ON THE CAPTURE OF NEGATIVE MUONS BY LIGHT NUCLEI

J. Leite Lopes*

Faculdade Nacional de Filosofia and Centro
Brasileiro de Pesquisas Físicas
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ABSTRACT

The capture of negative muons by protons can be regarded as taking place as a result of a direct coupling or via the pi-meson field. An attempt is made to check whether the static, cut-off pseudovector coupling model of the pi-meson-nucleon interaction can give rise to predictions on the capture of muons by nuclei in agreement with experiment. Chew theory leads to a capture rate 20 times smaller than that predicted by the direct coupling theory. Accurate calculations can be made for the case of hydrogen, deuterium or helium, for which experimental information would be highly desirable. Comparison with the observational data for carbon, oxygen and silicium, based on the

* On leave at Physics Department California Institute of Technology Pasadena, California, where this work has been performed.

ideal gas model for nuclei, indicates that they are consistent with the direct coupling theory. Meson theory gives a capture rate about 20 times too small.

INTRODUCTION

The experimental measurement of the capture rate of negative muons by protons would provide an invaluable source of information on the nature of the interaction between mu-mesons and nucleons.

Due to the fact that the decay of negative muons predominates over their nuclear absorption by nuclei with atomic number less than about 12, the experimental data refer up to now to medium and heavy nuclei[1]. The theoretical predictions of the capture rate for such nuclei are, however, very difficult to be made accurately at present due to our lack of knowledge of the nuclear wave functions.

As it is well known, two alternative views are possible for describing the interaction between muons and nucleons[2]. The most widely accepted prescribes a direct Fermi coupling between the nucleon and the muon-neutrino fields, in a manner analogous to beta decay. The alternative view is that of an interaction of muons and nucleons via the pi-meson field. The first interpretation received a great deal of attention when it was discovered that, within the experimental uncertainty, the same order of magnitude of the coupling constant could account for the beta decay of neutron and mu-meson as well as for the capture of muons by nuclei. On the other hand, the alternative interpretation of the interaction of mu-mesons with nucleons through the pion field is inevitable according to current quantum-mechanical principles. Since pi-mesons decay into mu-mesons and interact strongly with nucleons, the mechanism of capture of mu-mesons by protons via virtual pi-mesons cannot be discarded unless it gives a negligible contribution to the process or some as yet unknown rule forbids it.

The recent success of the cut-off meson theory for describing low energy pi-meson-nucleon scattering and related processes[3], led us to re-examine this problem. In this paper, we present results of calculations carried out under both interpretations, our aim being to check whether the Chew-Low meson theory would be capable of giving predictions in agreement with experiment. The calculations can be made accurately for the capture of mu-mesons by a free proton, by the deuteron and to a certain extent by the alpha particle. Chew theory predicts a capture rate about 20 times smaller than the direct coupling theory, for acceptable values of the coupling constants. An estimate based on the Fermi gas model for nuclei spherically symmetric in spin and isotopic spin space, indicates that the available experimental data are consistent with the direct coupling theory, not with the Chew theory which predicts lifetimes 20 times larger. The conclusion will be definite when measurements for capture by hydrogen become available.

<u>1. Capture by a proton</u>. The hamiltorian of a pi-meson field in interaction with a muon-neutrino field and an extended

nucleon is the following:

$$H = H_{\pi} + H_{\mu} + H_{\nu} + H_{\pi N} + H_{\pi\mu}, \qquad (1)$$

where H_{π}, H_{μ}, H_{ν} are the hamiltonians of free pi-mesons, free muons and neutrinos, respectively.

$$H_{\pi N} = F_o \tau_{\alpha} \int U(x)(\vec{\sigma} \cdot \vec{\nabla}) \varphi_{\alpha}(x) \, d^3x \qquad (2)$$

is the pseudo-vector interaction between pi-mesons and the extended nucleon with source $U(x)$, where $F_o = (4\pi)^{1/2} f_o \, m_{\pi}^{-1}$ and the sum over the subscript $\alpha = 1,2,3$ of isotopic spin space is implied; $\hbar = 1$, $c = 1$.

$$H_{\pi\mu} = i \, G_o \sqrt{2} \int \bar{\Psi}_{\mu} \gamma_5 \gamma_j \frac{\partial \varphi}{\partial x_j} \Psi_{\nu} \, d^3x + \text{herm.conj.} \qquad (3)$$

is the pseudo-vector interaction between the pi- and muon-neutrino fields. The muon-neutrino operators are labelled by the indices μ and ν; $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$ and $j = 0, 1, 2, 3$ is the ordinary space subscript. The representation has been taken in which $\gamma_5^+ = -\gamma_5$ and $a_j b_j = a_o b_o - \vec{a} \cdot \vec{b}$. Also $G_o = (4\pi)^{1/2} g_o m_{\pi}^{-1}$.

