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Ince's Limits for Confluent and Double-Confluent Heun Equations

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Abstract

We find pairs of solutions to a differential equation which is obtained as a special limit of a generalized spheroidal wave equation (this is also known as confluent Heun equation). One solution in each pair is given by a series of hypergeometric functions and converges for any finite value of the independent variable z, while the other is given by a series of modified Bessel functions and converges for $|z| > |z_0|$, where z_0 denotes a regular singularity. For short, the preceding limit is called Ince's limit after Ince who have used the same procedure to get the Mathieu equations from the Whittaker-Hill ones. We find as well that, when z_0 tends to zero, the Ince limit of the generalized spheroidal wave equation turns out to be the Ince limit of a double-confluent Heun equation, for which solutions are provided. Finally, we show that the Schrödinger equation for inverse fourth and sixth-power potentials reduces to peculiar cases of the double-confluent Heun equation and its Ince's limit, respectively.

1. Introduction

Firstly, we construct two linear differential equations whose solutions behave at infinity as the so-called subnormal Thomé solutions, in contrast to solutions of a confluent and a double-confluent Heun equations [1], from which the former equations are obtained by a limit process. Secondly, we provide solutions which afford the expected asymptotic behavior for these equations. Finally, we find that the Schrödinger equation with inverse fourth and sixth-power potentials reduces to particular instances of the double-confluent Heun equation and its Ince limit, respectively.

In the first place, let us introduce the two equations under consideration. Our starting point is the generalized spheroidal wave equation (GSWE) in the form used by Leaver [2], namely,

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\Big[B_3-2\eta\omega(z-z_0)+\omega^2z(z-z_0)\Big]U=0, (\omega\neq 0) \qquad (1)$$

where B_i, η and ω are constants (notice that, if $\omega=0$ and η is fixed, the equation may be transformed into a hypergeometric equation). The points $z=0$ and $z=z_0$ are regular singularities with indices $(0,1+B_1/z_0)$ and $(0,1-B_2-B_1/z_0)$, respectively, while the infinity is an irregular singularity in which the behavior of $U(z)$, inferred from the normal Thomé solutions [3], is given by

$$\lim_{z\to\infty}U(z)\sim e^{\pm i\omega z}z^{\mp i\eta-(B_2/2)}. \qquad (2)$$

Since its parameters are not specified, the above GSWE is equivalent to the confluent Heun equation [1], an equation that is more general than the original Wilson GSWE [4]. Furthermore, as noted by Leaver, for $z_0=0$ we obtain a double-confluent Heun equation (DCHE) having five parameters, rather than four as in other contexts [5, 6], namely,

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\Big[B_3-2\eta\omega z+\omega^2z^2\Big]U=0,\ \ (\omega\neq 0,\ \ B_1\neq 0), \qquad (3)$$

where the singular points $z=0$ and $z=\infty$ are both irregular. For $\omega=0$ and/or $B_1=0$ (with η fixed) this equation degenerates into confluent hypergeometric equations (see Appendix A). At infinity, the behavior of $U(z)$ is again given by (2), while at $z=0$ we find in the usual way [3] that

$$\lim_{z\to 0}U(z)\sim 1 \ \ \text{or} \ \ \lim_{z\to 0}U(z)\sim e^{B_1/z}z^{2-B_2}. \qquad (4)$$

The Leaver procedure also allows us to obtain solutions to the DCHE from solutions to the GSWE when z_0 goes to zero. The known Leaver-type solutions [2, 7] are appropriate to solve, for instance, the Teukolsky equations for the extreme upper limit of the rotation parameter [2], the time dependence of Dirac test fields in dust-dominated Friedmann-Robertson-Walker spacetimes and the Schrödinger equation with asymmetric double-Morse potentials [7]. They are suitable either for handling the Schrödinger equation with inverse fourth-power potentials, as we will see.

Now, to get the equations we are interested in, the Levear limit is combined with a limit that Ince [8] had used to derive the Mathieu equation from the Whittaker-Hill equation. The Ince limit is obtained by taking

$$\omega \to 0, \quad \eta \to \infty, \text{ such that } \quad 2\eta\omega = -q, \tag{5}$$

where q is a constant. Thus, the Ince limit of the GSWE is

$$z(z-z_0)\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + [B_3 + q(z-z_0)]\,U = 0, \ (q \neq 0). \tag{6}$$

This is a generalization of the Mathieu equation for, by setting

$$z_0 = 1, \ B_1 = -1/2, \ B_2 = 1, \ z = \cos^2(\sigma u), \ W(u) = U(z), \tag{7a}$$

we obtain the equation

$$\frac{d^2W}{du^2} + \sigma^2\,[2q - 4B_3 - 2q\cos(2\sigma u)]\,W = 0, \tag{7b}$$

that is, the Mathieu equation if $\sigma = 1$, and the modified Mathieu equation if $\sigma = i$ [9]. In fact, inserting $z_0 = 1$, $B_1 = -1/2$ and $B_2 = 1$ into Eq. (6), one recovers the algebraic Lindemann form for the Mathieu equation [10]. Nevertheless, the trigonometric form (7b) with $4B_3 = 2q - a$ is useful to verify that our solutions for the Ince limit of the GSWE give solutions already known for the Mathieu equation.

