | | | | | |
| 0 | 1.022 | 1.212 | 1.268 | 1.042 | 1.096 | 1.090 | 1.058 | 1.090 | 1.270 | 1.296 |
| .04 | .976 | 1.152 | 1.220 | .964 | 1.024 | 1.040 | .966 | 1.038 | 1.220 | 1.250 |
| .08 | .924 | 1.100 | 1.170 | .910 | .956 | .980 | .896 | .974 | 1.176 | 1.182 |
| .12 | .860 | 1.044 | 1.110 | .836 | .904 | .910 | .818 | .910 | 1.116 | 1.118 |
| .16 | .808 | .982 | 1.052 | .782 | .842 | .450 | .754 | .856 | 1.070 | 1.042 |
| .20 | .752 | .920 | .998 | .740 | .782 | .796 | .690 | .798 | 1.022 | .976 |
| .24 | .712 | .862 | .944 | .690 | .728 | .752 | .638 | .742 | .978 | .930 |
| .28 | .668 | .816 | .882 | .636 | .682 | .698 | .590 | .682 | .922 | .878 |
| .32 | .606 | .762 | .842 | .598 | .630 | .656 | .552 | .640 | .868 | .830 |
| .36 | .566 | .720 | .792 | .560 | .582 | .620 | .520 | .598 | .820 | .790 |
| .40 | .530 | .678 | .750 | .516 | .540 | .580 | .488 | .556 | .768 | .750 |
| .50 | .440 | .578 | .654 | .438 | .448 | .520 | .420 | .460 | .662 | .644 |
| .60 | .348 | .498 | .558 | .360 | .378 | .460 | .358 | .372 | .550 | .552 |
| .70 | .296 | .422 | .470 | .280 | .304 | .418 | .302 | .290 | .440 | .460 |
| .80 | .224 | .360 | .396 | .236 | .256 | .368 | .252 | .242 | .364 | .384 |
| .90 | .180 | .302 | .338 | .198 | .204 | .330 | .206 | .204 | .302 | .306 |
| 1.00 | .156 | .260 | .288 | .162 | .162 | .298 | .170 | .178 | .250 | .254 |
| 1.20 | .104 | .180 | .206 | .110 | .104 | .232 | .102 | .122 | .160 | .170 |
| 1.40 | .062 | .122 | .140 | .032 | .070 | .180 | .060 | .080 | .104 | .116 |
| 1.60 | .058 | .084 | .096 | .044 | .056 | .138 | .040 | .058 | .094 | .078 |
| 1.80 | .052 | .060 | .056 | .030 | .042 | .112 | .036 | .044 | .052 | .050 |
| 2.00 | .048 | .054 | .024 | .020 | .036 | .098 | .030 | .040 | .038 | .036 |
| $\log E_i =$ | 1.467 | 1.126 | .821 | .639 | 1.271 | 1.306 | 1.387 | 1.395 | 1.055 | 1.060 |
| $q =$ | | | | | − .541 | | | | | |
| $r =$ | | | | | 5 | | | | | |

| System | 2N | Nπ | 2π | KΣ | ΛK | 4N | 2N2π | 4π | NΣKπ | 2N3π | 5π |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\log(E/E_i)$ | | | | ΔL = | | | | | | | |
| 0 | .479 | .792 | .552 | .605 | .605 | 1.482 | 1.806 | 1.518 | 1.776 | 2.364 | 1.976 |
| .04 | .451 | .761 | .522 | .582 | .588 | 1.410 | 1.743 | 1.443 | 1.686 | 2.260 | 1.886 |
| .08 | .420 | .728 | .499 | .558 | .522 | 1.317 | 1.677 | 1.386 | 1.593 | 2.164 | 1.804 |
| .12 | .391 | .700 | .470 | .524 | .520 | 1.224 | 1.608 | 1.308 | 1.506 | 2.060 | 1.704 |
| .16 | .368 | .662 | .440 | .491 | .489 | 1.140 | 1.548 | 1.233 | 1.416 | 1.952 | 1.592 |
| .20 | .342 | .631 | .411 | .462 | .460 | 1.050 | 1.455 | 1.146 | 1.323 | 1.844 | 1.472 |
| .24 | .321 | .598 | .382 | .429 | .430 | .978 | 1.362 | 1.074 | 1.233 | 1.732 | 1.380 |
| .28 | .301 | .562 | .358 | .401 | .401 | .906 | 1.323 | .993 | 1.167 | 1.624 | 1.280 |
| .32 | .285 | .531 | .332 | .371 | .379 | .843 | 1.197 | .924 | 1.104 | 1.528 | 1.188 |
| .36 | .271 | .502 | .309 | .352 | .359 | .825 | 1.116 | .864 | 1.047 | 1.444 | 1.100 |
| .40 | .253 | .471 | .282 | .325 | .340 | .747 | 1.026 | .792 | .963 | 1.364 | 1.016 |
| .50 | .225 | .404 | .238 | .275 | .290 | .624 | .900 | .660 | .813 | 1.204 | .844 |
| .60 | .200 | .343 | .193 | .228 | .242 | .513 | .753 | .534 | .675 | 1.040 | .688 |
| .70 | .174 | .289 | .162 | .189 | .201 | .420 | .633 | .423 | .558 | .846 | .560 |
| .80 | .156 | .243 | .135 | .159 | .161 | .348 | .510 | .357 | .474 | .724 | .444 |
| .90 | .140 | .210 | .116 | .130 | .132 | .282 | .426 | .297 | .396 | .596 | .372 |
| 1.00 | .129 | .172 | .099 | .109 | .110 | .231 | .309 | .243 | .327 | .484 | .300 |
| 1.20 | .102 | .120 | .071 | .080 | .080 | .150 | .195 | .165 | .219 | .336 | .196 |
| 1.40 | .081 | .081 | .051 | .059 | .057 | .108 | .150 | .105 | .147 | .236 | .132 |
| 1.60 | .062 | .059 | .036 | .044 | .041 | .090 | .126 | .069 | .093 | .196 | .084 |
| 1.80 | .050 | .041 | .022 | .031 | .030 | .081 | .117 | .036 | .069 | .100 | .044 |
| 2.00 | .045 | .031 | .017 | .025 | .025 | .075 | .108 | .027 | .060 | .064 | .020 |
| $\log E_i$ = | 1.229 | .624 | .401 | 1.101 | 1.092 | 1.609 | 1.138 | .781 | 1.319 | 1.167 | .887 |
| q = | | | .196 | | | | -1.896 | | | -3.661 | |
| r = | | | 2 | | | | 8 | | | 11 | |