The interaction (3) gives rise to the decay $\pi \longrightarrow \mu + \nu$ of which the transition probability for pions at rest is:

$$\tau_{\pi\mu}^{-1} = g_o^2 m_{\pi} \left(\frac{m_{\mu}}{m_{\pi}}\right)^2 \left[1 - \left(\frac{m_{\mu}}{m_{\pi}}\right)^2 \right]^2. \qquad (4)$$

The experimental value of $\tau_{\pi\mu} = 2.55 \times 10^{-8}$ sec determines g_o:

$$g_o^2 = 0.18 \times 10^{-14}.$$

The smallness of g_o allows the treatment of $H_{\pi\mu}$ as a small perturbation of the remaining hamiltonian in (1).

The annihilation of a negative muon accompanied by the transformation of a proton into a neutron and the emission of an anti-neutrino is described by the matrix element $(\bar{\nu}N|H_{\pi\mu}|\bar{\mu}P)$. The Fourier development of φ, ψ_μ and ψ_ν:

$$\varphi_\alpha = \frac{1}{(2V)^{1/2}} \sum_k \frac{1}{\omega_k^{1/2}} [a_\alpha(\vec{k}) e^{i\vec{k}.\vec{x}} + b_\alpha^*(\vec{k}) e^{-i\vec{k}.\vec{x}}],$$

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2); \quad a(\vec{k}) = \frac{1}{\sqrt{2}}(a_1(\vec{k}) - ia_2(\vec{k})); \quad b^*(\vec{k}) = $$
$$= \frac{1}{\sqrt{2}}(a_1^* - ia_2^*);$$

$$\psi_{\mu,\nu} = \frac{1}{V^{1/2}} \sum_k \sum_{r=1,2} [c_r^{\mu,\nu}(\vec{k}) u_r^{\mu,\nu} e^{i\vec{k}.\vec{x}} + d_r^{+\mu,\nu} v_r^{\mu,\nu} e^{-i\vec{k}.\vec{x}}]$$

in (3) gives for the matrix element $(\bar{\nu}N|H_{\pi\mu}|\bar{\mu}P)$ between the initial state $d_r^{+\mu}(p_\mu)|P\rangle\psi_0$ and the final state $d_s^{+\nu}(p_\nu)|N\rangle\psi_0$ the following expression (ψ_0 is the vacuum state vector):

$$(\nu N|H_{\pi\mu}|\mu P) = (\bar{v}_r^\mu, M(\vec{k}) v_s^\nu) \langle N|a(k)|P\rangle + (\bar{v}_r^\mu, M(-\vec{k}) v_s^\nu) \langle N|b^*(-\vec{k})|P\rangle \qquad (5)$$

Here $|P\rangle$ and $|N\rangle$ are the wave functions which describe the physical proton and neutron (with their meson clouds):

$$(H_\pi + H_{\pi N}) |P\rangle = E_p |P\rangle,$$
$$(H_\pi + H_{\pi N}) |N\rangle = E_n |N\rangle$$

and $M(\vec{k}) = \frac{G_0}{V^{3/2}} \omega_k^{-1/2} \gamma_5 k\!\!\!/, \; k = p_\mu - p_\nu, \; k\!\!\!/ = k_\alpha \gamma_\alpha$.

The matrix elements $\langle N|a(\vec{k})|P\rangle$ and $\langle N|b^*(\vec{k})|P\rangle$ can be expressed in terms of the nucleon and pi-meson variables if one considers the commutation relations:

$$[H_\pi + H_{\pi N}, a(\vec{k})] = -\omega_k a(\vec{k}) + V_a(\vec{k}),$$

$$[H_\pi + H_{\pi N}, b^*(\vec{k})] = \omega_k b^*(\vec{k}) + V_b(\vec{k}),$$

$$V_a(\vec{k}) = \frac{F_o \tau^-}{(V\omega_k)^{1/2}} i(\vec{\sigma}.\vec{k}) v^*(\vec{k}); \quad V_b(\vec{k}) = -V_a(-\vec{k}); \; v(\vec{k}) =$$

$$= \int u(\vec{x}) e^{i\vec{k}.\vec{x}} d^3x.$$

One obtains:

$$\langle N|a(\vec{k})|P\rangle = \frac{\langle N|V_a(\vec{k})|P\rangle}{E_N - E_P + \omega_k}$$

$$\langle N|b^*(\vec{k})|P\rangle = \frac{\langle N|V_b(k)|P\rangle}{E_N - E_P - \omega_k}$$

