A CAUSAL INTERPRETATION OF THE PAULI EQUATION * (A)

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(November 15,1954)

In this paper, we develop a causal interpretation of the Pauli equation, in terms of the model of a fluid having a characteristically new property of "intrinsic angular momentum", arising from the spin of quasi-rigid bodies, of which the fluid is assumed to be constituted. The two-component spinor appearing in the Pauli equation is interpreted in terms of a specification of the density of the fluid, and of the three Euler angles representing the orientations of the bodies at each point in space. We then show that with these assumptions, the Pauli equation

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* Most of this work was accomplished when the latter author (J. T.) was at the Faculdade de Filosofia, Ciências e Letras, Universidade de São Paulo.

§ Submitted for publication to IL NUOVO CIMENTO together with a subsequent paper of the same title (B) by Bohm and Schiller.

implies that the spin angular momentum always points along the principal axis of the body. The equations of motion of the spin are then seen to take a rather natural form, involving a coupling between the translational and rotational motions of the bodies. This coupling is produced by a spin dependent addition to the "quantum potential" of a simple type that leads to a tendency for the spins of neighboring bodies to become parallel. The torques resulting from this term are such that the condition that the angular momentum shall always point along the principal axis of the body is preserved automatically by the equations of motion as a consistent subsidiary condition.

I. Introduction

In various previous papers[1-8], several new interpretations of the non-relativistic quantum theory without spin have been proposed. Although these new interpretations differ in various details, they all have in common that they explain the quantum theory in terms of continuous and casually determined motions of various kinds of entities such as fields and bodies, which are assumed to exist objectively at the microscopic level. Thus far, those interpretations which have been carried far enough to demonstrate in full detail their ability to explain causally all features of the Schrödinger equation without spin have followed one of two general lines of approach.

The first of these general lines, initiated originally by de Broglie [1], and later carried to its logical conclusions by one of the authors of the present paper[3], involves the notion that the Schrödinger wave function, $\psi(\underset{\sim}{x},t)$, represents an objectively real but qualitatively new kind of field of force that influences the motion of a body, which latter has a well defined location, $\xi(t)$, varying continuously and in a new causally determined way with the passage of time. This new field of force produces no important effects at the macroscopic level, but at the atomic level it is, as has been shown in various papers referred to above, able to explain the characteristic new quantum-mechanical properties of matter, which manifest themselves strongly only at this level.

The second general type of causal explanation of the quantum theory is along the lines of the hydrodynamic model; proposed originally by Madelung[2], and later extended by Takabayasi[4] and by Schönberg[7]. In this model, $|\psi|^2$ represents the density of a fluid, while $\nabla S/m$ represents its local stream velocity (where $\psi = R e^{iS/\hbar}$). In another paper[8], this model has been completed with the aid of the postulate that there is a stable particle-like inhomogeneity in the fluid that moves with the local stream velocity. This inhomogeneity plays a role analogous to that played by the body in the interpretations initiated by de Broglie, while the fluid plays a role analogous to that played by the ψ field.

In the present paper we shall find it convenient to work in terms of the hydrodynamic model, which we shall extend with the aid of various new postulates to the Pauli equation describing an electron with spin. We shall therefore now summarize a few important aspects of the hydrodynamic model without spin, in order to facilitate the description of the new features that are needed for the treatment of spin. We first write Schrödinger's equation in the well-known form

(1) $$\frac{\partial R^2}{\partial t} + i\, \mathrm{div}\left(R^2 \nabla S/m\right) = 0$$

(2). $$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \nabla^2 R = 0$$

In terms of the Madelung interpretation, equation (1) then represents the conservation of fluid, while (2) represents the equation which determines the velocity potential, S., in terms of the classical potential, V, and the "quantum potential",

$$U = -\frac{\hbar^2}{2m} \nabla^2 R/R = \frac{\hbar^2}{4m}\left[(\nabla \rho)^2/2\rho^2 - \nabla^2 \rho/\rho \right]$$

As shown by Takabayasi[4] and by Schönberg[7], the quantum potential can be interpreted in terms of a kind of internal stress in the fluid, which depends not, however, on ρ itself, as is the case with pressures and other internal stresses in an ordinary macroscopic fluid, but rather, on derivatives of ρ.

In one of the papers referred to previously[8], the additional assumption was made that the fluid is in a state of irregular fluctuation resembling turbulent motion, in which the actual density ρ, and the actual velocity $\underset{\sim}{v}$, fluctuate more or less at random around $|\psi|^2$ and $\nabla S/m$ respectively as means. In these fluctuations, it is permissible to assume that $\underset{\sim}{v}$ has vortex motions, which however average out to zero. As a result of these fluctuations, elements of fluid on any one of the mean lines of flow associated with the Madelung fluid are continually moving in an irregular way to other lines of flow, and this process tends to produce a more or less random "mixing" of the fluid. Like one of the fluid elements the body-like inhomogeneity, which is carried along by the fluid, follows a very irregular path. It is then quite easily shown that a statistical ensemble of such systems with an arbitrary probability density, $P(x)$, of inhomogeneities eventually decays into one with $P = |\psi|^2$. Thus, the irregular fluctuations in the motions of the Madelung fluid provide a model explaining in a natural way a possible physical origin for the statistical distributions of the quantum theory [9].

2. Introduction of "Intrinsic Spin" in Hydrodynamical Model

In the present paper (and in a subsequent one), we shall extend the hydrodynamic model to a treatment of the Pauli equation for a spinning electron. Now, the Pauli equation is [10]

$$i\hbar \frac{\partial \psi_a}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - ie\underset{\sim}{A}/c\right)^2 \psi_a + V\psi_a + \frac{e\hbar}{2mc}\left(\underset{\sim}{\sigma}\cdot\underset{\sim}{\mathcal{H}}\right)\psi_a \tag{3}$$

where ψ_a represents a two component spinor with index a, while A is the electromagnetic vector potential, V, the total external potential energy (electromagnetic and otherwise), and $\underset{\sim}{\mathcal{H}}$ the magnetic field.