On the other hand, the Ince limit of the DCHE – or Leaver limit of the Eq. (6) – is the equation

$$z^2\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + (B_3 + qz)\,U = 0, \ (q \neq 0, \ B_1 \neq 0) \tag{8}$$

which degenerates into simpler equations if $q = 0$ and/or $B_1 = 0$ (see Appendix A). Solutions are obtained for this special DCHE by taking the Leaver limit of solutions for the Eq. (6). By the way, we shall see that the Schrödinger equation for an inverse sixth-power potential is a particular case of Eq. (8), as stated in the first paragraph.

We emphasize that the Ince limits of the GSWE and DCHE, unlike the original GSWE and DCHE, require solutions behaving according to the subnormal Thomé solutions [3], that is,

$$\lim_{z\to\infty} U(z) \sim e^{\pm 2i\sqrt{qz}} z^{(1/4)-(B_2/2)}. \tag{9}$$

Despite this, our main mathematical issue consists in deriving pairs of series solutions to Eqs. (6) and (8) – having the behavior stipulated above at the singular points – from pairs of solutions to the GSWE. For this we shall again employ the Ince and Leaver limits. The solutions in each pair have the same series coefficients and these satisfy three-term recurrence relations.

In section 2, a pair of solutions for the Ince limit of the GSWE is obtained by taking the Ince limit (5) of a known pair of solutions for the GSWE. One solution is given by an expansion in series of hypegeometric functions and converges for any finite z; the other solution is given by an expansion in series of modified Bessel functions and converges for $|z| > |z_0|$. Other pairs are generated by using transformation rules. These rules result from variable substitutions that preserve the form of the differential equations but modify their parameters and/or arguments.

In section 3, we find pairs of solutions for the Ince limit of the DCHE by taking the Leaver limit ($z_0 \to 0$) of solutions for the Ince limit of the GSWE. Solutions in series of irregular confluent hypergeometric functions result from expansions in series of hypergeometric functions and converge for any finite z. The other solution in each pair is given by a series of modified Bessel functions and converges for $|z| > 0$.

In both of these sections we deal with solutions with and without a phase parameter ν. In general, this ν is introduced in order to assure the convergence of the series when there is no free constant in the differential equation, as in some scattering problems or in equations where z is a variable related to the time [2, 7]. Solutions with a phase parameter are two-sided in the sense that the summation index n runs from $-\infty$ to ∞. However, if there is an arbitrary parameter in the equation, we can truncate the series by requiring that $n \geq 0$. In this manner, we obtain ν in terms of parameters of the differential equation.

In section 4, we show that the Schrödinger equation with inverse fourth and sixth-power potentials in fact leads to the DCHE and its Ince limit. Some additional considerations are provided in section 5, while in Appendix A we discuss the degenerate cases of the DCHEs, in Appendix B we present an alternative derivation of the expansions in Bessel functions, and in Appendix C we rewrite the Leaver-type solutions for the DCHE in a form appropriate to solve the Schrödinger equation with an inverse fourth-power potential.

2. Ince's limits for the generalized spheroidal wave equation

In this section we use transformation rules that permit us to generate new solutions from a given solution for the Ince limit of the GSWE. The rules T_1, T_2 and T_3 below are derived from the ones valid for the GSWE [7] and can be checked by substitution of variables. If $U(z) = U(B_1, B_2, B_3; z_0, q; z)$ denotes one solution for Eq. (6), the effects of these rules are as follows

$$\begin{aligned} T_1U(z) &= z^{1+B_1/z_0}U(C_1, C_2, C_3; z_0, q; z), \quad z_0 \neq 0, \\ T_2U(z) &= (z - z_0)^{1-B_2-B_1/z_0}U(B_1, D_2, D_3; z_0, q; z), \quad z_0 \neq 0, \\ T_3U(z) &= U(-B_1 - B_2z_0, B_2, B_3 - qz_0; z_0, -q; z_0 - z), \end{aligned} \tag{10a}$$

where

$$\begin{aligned} &C_1 = -B_1 - 2z_0, \quad C_2 = 2 + B_2 + \tfrac{2B_1}{z_0}, \; C_3 = B_3 + \left(1 + \tfrac{B_1}{z_0}\right)\left(B_2 + \tfrac{B_1}{z_0}\right), \\ &D_2 = 2 - B_2 - \tfrac{2B_1}{z_0}, \; D_3 = B_3 + \tfrac{B_1}{z_0}\left(\tfrac{B_1}{z_0} + B_2 - 1\right). \end{aligned} \tag{10b}$$

We use only T_1 and T_2. The rule T_3 exchange the position of the regular singular points $z = z_0 \leftrightarrow z = 0$ and may be used to get an alternative representation for the solutions, but these are not proper for getting the limit $z_0 \to 0$.