Substitution of these expressions in (5) gives:

$$(\nu N|H_{\pi\mu}|\mu P) = -\frac{2G_o F_o i}{V} \langle N|\vec{\sigma}.\vec{p}|P\rangle \frac{(v_r^{\ \mu}, \gamma_5 \not{p} v_s^{\ \nu})}{p_\alpha^2 - m_\pi^2}, \qquad (6)$$

where $p_\alpha = (p_\nu^{\ \alpha} - p_\mu^{\ \alpha})$ is the (four) momentum transfer. Now the well-known theorem:

$$F_o \langle N|\,\vec{\sigma}.\vec{p}\tau^-|P\rangle = F(u_N^+ \vec{\sigma}.\vec{p}\tau^- u_P)$$

permits the replacement in our formula of F_o by the renormalized coupling constant F and the unknown matrix element $\langle N|\vec{\sigma}.\vec{p}\tau^-|P\rangle$ by the one computed with free particle spinors $(u_N^+ \vec{\sigma}.\vec{p}\tau^- u_p)$.

The transition probability for capture of a negative muon by a proton is then:

$$\tau_H^{-1} = 2\pi \int \frac{1}{2} \sum_{r,s=1,2} \frac{1}{2} \sum_{NP} |(\nu_s N | H_{\pi\mu} | \mu_r P)|^2 \times$$
$$\times \frac{E_\nu^2 V}{(2\pi)^3} |\varphi(0)|^2 V \, d\Omega_\nu,$$

where $\frac{1}{2} \sum_{rs} \frac{1}{2} \sum_{NP}$ means sum over the spin of the final neutron and neutrino and average of the spin of the initial proton and muon, E_ν is the energy of the neutrino, $|\varphi(0)|^2 = \frac{1}{\pi}(m_\mu e^2)^3$ is the probability to find the muon at the proton position (from the K-orbit).

Assuming that both the proton and mu-meson are at rest gives:

$$\tau_H^{-1} = 32 f^2 g_0^2 m_\pi \left(\frac{m_\mu}{m_\pi}\right)^9 (e^2)^3 \left[1 + \left(\frac{m_\mu}{m_\pi}\right)^2\right]^{-2}. \qquad (7)$$

In this formula, as well as in the matrix element (6), we have replaced the fourier transformed of the source function, v(k), by I. This results from the fact that the cut-off momentum in the Chew-Low theory is of the order of the nucleon rest energy. If one assumes a Gaussian distribution for the source, it may be practically replaced by the unity.

From (4) and (7) we obtain:

$$\frac{\tau_{\pi\mu}}{\tau_H} = 32 f^2 \left(\frac{m_\mu}{m_\pi}\right)^7 (e^2)^3 \left[1 - \left(\frac{m_\mu}{m_\pi}\right)^4\right]^{-2}.$$

2. Muon capture by a light nucleus.

The preceding calculations can be easily generalized when a muon is captured by a nucleus.

Let $|\eta\rangle$ be the wave function which describes the state of a nucleus with energy E:

$$H|\eta\rangle = E|\eta\rangle .$$

H is formed of the sum of the kinetic energies of the (slowly moving) nucleons and of the pi-meson field hamiltonian and its interaction with the nucleons.

The latter is:

$$\sum_{i=1} F_o \tau_\alpha^i \int U(x_i)(\vec{\sigma}_i.\vec{\nabla}_i)\varphi_\alpha(x_i)d^3x_i,$$

where the sum is extended over the nucleons and summation over the isotopic spin subscript is understood. We want the matrix elements $\langle\eta'|a(\vec{k})|\eta\rangle$ and $\langle\eta'|b^*(\vec{k})|\eta\rangle$ which occur in the expression (analogous to (5)) for the amplitude $(\bar{\nu}\eta'|H_{\pi\mu}|\bar{\mu}\eta)$ of capture of a negative muon by a nucleus in state $|\eta\rangle$ which then goes over into state $|\eta'\rangle$:

$$(\bar{\nu}\eta'|H_{\pi\mu}|\bar{U}\eta) = (\bar{v}_r^\mu, M(\vec{k})v_s^\nu)\langle\eta'|a(\vec{k})|\eta\rangle + (\bar{v}_r^\mu, M(-\vec{k})v_s^\nu)\langle\eta'|b^*(-\vec{k})|\eta\rangle .$$

They are obtained from the commutation relation:

$$H,a(\vec{k}) = -\omega_k a(\vec{k}) + \sum_{j=1}^{A} V_j(\vec{k});\ V_j(\vec{k}) =$$

$$= \frac{iF_o}{(V\omega_k)^{1/2}}(\vec{\sigma}_j.\vec{k})\tau_j\psi_j^*(\vec{k}),$$

and another one with $b^*(\vec{k})$.