As in the case of the Madelung model of Schrödinger's equation we must define a fluid density and a fluid velocity. Now, it is well known that the Pauli equation admits a conserved charge and current given (except for a factor of e) by

$$\rho = \psi^* \psi \tag{4a}$$

$$\underset{\sim}{\vec{j}} = \frac{\hbar}{2mi}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) - \frac{e}{c}\underset{\sim}{A} \tag{4b}$$

Writing $\underset{\sim}{j} = \rho \underset{\sim}{v}$, we may then define the velocity as

$$\underset{\sim}{v} = \frac{\hbar}{2mi} \frac{(\psi^* \nabla \psi - \psi \nabla \psi^*)}{\psi^* \psi} - \frac{e}{c} \underset{\sim}{A} \tag{4c}$$

It is then evidently consistent to assume that the fluid density is given by $\rho = \psi^* \psi$, since the conservation equation obtained from the Pauli equation can then be interpreted as describing the conservation of fluid.

ρ and $\underset{\sim}{v}$ do not, however, permit a complete definition of the physical meaning of the ψ function; for since ψ is a spinor, it contains four independent quantities, as indicated below

$$\psi = \begin{pmatrix} a + ib \\ c + id \end{pmatrix} \tag{5}$$

where the a,b,c,d are <u>real</u>. On the other hand, (4a) defines only $\rho = a^2 + b^2 + c^2 + d^2$ while (4b) defines only the derivatives of the a,b,c,d.

In order to define all parts of ψ more directly in terms of physical properties of the fluid, we shall therefore have to assume some new properties for the fluid. Now, since the Pauli equation deals with the electron spin, it seems natural to assume that our fluid should have a new property, which we may call in <u>intrinsic angular momentum</u> connected with the spin. By this, we mean that the total angular momentum density of the fluid should include, in addition to the "orbital" contribution of $m\rho(\underset{\sim}{r} \times \underset{\sim}{v})$ an additional contribution depending on some parameters connected with the internal motions of each fluid element. To obtain a possible physical picture of where such an intrinsic angular momentum could come from, we may suppose, for the sake of discussion, that the fluid is constituted of molecules, and that the spin motion of the constituent molecules may contribute to the total angular momentum of the system. On the other hand, the same result could be achieved in a much more general way. For if the fluid has an inhomogeneous structure, then these inhomogeneities will in general have a certain inertia. If a fluid element is turning, the inhomogeneities will turn with it and contribute to the total angular momentum. Such inhomogeneities might, for example, be stable or semi-stable pulse-like structures formed in the fluid itself, or they might be small highly localized vortices or eddies. As far as our purposes in this paper are concerned, however, the origin of the intrinsic angular momentum of the fluid is irrelevant. All that is relevant is the assumption that, for one reason or another, such an intrinsic angular momentum exists. In order to have a convenient model in terms of which we can work, however, we shall assume in this paper that the intrinsic angular momentum is due to the turning of very small quasi-rigid bodies of which the fluid is supposed to be constituted (just as ordinary macroscopic fluids are constituted of molecules). We shall see that this simplified model is adequate for giving a causal explanation of the Pauli equation. On the other hand, there is no reason inherent in the model why the bodies must always be regid; or, indeed, even why the inhomogeneities must necessarily be regarded as arising in distinct bodies out of which the fluid is supposed to be constituted. Nevertheless, under the conditions in which the Pauli equation applies, we shall assume that the fluid acts so nearly as if it were composed of distinct rigid spinning bodies, that we may use this model as a simplifying abstraction, as one, for example, frequently simplifies the treatment of atoms by replacing them by idealized mass points even though they really have finite sizes.

3. Kinematic Description of Rotations

We shall now proceed to develop a kinematic description of the rotations of a body, which is particularly easy to apply to the Pauli equation, because it works in terms of spinors of the same kind as appear in this equation.

The first problem is to specify the state of rotation of a body. This can be done in terms of the three Euler angles, θ, ϕ and ψ, where θ represents the angle of the first principal axis with a Z axis fixed in space, ϕ represents the angle that the projection of this axis makes on the X-Y plane, and ψ represents the angle of rotation about the first principle axis relative to the intersection of the plane of the second and third principal axes with the X-Y plane.

We wish now, however, to connect these angles to a spinor. To do this, let us imagine that we always start with the body in a standard orientation, in which its first principal axis is directed along the Z axis, which is fixed in space. We then make a rotation, $R(\theta, \phi, \psi)$ which carries the body from its standard orientation to its actual orientation. This rotation can be carried out in three steps. First, we make a rotation, $R_1(\psi)$ through an angle ψ about the Z axis fixed in space (which is at this time also the first principal axis of the body). Then we make a rotation $R_2(\theta)$ about the X axis, of an angle, θ. Then we make a rotation $R_3(\phi)$ again about the Z axis fixed in space (but this time the principal axis of the body is no longer parallel to the Z axis.

If the reader will draw a diagram, he will readily convince himself that these rotations give just those described by the Euler angles[11].

Let us now carry out these operations mathematically. We shall assume that the state of the body at rest is to be represented by what we shall call the standard unit spinor[12] $\beta_0 = \begin{pmatrix}1\\0\end{pmatrix}$. When we carry out the rotations described above, we shall see that a rotation involving arbitrary angles, θ, ϕ, ψ can lead to an arbitrary unit spinor (such that $\beta^*\beta = 1$). Thus, any unit spinor can be interpreted as a specification of the angles of rotation of a body. In terms of spinor notation, the first rotation $R_1(\psi)$, is represented by the matrix

$$e^{i\sigma_z \psi/2} = \cos \psi/2 + i\sigma_z \sin \psi/2$$

Applying this to the standard unit spinor, we get

$$R_1(\psi)\beta_0 = e^{i\sigma_z \psi/2}\begin{pmatrix}1\\0\end{pmatrix} = e^{i\psi/2}\begin{pmatrix}1\\0\end{pmatrix} \tag{6}$$

We now apply the rotation, $R_2 = e^{i\sigma_x \theta/2} = \cos\frac{\theta}{2} + i\sigma_x \sin\frac{\theta}{2}$