One obtains:

$$\langle\eta'|a(\vec{k})|\eta\rangle = \frac{\langle\eta'|\sum_{i=1}^{A} V_i(\vec{k})|\eta\rangle}{E' - E + \omega_k},$$

$$\langle \eta' | b^*(\vec{k}) | \eta \rangle = \frac{\langle \eta' | \sum_{i=1}^{A} v_i(\vec{k}) | \eta \rangle}{E' - E - \omega_k}$$

The matrix element analogous to (6) is then:

$$(\bar{\nu}\,\eta' | H_{\pi\mu} | \bar{\mu}\,\eta) = -\frac{2G_o\,F_o\,i}{V} \langle \eta' | \sum_{i=1}^{A} (\vec{\sigma}_i \cdot \vec{p}) \tau_i^- \, v_i^{*}(\vec{k}) | \eta \rangle \times$$

$$\times \quad \frac{(\bar{v}_r^{\,\mu}, \gamma_5 \not{p} v_s^{\,\nu})}{p_\alpha^{\,2} - m_\pi^{\,2}} .$$

Here we have neglected the effect of the nuclear Coulomb field on the muon wave function which was represented by a plane wave. The formula should, however, be sufficiently accurate for light nuclei if we use the effective atomic number calculated by Wheeler[4].

In the preceding expression we shall replace F_o by the renormalized Chew coupling constant F. This amounts to introducing the impulse approximation. The interaction of a nuclear proton with the muon is complicated by the presence of other nucleons with which it interacts. However, the time during which the interaction with the muon takes place is small compared to the past and future history of the nucleus. We therefore assume that only a small error is committed if we take the proton in question as free during that time. The effect of this assumption will be to replace F_o by the effective renormalized coupling constant relative to a free nucleon. This approximation is also equivalent to assuming that a proton which absorbs a pion emitted by the muon does not exchange pions with the neighboring nucleons for a short time before and after the arrival of the virtual pi-meson.

The transition probability is:

$$\tau^{-1} = 2\pi \sum_{E'} \int \tfrac{1}{2}\, m_\mu^2 \left(1 - \frac{\vec{P}_\nu \cdot \vec{P}_\mu}{E_\nu\, E_\mu}\right) \frac{E_\nu^2}{(2\pi)^3} \frac{1}{\pi} (Zm_\mu e^2)^3 \times$$

$$\times \frac{4\, G_o^2\, F_o^2}{(p_\alpha^2 - m_\pi^2)^2} \langle M \rangle^2 d\Omega_\nu ,$$

where $\sum_{E'}$ is the sum over the final nuclear states and $\langle M\rangle^2$ is the sum over the final nuclear spins and average over initial nuclear spin of the absolute square of:

$$M = \langle \eta' | \sum_{j=1} (\sigma_j \cdot p) \tau_j^- \, e^{-i\vec{k}\cdot\vec{x}_j} | \eta \rangle .$$

Here again we neglect the nucleon extension and thus replace $v_j(k)$ by $\exp(-ik\cdot x_j)$.

$$\tau^{-1} = \frac{8}{\pi} g_o^2 f^2 m_\pi \left(\frac{m_\mu}{m_\pi}\right)^5 (Ze^2)^3 \sum_{E'} \int \Omega_\nu \left(1 - \frac{\vec{p}_\nu \cdot \vec{p}_\mu}{E_\nu E_\mu}\right) \times$$

$$\times \frac{E^2}{[(E_\nu - E_\mu)^2 - c^2(\vec{p}_\nu - \vec{p}_\mu)^2\, m_\pi^2]^2} \langle M\rangle^2 d\Omega_\nu .$$

3. Case of the deuteron

$\langle M\rangle^2$ can be calculated in the case of the deuteron. The ground state wave function is:

$$|\eta\rangle = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} [p(1)n(2) - p(2)n(1)]^3 \chi_m(1,2) u_D(r),$$

where p and n are the proton and neutron components of the isotopic spin wave function, $^3\chi_m(1,2)$ is the triplet spin wave function, $u_D(r)$ is the radial function.

After capture of the muon, the deuteron transforms into a di-neutron which can have spin 0 or 1. The two possible final wave functions are thus:

$$|\eta'_s\rangle = \frac{1}{(2\pi)^{3/2}}\, n(1)n(2)\,^1\chi_0(1,2)u_s(r)e^{i\vec{P}\cdot\vec{R}},$$

$$|\eta'_a\rangle = \frac{1}{(2\pi)^{3/2}}\, n(1)n(2)\,^3\chi_m(1,2)u_a(r)e^{i\vec{P}\cdot\vec{R}}.$$

$^1\chi_0$ is the singlet spin wave function, $\vec{P}$ is the center of mass momentum, $\vec{R} = \frac{1}{2}(\vec{r}_1+\vec{r}_2)$, $\vec{r} = \vec{r}_1-\vec{r}_2$ and the subscripts s and a stand for symmetric and antisymmetric wave functions in space coordinates.