We get

$$R_2R_1\beta_0 = e^{i\psi/2}\begin{pmatrix}\cos\theta/2 & i\sin\theta/2\\ 2\sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = e^{i\psi/2}\begin{pmatrix}\cos\theta/2\\ i\sin\theta/2\end{pmatrix} \tag{7}$$

Applying $R_3 = e^{i\sigma_z\phi/2}$, we get

$$\beta = R\beta_0 = \begin{pmatrix}\cos\theta/2 & e^{i(\psi+\varphi)/2}\\ i\sin\theta/2 & e^{i(\psi-\phi)/2}\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \tag{8}$$

Writing

$$\beta = \begin{pmatrix}b_1 + ib_2\\ b_3 + ib_4\end{pmatrix} \tag{8a}$$

We get

$$b_1 = \cos\frac{\theta}{2}\cos\frac{(\psi+\phi)}{2} \qquad b_2 = \cos\frac{\theta}{2}\sin\frac{(\psi+\phi)}{2}$$

$$b_3 = -\sin\frac{\theta}{2}\sin\frac{(\psi+\phi)}{2} \qquad b_4 = \sin\frac{\theta}{2}\cos\frac{(\psi-\phi)}{2} \tag{8b}$$

The b_i evidently satisfy the relation,

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1 \tag{9}$$

The b_i are just the well-known Cayley-Klein parameters of the rotation group[13], of which evidently only three are independent. By specifying the b_i we can solve for the Euler angles θ, ϕ, ψ. Indeed, the specification is two valued, in the sense that there are <u>two</u> values of the b_i for each set of Euler angles.

Thus, we see that a unit spinor can be interpreted as defining the orientation of a body in space. We shall therefore tentatively interpret the spinor appearing in the Pauli equation in this way, and then show that a consistent interpretation can so be obtained[14].

4. <u>Canonical Relations for the Cayley-Klein Parameters</u>

Now, we shall set up a <u>classical</u> canonical formalism, in which the Pauli equation is used to define the equations of motion of the Cayley-Klein parameters, and therefore the equations governing the rotations of the bodies making up the fluid.

Our first step is to write down a classical Hamiltonian from which the Pauli equation can be derived. This Hamiltonian is formally the same function of the spinor ψ as that which appears in the Hamiltonian from which the Pauli equation is derived in the usual form of the quantum theory. This is

$$H = \int \left[\frac{\hbar^2}{2m} \left| (\nabla - ie\underset{\sim}{A}/c\,\Psi \right|^2 + V\Psi^*\Psi + \frac{e\hbar}{2mc}\Psi^+ \underset{\sim}{\sigma}\cdot\underset{\sim}{\mathcal{H}}\Psi \right] d\underset{\sim}{x} \quad (10)$$

The above Hamiltonian will lead to the correct equation for pro- vided that we assume the classical Poisson bracket relations

$$[\Psi_a(\underset{\sim}{x}'), \Psi_b(\underset{\sim}{x})] = \frac{\delta_{ab}}{i\hbar}\delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad (11)$$

For then we obtain just the Pauli equation

$$i\hbar\frac{\partial \Psi_a}{\partial t} = i\hbar[H, \Psi_a] = -\frac{\hbar^2}{2m}(\nabla - ie\underset{\sim}{A}/c)^2\Psi_a + V + \frac{e\hbar}{2mc}\left[(\underset{\sim}{\sigma}\cdot\underset{\sim}{\mathcal{H}})\Psi\right]_a$$

where we have used the fact that the Poisson bracket of Ψ_a with an integral such as H is defined with the aid of the well-known functional derivatives. Thus

$$[H,F] = \frac{\delta H}{\delta\Psi}\frac{\delta F}{\delta\Psi^*} + \frac{\delta H}{\delta\Psi_2}\frac{\delta F}{\delta\Psi_2^+} + \frac{\delta H}{\delta\Psi_1^*}\frac{\delta F}{\delta\Psi_1} + \frac{\delta H}{\delta\Psi_2^*} + \frac{\delta F}{\delta\Psi_2} \quad (12)$$

In order to simplify the treatment, we now split the whole space into volume elements so small that Ψ does not change appreciably with- in them. We may then define $\Psi(\underset{\sim}{x}_m)$ as the mean value of Ψ in such a region, centering on the point $\underset{\sim}{x}_m$. By integrating equ. (11) over small regions of $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ centered respectively at $\underset{\sim}{x}_m$ and $\underset{\sim}{x}_n$ and dividing by $(\Delta v)^2$ we get

$$\left[\Psi_a^*(x_m), \Psi_b(x_n)\right] = \frac{\delta_{ab}}{i\hbar} \frac{\delta(\underset{\sim}{x}_m, \underset{\sim}{x}_n)}{(\Delta v)}$$

We shall now find it convenient to introduce a new spinor

$$\chi = \sqrt{\hbar \Delta v}\ \Psi \tag{13}$$

We obtain

$$\left[\chi_a^*(\underset{\sim}{x}_m), \chi_b(\underset{\sim}{x}_n)\right] = i\,\delta_{ab}\,\delta(\underset{\sim}{x}_m - \underset{\sim}{x}_n) \tag{14}$$

We then write

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 + i a_2 \\ a_3 + i a_4 \end{pmatrix} \tag{15}$$

From the Poisson bracket relations, (14), we then deduce that

$$\begin{aligned} &[a_1, a_2] = 1 \qquad ; \qquad [a_3, a_4] = 1 \\ &[a_1, a_3] = [a_1, a_4] = [a_2, a_3] = [a_2, a_4] = 0 \end{aligned} \tag{16}$$

Thus a_1 is the momentum canonically conjugate to a_2 and a_3 is canonically conjugate to a_4. The a_i are evidently proportional to the Cayley-Klein parameters b_i (given by equ. (8a)). Thus, we have found the canonical relations among the a_i, implied by the Pauli equation.

5. Definition of Spin Angular Momentum

In order to motivate further steps in the interpretation of the

Pauli equation, let us recall that in the usual quantum mechanics, the total spin is given by $\frac{\hbar}{2}\int \Psi^* \underset{\sim}{\sigma} \Psi \, d\underset{\sim}{x}$ This suggests that $\frac{\hbar}{2}\Psi^* \underset{\sim}{\sigma} \Psi$ is a spin density, and that $\underset{\sim}{S} = \frac{\hbar}{2}(\Psi^* \underset{\sim}{\sigma} \Psi)\Delta v$ is the total spin in the small element of volume Δv. Expressing Ψ in terms of χ through equ. (13), we get

$$\underset{\sim}{S} = \frac{1}{2}(\chi^* \underset{\sim}{\sigma} \chi) \tag{17}$$

The total number of particles in this region is just

$$\Delta N = \Psi^* \Psi \Delta v = \chi^* \chi / \hbar \tag{18}$$

Thus, the spin per particle is

$$\underset{\sim}{s} = \frac{\hbar}{2} \frac{\chi^* \underset{\sim}{\sigma} \chi}{\chi^* \chi} = \frac{\hbar}{2} \frac{\Psi^* \underset{\sim}{\sigma} \Psi}{\Psi^* \Psi} \tag{19}$$

We see then that the maximum spin per body in a given direction is $\hbar/2$. This is in agreement with what one gets for the spin "observable" in the usual quantum theory[15].

With the aid of the P.B. relations (11), it can be shown by means of a simple calculation that the components of the spin vectors $\underset{\sim}{S}$, satisfy the cyclical relations, characteristic of angular momenta

$$[S_x, S_y] = S_z \tag{20}$$

Thus, the Poisson-Bracket relations obtained in the derivation of the Pauli equations from a hamiltonian are just what are needed to lead to the correct P.B. relations for the angular momenta defined in equ. (17).

In the present theory, the basic spinor

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 + i a_2 \\ a_3 + i a_4 \end{pmatrix},$$

contains only <u>two</u> canonically independent variables. We shall now see the physical meaning of this reduction of the number of basic variables to two. To do this, we find it convenient to go to a new set of canonically independent variables which we take to be two of the Euler angles ψ and ϕ. In terms of the a_i these are (as can easily be seen from eq. (8b)

$$\psi = \tan^{-1}\frac{a_2}{a_1} + \tan^{-1}\frac{a_4}{a_3}\ ; \quad \phi = \tan^{-1}\frac{a_2}{a_1} - \tan^{-1}\frac{a_4}{a_3} \tag{21}$$

By means of a simple calculation, we prove that the quantity canonically conjugate to $-\frac{\psi}{2}$ is $\rho\hbar$, which is proportional to the total number of particles in the region (See equ.(18)), while the quantity canonically conjugate to $-\frac{\phi}{2}$ is S_z, which is equal to the z component of the total angular momentum of these particles. Thus, we have

$$\hbar\left[\rho, -\frac{\psi}{2}\right] = \left[\left(\frac{a_1^2+a_2^2+a_3^2+a_4^2}{2}\right), \left(\tan^{-1}\frac{a_2}{a_1} + \tan^{-1}\frac{a_4}{a_3}\right)\right] = 1 \tag{22a}$$

$$\left[S_z, -\frac{\phi}{2}\right] = \left[\left(\frac{a_1^2+a_2^2-a_3^2-a_4^2}{2}\right), \left(\tan^{-1}\frac{a_2}{a_1} - \tan^{-1}\frac{a_4}{a_3}\right)\right] = 1 \tag{22b}$$

But now we can show by a simple calculation that ϕ is also equal to the angle ϕ' made by the projection of the angular momentum vector, $\underset{\sim}{S}$ on the X-Y plane. Thus

$$\tan\phi' = \frac{\psi^* \sigma_y \psi}{\psi \sigma_x \psi} = i\,\frac{\begin{pmatrix} a_1 & -ia_2 \\ a_3 & -ia_4 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & +ia_2 \\ a_3 & +ia_4 \end{pmatrix}}{\begin{pmatrix} a_1 & -ia_2 \\ a_3 & -ia_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & +ia_2 \\ a_3 & +ia_4 \end{pmatrix}} = \frac{a_2a_4 - a_1a_3}{a_1a_3 + a_2a_4}$$

Now, from eq. (21)

$$\tan\phi = \frac{a_2/a_1 - a_3/a_4}{1 + \dfrac{a_2}{a_1}\dfrac{a_3}{a_4}} \qquad \frac{a_2a_4 - a_1a_3}{a_1a_3 + a_2a_4}$$

Thus we see that $\phi' = \phi$. Moreover, the co-latitude angle, θ', of the angular momentum vector, is defined by

$$\cos\theta' = \frac{S_z}{S} = \frac{a_1^2 + a_2^2 - a_3^2 - a_4^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

But from eq. (8b), we can calculate the angle, θ, made by the prin - cipal axis (1) of the body with the z axis. This is given by

$$\cos\theta = \frac{a_1^2 + a_2^2 - a_3^2 - a_4^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

Thus, $\theta' = \theta$.

We conclude then that in the Pauli theory, the angular momentum $\underset{\sim}{S}$, is always pointing along the principal axis (1) of the body. Such a special orientation of the angular momentum cannot be maintained for the most general kind of torque that may act on the body. In the following section, and in a subsequent paper, we shall see, however, that the Pauli equation implies a special kind of "quantum-mechanical torque" that permits this condition to be maintained as a consistent subsidiary condition.

The special relation between the direction of $\underset{\sim}{S}$ and the principal axis (1) is what permits us to reduce the number of independent canonical variables in the theory. When this relation exists, our theory becomes equivalent to a theory of the angular momentum of a point dipole, which has already been treated by several other authors.[16,17,18] In the subsequent paper we shall see, however, that the theory developed here can be generalized to treat cases in which $\underset{\sim}{S}$ does not necessarily point along the principal axis.

6. Relationship Between Spin and Velocity

Let us now recall that eq. (4c) defines the velocity of a particle in terms of our spinor. Since the spinor is already interpreted in terms of the orientation of a spinning body, eq. (4c) clearly implies a certain relationship between the spin and the velocity. We shall now study this relationship and show that it can be understood in terms of a reasonable physical model.