The matrix elements corresponding to the two types of transition are:

$$M_s = \frac{1}{\sqrt{2}}\langle ^1\chi_0|\ (\vec{\sigma}_1-\vec{\sigma}_2)\ \cdot \vec{p}\,|^3\chi_m\rangle\ I_s,$$

$$M_a = \frac{1}{\sqrt{2}}\langle ^3\chi_{m'}|\ (\vec{\sigma}_1+\vec{\sigma}_2)\ \cdot \vec{p}\,|^3\chi_m\rangle\ I_a.$$

where

$$I_s = \int u_s^*(r)u_D(r)e^{\frac{i}{2}\vec{p}\cdot\vec{r}}\,d^3r.$$

$$I_a = \int u_a^*(r)u_D(r)e^{-\frac{i}{2}\vec{p}\cdot\vec{r}}\,d^3r.$$

For a muon at rest the transition probability is:

$$\tau_D^{-1} = \frac{2}{\tau_H}\sum_E \frac{g(E)}{g(0)}\,[2|I_a|^2 + |I_s|^2]. \qquad (8)$$

$E = m_\mu - E_\nu - \frac{p_\nu^2}{4M}$ is the nuclear excitation energy and:

$$g(E) = \frac{(m_\mu - E)^4}{(2m_\mu E - m_\mu^2 - m_\pi^2)^2}.$$

To evaluate the sum in (8), develop g(E) in powers of E. The first and second terms depend on the following integrals:

$$|I_a|^2 = \iint u^*(r)u_D(r)u(r')u_D^*(r')\sin(K\cdot r)\sin(K\cdot r')\,d^3r\,d^3r', \quad u = u_s + u_a$$

$$|I_s|^2 = \iint u^*(r)u_D(r)u(r')u_D^*(r')\cos(K.r)\cos(K.r')\,d^3r\,d^3r',$$

$$K = \frac{p}{2}.$$

The sum can be obtained if we assume that K does not depend on E; this amounts to assuming an average excitation energy small compared to the muon rest energy which is known experimentally to be true. One gets then:

$$\sum_E |I_a|^2 = \int u_D^*(r)u_D(r)\sin^2(K.r)\,d^3r = J$$

$$\sum_E |I_s|^2 = \int u_D^*(r)u_D(r)\cos^2(K.r)\,d^3r = 1 - J,$$

$$\sum_E E\,|I_a|^2 = \frac{p^2}{4M}(1-J) + \langle V_f - V_D\rangle_s,$$

$$\sum_E E\,|I_a|^2 = \frac{p^2}{4M}\,J + \langle V_f - V_D\rangle_c,$$

where

$$\langle V_f - V_D\rangle_s = \sum_E \iint u^*(r)u(r')(V_f - V_D)u_D^*(r')u_D(r)\sin(K.r)\sin(K.r')\times$$

$d^3r\,d^3r'$, and $\langle V_f - V_D\rangle_c$ is the same expression as $\langle V_f - V_D\rangle_s$ with cosines in place of sines; V_D is the deuteron potential energy, V_f the potential of the final di-neutron. We neglect V_f and write:

$$\tau_D^{-1} = 2\tau_H^{-1} \{ 1 + J + \frac{g'(0)}{g(0)} [\frac{p^2}{4M}(2-J) - \int u_D^* u_D V_D (1+\sin^2 K.r) d^3r]\}.$$

For evaluation of the integrals we take the deuteron wave function[6]

$$u_D(r) = \left[\frac{\alpha}{2\pi(1-\alpha\rho)}\right]^{1/2}\left(\frac{e^{-\alpha r}}{r} - \frac{e^{-\beta r}}{r}\right),$$

where

$$\rho = \frac{4}{\alpha+\beta} - \frac{1}{\beta}; \quad \beta = 7\alpha; \quad \alpha = 45.5 \text{ Mev},$$

and for the deuteron potential energy [7]

$$V_D = V_o\, e^{-r/a}, \; V_o = 214\, m_e c^2, \; a = 0.251(e^2/m_e c^2).$$

One then obtains

$$J = \frac{1}{2(1-\alpha\rho)}\{1 - \alpha\rho + \frac{\alpha}{K}\,[\tan^{-1}\frac{\alpha}{K} + \tan^{-1}\frac{\beta}{K} - 2\tan^{-1}\frac{\alpha+\beta}{2K}]\},$$