We begin by expressing our spinor (with the aid of eq. (7)) as

$$\psi = R\begin{pmatrix} \cos\theta/2 \, e^{i(\psi+\phi)/2} \\ i\sin\theta/2 \, e^{i(\psi-\phi)/2} \end{pmatrix} \qquad \text{Where R is real}$$

Using this value of ψ in eq. (4c), we readily get

$$\underset{\sim}{v} = \frac{\hbar}{2m}\left(\nabla\psi + \cos\theta\,\nabla\phi\right) - \frac{e}{c}\underset{\sim}{A} \tag{23}$$

The above, is however an expression of the velocity in a form that has already long been familiar in classical hydrodynamics. Indeed, in classical hydrodynamics, it is shown [19] that an arbitrary velocity field can always be expressed as

$$\underset{\sim}{v} = \nabla S + \xi \nabla \eta - \frac{e}{c} \underset{\sim}{A} \tag{24}$$

where S, ξ, η are scalars, called the Clebsch parameters.

The curl of the velocity is then

$$\nabla \times \underset{\sim}{v} = \nabla \xi \times \nabla \eta - \frac{e}{c} \nabla \times \underset{\sim}{A} \tag{25}$$

(This is quite a different expression from the more common form $\underset{\sim}{v} = \nabla S' + \nabla \times \underset{\sim}{B} - \frac{e}{c} \underset{\sim}{A}$ in the sense that S' and S are different functions, since $\mathrm{div}(\xi \nabla \eta)$ is not in general equal to zero).

The Clebsch parameters have recently been used by Dirac[20] and others[21] to treat a theory of a classical electrified fluid ether. They have also been used by Takabayasi[4] to treat possible vortices in the Medelung fluid, and by Schönberg[7] in a more general context.

In order to establish a basis of comparison with the Pauli theory we shall first treat the ordinary classical hydrodynamics of a charged fluid, capable of maintaining a pressure gradient, in terms of the Clebsch parameters. One of the advantages of these parameters is that they permit the formulation of hydrodymamics in terms of a variational principle. Indeed, if ρ is the density of the fluid, then the Hamiltonian is just equal to the total kinetic plus potential energy of the fluid (including the potential energy due to compression).

$$H_c = \int \left[\frac{\rho}{2m}\left(\nabla S + \xi \nabla \eta - \frac{e}{c}\underset{\sim}{A}\right)^2 + 2e\phi + f(\rho) \right] d\underset{\sim}{x} \qquad (26)$$

where $\underset{\sim}{A}$ is the vector potential, ϕ, the scalar potential, $\frac{e}{m}$ the ratio of charge to mass for an element of the fluid in question, and $f(\rho)$ defines the pressure through $\rho = \partial f/\partial \rho$

Now if we adopt essentially the same Poisson Brackets as were derived from the Pauli theory, i.e.,

$$[\rho(\underset{\sim}{x}), S(\underset{\sim}{x}')] = \delta(\underset{\sim}{x} - \underset{\sim}{x}')\,;\ [\rho(\underset{\sim}{x})\xi(\underset{\sim}{x}), \eta(\underset{\sim}{x}')] = \delta(x - x') \qquad (27)$$

then we obtain the following equations of motion

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \left(\nabla S - \frac{e}{c}\underset{\sim}{A} + \xi \nabla \eta\right) = \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underset{\sim}{v} = 0 \qquad (28a)$$

$$\frac{\partial S}{\partial t} - \xi(\underset{\sim}{v}\cdot\nabla)\eta + \frac{\left(\nabla S + \xi\nabla\eta - \frac{e}{c}\underset{\sim}{A}\right)^2}{2m} + \frac{e}{m}\phi + \partial f/\partial\rho = 0 \qquad (28b)$$

$$\frac{\partial \xi}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\xi = \frac{d\xi}{dt} = 0 \qquad (28c)$$

$$\frac{\partial \eta}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\eta = \frac{d\eta}{dt} = 0 \qquad (28d)$$

Combining (28b) and (28d) we get

$$\frac{\partial S}{\partial t}+\xi\frac{\partial\eta}{\partial t}+\frac{(\nabla S+\xi\nabla\eta-\frac{e}{c}\underset{\sim}{A})^2}{2m}+\frac{e}{m}\varphi+\frac{\partial f}{\partial\rho}=0 \tag{28e}$$

Eq. (28a) is just the equation of conservation of fluid. Eq. (28b) (and therefore also (28e)), is a generalization of the equation for the velocity potential, S, which enables us to take veracity into account (through ξ and η). Eqs. (28c) and (28d) are very interesting, for they say that <u>if we follow a moving fluid element,</u> ξ and η are constants of the motion (but not at a point fixed in space). Thus, the surfaces ξ (x,y,z,t) = const. and η = η(x,y,z,t) = const. define the tubes of flow of the fluid.

To understand the motion in more detail, let us focus our attention on eq. (28e). This is analogous to a Hamiltonian-Jacobi equation but the equation involves, in addition to the electromagnetic potentials, and the pressure potential ($\frac{\partial f}{\partial\rho}=h$), a set of equivalent potentials that are functions of the Calebsch parameters[22]. In fact, the Clebsch parameters contribute an addition of $\xi\frac{\partial\eta}{\partial t}$ to the scalar potential and $-\xi\nabla\eta$ to the vector potential. We may thus conclude that formally they contribute to the equation of motion of a fluid element a "pseudo Lorentz force", given by

$$\underset{\sim}{F}' = (\underset{\sim}{\mathcal{E}}'+\underset{\sim}{v}\times\underset{\sim}{\mathcal{H}}') \tag{29}$$

where the "pseudo electric" and "pseudo magnetic" fields are given by

$$\mathcal{E}' = -\nabla\left(\xi\frac{\partial\eta}{\partial t}\right)+\frac{\partial}{\partial t}\left(\xi\nabla\eta\right) = \frac{\partial\xi}{\partial t}\nabla\eta-\frac{\partial\eta}{\partial t}\nabla\xi$$

$$\mathcal{H}' = -\nabla\times(\xi\nabla\eta) = -\nabla\xi\times\nabla\eta$$

Thus, for the additional force, we get

$$F' = \frac{\partial \xi}{\partial t} \nabla \eta - \frac{\partial \eta}{\partial t} \nabla \quad - v \times (\nabla \xi \times \nabla \eta) \tag{30}$$

Writing

$$- v \times (\nabla \xi \times \nabla \eta) = (\underset{\sim}{v} \cdot \nabla \xi) \nabla \eta - (\underset{\sim}{v} \cdot \nabla \eta) \nabla \xi$$

we obtain

$$F' = \left(\frac{\partial \xi}{\partial t} + (\underset{\sim}{v} \cdot \nabla) \xi\right) - \left(\frac{\partial \eta}{\partial t} + (\underset{\sim}{v} \cdot \nabla) \eta\right) = \frac{d\xi}{dt} \nabla \eta - \frac{d\eta}{dt} \nabla \xi$$

Since from (28c) and (28d) we have $\frac{d\xi}{dt} = 0$ and $\frac{d\eta}{dt} = 0$, we get $F' = 0$. Thus the Clebsch parameters formally add to the vector potential a set of quantities that produce a pseudo-electromagnetic field for which the pseudo-Lorentz-force vanishes. Therefore the Clebsch parameters appearing in equation (28e) do not alter the equations of motion of a fluid element [23]. What they do is to enable us to consider in a canonical formalism an ensemble of trajectories for the fluid elements which have a vorticity that does not come from the effects of the electromagnetic vector potential, $\underset{\sim}{A}$. Thus, they make possible a generalization of the Hamilton-Jacobi theory; for in the latter, we consider only ensembles in which $m\underset{\sim}{v} - \frac{e}{c}\underset{\sim}{A}$ is derivable from a potential (equal to the action function).

Let us now return to the Pauli theory. Our first problem will be to compare the classical Hamiltonian (26) with the Pauli hamiltonian (10) By means of a simple calculation, we obtain the result that the Pauli hamiltonian can be written as the sum of the following three terms:

$$H = H_T + H_q + H_{sp}$$

where we have

$$H_T = \int \frac{\hbar^2}{2m} \rho' \left(\frac{\nabla \Psi}{2} + \cos\theta \frac{\nabla \varphi}{2} - \frac{e}{c} \underset{\sim}{A} \right)^2 d\underset{\sim}{x} \tag{31a}$$

$$H_q = \int \frac{\hbar^2}{8m} \frac{(\nabla \rho')^2}{\rho'} \, d\underset{\sim}{x} \tag{31b}$$

$$H_{sp} = \int \rho' \Big[\frac{\hbar^2}{8m} \left((\nabla\theta)^2 + \sin^2\theta (\nabla\varphi)^2 \right) + \frac{e\hbar}{2mc} \Big(\cos\theta H_z + \tag{31c}$$

$$+ \sin\theta \cos\varphi H_x + \sin\theta \sin\varphi H_y \Big) \Big] d\underset{\sim}{x}$$

and where $\underset{\sim}{H}$ is the magnetic field.

In the above equations, we have, as in eq. (4), defined

$$\rho' = |\Psi|^2 = \rho$$

where ρ' is the density of bodies.

In order to compare the Pauli hamiltonian to the simple hamiltonian, H_c, of classical hydrodynamics, we note that ρ' is canonically conjugate to $-\frac{\Psi}{2}$, $(\rho' \cos\theta)$ to $-\frac{\phi}{2}$. Thus, if we replace ρ by ρ', and write

$$\xi = \cos\theta \;, \quad \frac{\eta}{\hbar} = -\frac{\varphi}{2} \;, \quad \frac{S}{\hbar} = -\frac{\Psi}{2} \tag{32}$$

then H_T becomes equal to kinetic energy term in eq. (26).

The terms that are left over in the Pauli equation are then H_q and H_{sp}. We shall see in eq. (34b) that H_q leads to the "quantum-potential" term of the Madelung fluid. Thus, the two terms, $H_T + H_q$ represent the energy of a charged Madelung fluid, in which fortex flow as well as potential flow are taking place.

As for the third term, H_{sp}, it is clear first of all that it contains a part describing the energy of a spinning dipole in a magnetic field $\underset{\sim}{H}$. The magnetic moment of this dipole is $e\hbar/2mc$.

If we assume that our bodies have a radius of the order of the classical electronic radius, $r_o = e^2/mc^2 = 2.8 \times 10^{-13}$cm, then the magnetic moment is $\mu = \lambda e r_o v/c$, where v is the velocity of rotation of the body at its periphery, and where λ is a constant of the order of unity depending on the charge distribution in the body. We then obtain

$$\frac{v}{c} = \frac{\hbar}{\lambda m c r_o} = \frac{137}{\lambda}$$

Thus, the body must, at its periphery, be moving much faster than light. In a non-relativistic theory, such as we are discussing in the present paper, this of course leads to no difficulties. In a later paper, concerned with the extension of the theory to the Dirac equation, we shall see that the spin motion becomes closely connected with the mass motion of the fluid, so that a spin angular momentum of eh/2mc for the body is possible, even when the speed of motion of its periphery does not exceed the speed of light, because the fluid motions are further connected with the body rotations by a spin-orbit coupling, no present in the non-relativistic theory.

As for the remaining part of H_{sp}, this represents an interaction between the directions of the spins of neighboring bodies, which functions essentially as a spin-dependent addition to the "quantum potential". To obtain a more definite model for this part of the spin energy, we may let δA represent the total angle between two spin vectores, separated by a distance, $\delta \underset{\sim}{x}$. Then, by going to spherical polar coordinates in the space of the spin vectors, we can show that

$$|\delta A|^2 = |\nabla\theta\cdot\delta \underset{\sim}{x}|^2 + \sin^2\theta\,|\nabla\varphi\cdot\delta \underset{\sim}{x}|^2$$