$$\int u_D^2 V_D d^3r = -\frac{2\alpha V_o}{1-\alpha\rho}\{\frac{1}{2\alpha+\frac{1}{a}} + \frac{1}{2\beta+\frac{1}{a}} - \frac{2}{\alpha+\beta+\frac{1}{a}}\},$$

$$\int u_D^2 V_D \sin^2(K.r) d^3r = -\frac{\alpha V_o}{2(1-\alpha\rho)}\{\frac{1}{\alpha+\frac{1}{2a}} + \frac{1}{\beta+\frac{1}{2a}} -$$

$$-\frac{4}{\alpha+\beta+\frac{1}{a}} + \frac{1}{K}[\tan^{-1}\frac{\alpha+2/a}{K} + \tan^{-1}\frac{\beta+2/a}{K} - 2\tan^{-1}\frac{\alpha+\beta+1/a}{2K}]\}.$$

The numerical result is

$$\tau_D^{-1} = 1.214\,\tau_H^{-1}, \quad \tau_D = 0.81\,\tau_H.$$

Physically the fact that the transition probability for the deuteron is a little larger than that for the proton can be understood from the finite extension of the deuteron within which the proton is spread. The average excitation energy of the di-neutron is 13.5 Mev.

4. Spherical nuclei.

We need to evaluate

$$\sum_{E} |\langle \eta' | \sum_{j=1} (\vec{\sigma}_j \cdot \vec{p}) \tau_j^- e^{-i\vec{k}\cdot\vec{x}_j} | \eta \rangle|^2$$

Under the assumption that the average nuclear excitation energy is small, we carry out this sum with k independent of E' and obtain

$$\sum_{E'} |M|^2 = \langle \eta | \sum_{\ell,m} (\vec{\sigma}_\ell \cdot \vec{p})(\vec{\sigma}_m \cdot \vec{p}) \tau_\ell^+ \tau_m^- e^{-i\vec{k}\cdot(\vec{x}_m - \vec{x}_\ell)} | \eta \rangle$$

We shall consider nuclei which have spin zero and equal number of neutrons and protons. The spherical symmetry in isotopic spin space reduces the above matrix element to the following one:

$$\sum_{E'} |M|^2 = \langle \eta | \sum_{\ell,m} (\vec{\sigma}_\ell \cdot \vec{p})(\vec{\sigma}_m \cdot \vec{p}) \tau_{z\ell} \tau_{zm} \cos \vec{k}\cdot(\vec{x}_\ell - \vec{x}_m) | \eta \rangle .$$

Call $$|\eta\rangle = \Psi(x_1 x_2 \ldots x_A)$$

the wave function of the nuclear ground state, the x's representing the space, spin and isotopic spin vari bles of the nucleons. Let $\rho_1(\vec{x}_i, \vec{x}_j)$ denote the probability density of finding a proton with spin up at $\vec{x}_i$ and another proton with spin up at $\vec{x}_j$. Call $\rho_2(\vec{x}_i \vec{x}_j)$, $\rho_3(\vec{x}_i \vec{x}_j)$, $\rho_4(\vec{x}_i \vec{x}_j)$ the analogous quantities for a proton up at $\vec{x}_i$ and a proton down at $\vec{x}_j$, a proton up at $\vec{x}_i$ and a neutron up at $\vec{x}_j$, a proton up at $\vec{x}_i$ and a neutron up at $\vec{x}_j$, **a proton up at $\vec{x}_i$ and a neutron down at $\vec{x}_j$, respec**tively. We have

$$P_1^- = \int \rho_1(\vec{x}_i \vec{x}_j) \cos \vec{k} \cdot (\vec{x}_j - \vec{x}_i) d^3x_i d^3x_j =$$

$$= \int \Psi^* \frac{1+\tau_{zi}}{2} \frac{1}{2}(1+\tau_{zj}) \frac{1}{2}(1+\sigma_{zi}) \frac{1}{2}(1+\sigma_{zj}) \Psi \cos \vec{k}\cdot(\vec{x}_j - \vec{x}_i) \times$$

$$\times d^3x_1 \ldots d^3x_A$$

whence (taking into account the spherical symmetry in spin and isotopic spin space):

$$P_1 = \frac{1}{8}\{\langle \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle + \langle\sigma_{zi}\sigma_{zj} \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle +$$

$$+\langle\tau_{zi}\tau_{zj} \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle +\langle\sigma_{zi}\sigma_{zj}\tau_{zi}\tau_{zj} \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle\}$$

where the brackets $\langle\ \rangle$ indicate expectation value in the nuclear ground state.