Thus, the energy, H_{sp} is just an average of $|\delta A|^2$, taken over a small region that surrounds the body and weighted with the density, ρ. We can therefore interpret H_{sp} physically as the result of an assumed short-range interaction between bodies, which tends to line up the direction of their spins. For example, the energy of two neighboring bodies could be proportional to

$$U_{\text{Interaction}} = -\cos\delta A \cong -1 + (\delta A)^2/2 \tag{33}$$

Let us now obtain the equations of motion that follow from our hamiltonian. To do this, it will be convenient to write H as

$$H = \int \frac{\rho'}{2m}\left(\nabla S + \xi\nabla\eta - \frac{e}{c}\underset{\sim}{A}\right)^2 d\underset{\sim}{x} + \frac{\hbar^2}{8m}\int\frac{(\nabla\rho')^2}{\rho'}d\underset{\sim}{x} + \int\rho' H_s\, d\underset{\sim}{x}$$

where

$$H_s = \frac{\hbar^2}{2m}\left[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2\right] + \frac{e}{mc}\underset{\sim}{s}\cdot\underset{\sim}{H} = \frac{\hbar^2}{2m}\left[\frac{(\nabla\xi)^2}{1-\xi^2} + (1-\xi^2)(\nabla\eta)^2\right] + \frac{e}{mc}\underset{\sim}{s}\cdot\underset{\sim}{H}$$

The equations of motion then become

$$\frac{\partial\rho'}{\partial t} + \operatorname{div}\rho'\underset{\sim}{v} = 0 \tag{34a}$$

$$\frac{\partial S}{\partial t} - \frac{\xi(\underset{\sim}{v}\cdot\nabla)\eta}{m} + \frac{mv^2}{2} - \frac{\hbar^2}{4m}\left(\frac{\nabla^2\rho'}{\rho'} - \frac{1}{2}\frac{(\nabla\rho')^2}{(\rho')^2}\right) + H_s = 0 \tag{34b}$$

$$\frac{d\xi}{dt} = -\frac{\delta H_s}{\delta\eta} \tag{34c}$$

$$\frac{d\eta}{dt} = \frac{\delta H_s}{\delta \xi} \tag{34d}$$

Eq. (34a) is then just the conservation equation while eq. (34b) is the pseudo Hamilton-Jacobi equation for a fluid with the usual quantum potential,

$$-\left(\frac{\hbar^2}{2m}\frac{\nabla^2 \rho'}{\rho'} - \frac{1}{2}\frac{(\nabla \rho')^2}{\rho'^2}\right.$$

and a spin dependent quantum potential,

$$H_s - \xi(\underset{\sim}{v}\cdot\nabla)\eta$$

Eqs. (34c) and (34d), which tell us how the spin directions change with time are in precisely the same form as that of the equations expressing the torque acting on a point dipole, derived previously by Schönberg [18] in a purely classical problem.

We may now, however, obtain another instructive form of the equations for the spin variables by noting that H_{sp} can be written in the form

$$H_{sp} = \frac{\hbar^2}{2m}\sum_{i,j}\left(\frac{\partial s_i}{\partial x_j}\right)^2 + \frac{e}{mc}\,\underset{\sim}{s}\cdot\underset{\sim}{H} \tag{35a}$$

Noting that $\rho = \frac{2|S|}{\hbar}$, we also obtain the expression

$$H_q + H_{sp} = \int\left[\frac{\hbar^2}{2m}\sum_{i,j}\left(\frac{\partial S_i}{\partial x_j}\right)^2\Big/\rho + \frac{e}{mc}(\underset{\sim}{S}\cdot\underset{\sim}{H})\right]d\underset{\sim}{x} \tag{35b}$$

We can obtain the equations of motion for $\underset{\sim}{s}$, by writing $\underset{\sim}{s} = \underset{\sim}{S}/\rho$ and using the P.B. relations for the angular momentum. We get (with the aid of eq. (35a))

$$\frac{d\underset{\sim}{s}}{dt} = -\frac{\underset{\sim}{s}}{\rho}\times\sum_i\frac{\partial}{\partial x_i}\left(\rho\frac{\partial \underset{\sim}{s}}{\partial x_i}\right) + \frac{e}{mc}(\underset{\sim}{s}\times\underset{\sim}{H}) \tag{36}$$

Thus, the spin vector precesses about the magnetic field, with angular frequency, $\frac{eH}{mc}$ and there is an additional precession with angular velocity

$$-\frac{1}{\rho}\sum_i \frac{\partial}{\partial x_i}\left(\rho \frac{\partial s}{\partial x_i}\right)$$

which results from the torques produced by the neighboring spins. In some ways, one may think of this extra torque as due to a kind of "quantum-mechanical" addition to the magnetic field. But to make the analogy correct, one must think of the magnetic field in a polarizable medium where each dipole interacts strongly with its neighbors.

Finally, we shall discuss the motion of an element of fluid. This is best done in terms of the expression for the energy-momentum-stress tensor. To derive this tensor, we start with the lagrangian leaving out the vector and scalar potentials, which do not alter the problem in any essential way

$$L = \int \rho'\left[\left(\frac{\partial S}{\partial t} + \xi \frac{\partial \eta}{\partial t}\right) - \frac{\hbar^2}{8m}(\nabla\psi + \xi\nabla\eta)^2 - \frac{\hbar^2(\nabla\rho')^2}{8m(\rho')^2} - \frac{\hbar^2}{8m}\left((\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2\right)\right] d\underset{\sim}{x} \qquad (37)$$

It can easily be shown that this lagrangian leads to the correct canonical relations (27) between ξ and η, and between ρ' and S., and that it leads to the Pauli hamiltonian (10).

Now the canonical energy-momentum stress tensor is given by[24]

$$T_{ij} = \sum_\alpha \frac{\partial L}{\partial\left(\frac{\partial f_\alpha}{\partial x_i}\right)} \frac{\partial f_\alpha}{\partial x_j}\ ; \quad T_{oj} = \sum_\alpha \frac{\partial L}{\partial\left(\frac{\partial f_\alpha}{\partial t}\right)} \frac{\partial f_\alpha}{\partial x_j}\ ; \quad T_{oo} = \sum \frac{\partial L}{\partial\left(\frac{\partial f_\alpha}{\partial t}\right)} \frac{\partial f_\alpha}{\partial t}$$

where the f_α represent all the possible field quantities. T_{oj} represents the momentum density, and T_{oo} the energy density. We readily obtain

$$T_{00} = \rho'\left(\frac{\partial S}{\partial t} + \xi \frac{\partial \eta}{\partial t}\right)$$

$$T_{0j} = \rho'\left(\frac{\partial S}{\partial x_j} + \xi \frac{\partial \eta}{\partial x_j}\right) \tag{38}$$

$$T_{ij} = \frac{\rho'}{m}\left(\frac{\partial S}{\partial x_i} + \xi \frac{\partial \eta}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_j} + \xi \frac{\partial \eta}{\partial x_j}\right) + \frac{\hbar^2}{4m\rho}\frac{\partial \rho}{\partial x}\frac{\partial \rho}{\partial x_j} + \frac{\hbar^2}{4m}\rho\left(\frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_j} + \sin^2\theta \frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial x_j}\right)$$

where ξ and η are given by eq. (32).