From similar expressions for P_2, P_3 and P_4 one obtains:

$$\langle\sigma_{zi}\sigma_{zj}\tau_{zi}\tau_{zj} \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle = 2\{P_1-P_2-P_3+P_4\} .$$

The sum of this expression over the pairs of nucleons is then

$$\sum_{ij} \langle\sigma_{zi}\sigma_{zj}\tau_{zi}\tau_{zj} \cos \vec{k}\cdot(\vec{x}_j-\vec{x}_i)\rangle = 2\left(\frac{A}{4}\right)^2\{P_1-P_2-P_3+P_4\} .$$

The exclusion principle is taken care of by the density $\rho_1(x_i x_j)$. This density has also a δ-singularity for $x_i=x_j$ which accounts for the case when the two protons with spin up are the same particle.

The transition probability for spherical light nuclei is the following:

$$\tau^{-1} = Z^4 \frac{(1 - E/\mu c^2)^4}{\left\{1+ \left(\frac{\mu}{m_\pi}\right)^2 \left(1 - \frac{2E}{\mu c^2}\right)\right\}^2} \left[1 + \left(\frac{\mu}{m_\pi}\right)^2\right]^2 \frac{Z}{2}\,(P_1-P_2-P_3+P_4)\tau_H^{-1}$$

where E is the average nuclear excitation energy.

5. Helium

For the alpha particle one has

$$P_1 = 1, \quad P_2=P_3=P_4=\int\psi^*\psi \cos \vec{k}\cdot(\vec{x}_1-\vec{x}_2)\; d^3x_1 \ldots d^3x_4 ,$$

hence

$$\tau_\alpha^{-1} = \frac{16(1-E/\mu c^2)^4}{\left\{1+\left(\frac{m_\mu}{m_\pi}\right)^2(1-2E/\mu c^2)\right\}^2}\left[1+\left(\frac{m_\mu}{m_\pi}\right)^2\right]^2 \tau_H^{-1}\int \psi^*\psi(1-\cos \vec{k}\cdot(\vec{x}_1-\vec{x}_2))\times d^3x_1 \dots d^3x_4$$

We evaluate the integral by taking an oscillator-type wave function for :

$$\psi = (\alpha^{3/2}\pi^{-3/4})^4 \, e^{-\frac{1}{2}\alpha^2(x_1^2+x_2^2+x_3^2+x_4^2)}$$

which gives

$$\int \psi^*\psi\,(1 - \cos \vec{k}\cdot(\vec{x}_1-\vec{x}_2))\,d^3x_1 \dots d^3x_4 = 1 - e^{-k^2/2\alpha^2}.$$

To determine $\alpha = \left(\frac{M\omega}{\hbar}\right)^{1/2}$ we use the virial theorem

$$\frac{1}{2} M\omega^2 R^2 = \frac{1}{2}\frac{3}{2}\hbar\omega$$

with the nuclear radius for the expectation value of r^2 in the 1s level. With E = 11 Mev we obtain τ_α = 0.022 sec for f^2 = 0.08.

6. Fermi gas model for heavier nuclei

In order to compare with present experimental data, we need the lifetime for muon capture by such nuclei as carbon, oxigen and silicium. The P's can be computed in the Fermi gas model for nuclei. In this model we have

$$\rho_1(\vec{x}_1\vec{x}_2) = \frac{2}{Z}\frac{\delta(\vec{x}_1-\vec{x}_2)}{V} - \frac{9}{V^2}\frac{1}{(K_o r_{12})^4}\left[\frac{\sin K_o r_{12}}{K_o r_{12}} - \cos K_o\, r_{12}\right]^2$$

where $K_o = \left(\frac{9\pi}{8}\right)^{1/3} r_o^{-1}$ is the radius of the momentum space Fermi sphere, V is the nuclear volume $\frac{4}{3}\pi r_o^3 A$, $r_{12} = x_1 - x_2$

We obtain

$$\frac{Z}{2}(P_1-P_2-P_3+P_4) = kr_o\left\{\frac{3}{4}\left(\frac{8}{9\pi}\right)^{1/3} - \frac{1}{18\pi}(kr_o)^2\right\}$$

The average excitation energy of a nucleus is approximately given by

$$\overline{E} = \frac{\hbar^2 k^2/2M}{\frac{Z}{2}(P_1-P_2-P_3+P_4)} \sim 16 \text{ Mev.}$$

7. Direct coupling theory

The interaction hamiltonian for the capture of muon by a nucleus, in the direct coupling theory is the following

$$H_S+H_T = g_S\int\sum_{i=1}^{A} \psi_F^*(x_1\ldots x_i\ldots x_A)\tau_i^- \psi_I(x_1\ldots x_i\ldots x_A)\varphi_\mu^*(x_i)\times$$
$$\times\beta\varphi(x_i)+g_T\int\sum_i \psi_F^*\beta_i\vec{\sigma}_i\tau_i^-\psi_I\cdot\varphi_\mu^*(x_i)\beta\vec{\sigma}\varphi_\nu(x_i).$$

for a combination of scalar and tensor couplings:

$$H_V+H_A = g_V\int\sum_i \psi_F(x_1\ldots x_i\ldots)\tau_i^-\psi_I(x_1\ldots x_2\ldots)\varphi_\mu(x_i)\varphi_\nu(x_i)+g_A\times$$
$$\times\int\sum_i\psi_F^*\vec{\sigma}_i\tau_i\psi_I\varphi_\mu^*(x_i)\vec{\sigma}\varphi_\nu(x_i)$$

for a combination of vector and axial vector couplings. ψ_F and ψ_I are the wave functions of the initial and final nuclear states.

A summation procedure analogous to the one used in 4 gives us

$$\tau_S^{-1}+\tau_T^{-1} = Z^4(1-E/\mu c^2)^2\left[(P_1+P_2-P_3-P_4)\frac{1}{\tau_S^H}+(P_1-P_2-P_3+P_4)\frac{1}{\tau_A^H}\right]\frac{Z}{2},$$

$$\tau_V^{-1}+\tau_A^{-1} = Z^4(1-E/\mu c^2)^2\left[(P_1+P_2-P_3-P_4)\frac{1}{\tau_V^H}+(P_1-P_2-P_3+P_4)\frac{1}{\tau_A^H}\right]\frac{Z}{2},$$

where

$$(g_S^2\tau_S^H)^{-1} = (g_V^2\tau_V^H)^{-1} = (3g_A^2\tau_A^H)^{-1} = (3g_T^2\tau_T^H)^{-1} = \frac{1}{2\pi^2}\frac{\mu c^2}{\hbar}\left(\frac{\mu c}{\hbar}\right)^4\left(\frac{e^2}{c\hbar}\right)^3\times$$
$$\times\frac{1}{(\hbar c)^2}.$$

Both combinations lead to predictions in agreement with experiment if the effective coupling constant, $(g_S^2+3g_T^2)^{1/2}$ or $(g_V^2+3g_A^2)^{1/2}$ is about 3.10^{-49} erg. cm^3. If it is assumed that $g_S = g_T$ or that $g_V = g_A$ then the value of each of these constants is $1.5\cdot10^{-49}$ erg. cm^3. Table I lists the results for the lifetime of muon capture by the elements indicated in the first column. The following column gives the values of τ corresponding to the value $f^2 = 0.08$ of the Chew-Low coupling constant. In the third column we list the lifetimes according to the direct coupling theory for $(g_S^2+3g_T^2)^{1/2} = 3.10^{-49}$ erg. cm^3 and average excitation energy (for carbon through calcium) of 11 Mev; in the fourth column the average excitation energy for these nuclei was taken to be 20 Mev. The last column gives the experimental results.

One sees that the capture rate in the static, cut-off pseudo-vector coupling model of the meson theory is too small by a factor 20. It is true that the comparison with observation was made possible by recourse to the ideal gas model for nuclei. It is to be expected that the true value of the nuclear matrix element will not be very different from the value given here. However, the comparison with capture by hydrogen, deuterium or helium would provide an invaluable check for the direct coupling theory. Thus the ratio of the number of muons captured in hydrogen to the number of muons which decay in electrons is about $\frac{1}{40,000}$ in Chew theory whereas the direct coupling predicts a ratio of about $\frac{1}{2,000}$.

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TABLE I

Muon Capture Lifetimes (in seconds)

	Chew-Low Theory $f^2 = 0.08$	Direct Coupling Theory $g_{eff} = 5.10^{-49}$ erg cm^3	Experiment
$_1H^1$	0.080	$0.44.10^{-2}$	
$_1H^2$	0.064		
$_2He^4$	0.022	$0.11.10^{-2}$	
$_6C^{12}$	$3.10.10^{-4}$	$0.15.10^{-4*}$ $0.20.10^{-4**}$	$0.18 \pm 0.05.10^{-4+}$
$_8O^{16}$	$10.60.10^{-5}$	$0.51.10^{-5*}$ $0.70.10^{-5**}$	$1.35 \pm 0.76.10^{-5+}$
$_{14}Si^{28}$	$15.08.10^{-6}$	$0.72.10^{-6*}$ $0.98.10^{-6**}$	$0.83 \pm 0.21.10^{-6+}$
$_{20}Ca^{40}$	$5.14.10^{-6}$	$0.25.10^{-6*}$ $0.34.10^{-6**}$	$0.41 \pm ?\ .10^{-6++}$

* Average nuclear excitation energy = 11 Mev.

** Average nuclear excitation energy = 20 Mev.

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