The energy per particle is then $\frac{\partial S}{\partial t} + \xi \frac{\partial \eta}{\partial t}$, which agrees with what we obtain with the "Hamilton-Jacobi" equation (28e). The momentum per particle is just $\underset{\sim}{p} = \nabla S + \xi \nabla \eta$, in agreement with what we have been assuming. The stress tensor, T_{ij} contains three parts. The first part is just $m\rho v_i v_j$, which is just the usual term representing the effects of mass motion. The second part corresponds to the quantum potential and the third part to the effects of the interactions between the spins of neighboring particles. The equation of conservation of momentum can then be written as

$$\frac{\partial}{\partial t}(\rho v_i) + \sum_j \frac{\partial T_{ij}}{\partial x_j} = 0$$

With the aid of the equation for conservation of particles, and the relation

$$\sum_j \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\frac{(\nabla^2 \rho^{1/2})}{\rho^{1/2}} = \frac{\partial U}{\partial x_i}.$$

we obtain

$$\frac{d\underset{\sim}{p}}{dt} = m\left(\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v}\right) = -\nabla U - \frac{1}{\rho}\frac{\partial}{\partial x_j}(\rho S_{ij}) \tag{39}$$

where

$$S_{ij} = \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} + \sin^2\theta \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}$$

We see then that the fluid element moves under the action of the "quantum-mechanical" stress tensor

$$\frac{\hbar^2}{4m\rho'}\frac{\partial\rho'}{\partial x_i}\frac{\partial\rho'}{\partial x_j}$$

which leads to the quantum potential, and an additional quantum-mechanical contribution to the stress tensor, arising from the interaction of neighboring spins.

1 L. de Broglie, Compt. Rend.; 183 447 (1926); 184, 273, (1927); 185, 380 (1927).

2 E. Madelung, z.f. Physik; 40, 332 (1926).

3 D. Bohm, Phys. Rev. 85, 166, 180 (1952); 89, 1458 (1953).

4 T. Takabayasi, Prog. Theor. Phys. 8, 143 (1952); 9, 187 (1953).

5 I. Fenyes; z.f. Physik 132, 81 (1952).

6 W. Weizel, z.f. Physik 134, 264 (1953); 135, 270 (1953).

7 Mario Schönberg, Nuovo Cimento XI, 674 (1954).

8 D. Bohm and J. P. Vigier, Phys. Rev. To be published Oct. 1, 1954.

9 In a previous paper (see third paper of reference (3)) a similar theorem was proved for the model in which the ψ function is assumed to represent a field of force. In this case, the random fluctuations come from perturbations arising outside the system under investigation.

10 We do not include the spin orbit coupling term $\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\times\underset{\sim}{\mathcal{E}}$ because a consistent treatment of this term requires a relativistic theory, since it is of the same order of magnitude as the Thomas precession. This term must be dealt with in terms of a causal interpretation of the Dirac equation.

11 For a clear diagram of the Euler angles, see H. Goldstein, Classical Mechanics, Addison Wesley Press, 1950, p. 107.

12 It will become clear that the spinor that we associate to the standard orientation is arbitrary, and that all that is important is the relationship among spinors corresponding to different orientations. We have chosen the above standard spinor for

the sake of simplicity.

13 See Goldstein, Chap. 4. Note that we have rotated from body to space axes.

14 Spinors have already been used to describe the orientation of a rigid body. See, for example, H.B.G. Casimir, Rotation of a Rigid Body in Quantum Mechanics, J. B. Walters, The Hague, 1931.

15 In our theory, the spin vector is a <u>continuous</u> variable, with arbitrary projections on any axis, and with a total magnitude of $s^2 = \hbar^2/4$ (as can easily be proved by using the Pauli identities). On the other hand, the spin "observables" appearing in the usual quantum theory may have only certain discrete projections on any given axis, while the magnitude of the "observable" for the total spin is $\frac{3}{4}\hbar^2$. The origin of this difference arises in the circumstance that in the model that we are proposing here, the spin "observable" is, as we shall see later, a kind of statistical or over-all property of the motions of the bodies in the fluid. Thus, there is no reason why it should be identical with the angular momentum of one of the bodies in the fluid, although there will in general of course exist a certain connection between the "observable" and the spins of the bodies, which we shall discuss in more detail in a subsequent paper.

16 Uhlenbeck and Goudsmidt, Nature, <u>117</u>, 264 (1926).

17 H. A. Kramers, Quantentheorie des Elektrons, Akademische Verlags-Gesellschaft, 1938.

18 C. M. Lattes, M. Schenberg, and Walter Schutzer, Anais da Academia Brasileira de Ciências, XIX, Sept. 30, 1947, Nº 3.

19 See, for example, H. Lamb, <u>Hydrodynamics</u>, Cambridge Univ. Press, 1953, P. 248.

20 P.A.M. Dirac, Proc. Royal Soc., A, <u>209</u>, 291 (1951); <u>212</u>, 330 (1952); <u>223</u>, 439 (1954).

21 See, for example, C. Buneman, Proc. Camb. Phil. Soc., <u>50</u>, 77 (1954).

22 We are using here an argument due originally to M. Schönberg. See Nuovo Cimento XI, 674, 1954.

23 Note that the fact that $\frac{d\xi}{dt} = \frac{d\eta}{dt} = 0$ plays an essential role in making this result possible. Later, in connection with spin theory, we are going to have non-zero values for these derivatives; and then, the Clebsch parameters will imply additional terms in the force on a particle.

24 See, for example, G. Wentzel, <u>Quantum Theory of Fields</u> Interscience Publishers, (1949).