In section 2.1 we derive two pairs of solutions for the Ince limit of the GSWE – denoted by $(U^{0}_{i\nu}, U^{\infty}_{i\nu})$, $i = 1, 2$ – with a phase parameter ν. The superscript 'zero' indicates that the series converges in any finite part of the complex plane, while the superscript 'infinity' indicates convergence for $|z| > |z_0|$. The second pair of solutions results from the first by means of the rule T_2. In section 2.2, we truncate these series by taking $n \geq 0$ and obtain four pairs of solutions without phase parameter.

2.1. Solutions with a phase parameter

Denoting by b_n the series coefficients of a solution, their recurrence relations will have the general form

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \ (-\infty < n < \infty) \tag{11a}$$

where α_n, β_n, γ_n and b_n depend on a phase parameter ν which may be determined from a characteristic equation given as a sum of two infinite continued fractions, namely,

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\cdots + \frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\cdots. \tag{11b}$$

For a specific pair of solutions we add a superscript in each of these quantities.

The first pair of solutions for the Ince limit of the GSWE comes from the following pair of solutions of the GSWE [7]

$$\begin{aligned} U^{0}_{1\nu}(z) &= e^{i\omega z}\sum_{n=-\infty}^{\infty} b^{(1)}_{n} F\left(\frac{B_2}{2} - n - \nu - 1, n + \nu + \frac{B_2}{2}; B_2 + \frac{B_1}{z_0}; 1 - \frac{z}{z_0}\right), \\ U^{\infty}_{1\nu}(z) &= e^{i\omega z} z^{1-(B_2/2)} \sum_{n=-\infty}^{\infty} b^{(1)}_{n} (-2i\omega z)^{n+\nu} \Psi(n + \nu + 1 + i\eta, 2n + 2\nu + 2; -2i\omega z), \end{aligned} \tag{12a}$$

where $F(a, b; c; y)$ and $\Psi(a, b; y)$ denote, respectively, the hypergeometric functions and the irregular confluent hypergeometric functions [11, 12]. The solution $U^{0}_{1\nu}$ converges for any finite z, whereas $U^{\infty}_{1\nu}$ converges for $| z |>| z_0 |$. In the recurrence relations (11a) for $b^{(1)}_n$ we have

$$\begin{aligned} \alpha^{(1)}_n &= i\omega z_0 \frac{\left(n+\nu+2-\frac{B_2}{2}\right)\left(n+\nu+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)(n+\nu+1-i\eta)}{2(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)}, \\ \beta^{(1)}_n &= -B_3 - \eta\omega z_0 - (n + \nu + 1 - \tfrac{B_2}{2})(n + \nu + \tfrac{B_2}{2}) - \frac{\eta\omega z_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{(n+\nu)(n+\nu+1)}, \\ \gamma^{(1)}_n &= -i\omega z_0 \frac{\left(n+\nu+\frac{B_2}{2}-1\right)\left(n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right)(n+\nu+i\eta)}{2\left(n+\nu-\frac{1}{2}\right)(n+\nu)}. \end{aligned} \tag{12b}$$

Note that ν cannot be integer or half-integer in order to avoid vanishing denominators in the coefficients of the recurrence relations. Moreover, for an integer or half-integer ν, we would have two equal hypergeometric or confluent hypergeometric functions (for different values of n), contrary to the hypothesis that all the terms of the series are independent. On the other hand, the hypergeometric functions are not defined if $B_2 + (B_1/z_0)$ is zero or a negative integer. Nonetheless, a transformation rule supplies another solution which is valid for these values of $B_2 + (B_1/z_0)$.

The three-term recurrence relations (11a) constitute a infinite system of homogeneous linear equations for which nontrivial solutions for the coefficients b_n demand that the determinant of respective tridiagonal matrix vanishes. Equivalently, the characteristic equation must be satisfied and this is a condition necessary also to assure the convergence of the series by means of a Poincaré-Perron theorem [13]. However, there are two possibilities to satisfy this requirement.

On the one hand, if there is some free constant in the differential equation, that constant must be determined so that the characteristic equation is fulfilled for the admissible values of ν (that is, neither integer nor half-integer). In this case, the freedom of choosing ν may be used in two different ways: (i) to obtain two-sided solutions ($-\infty < n < \infty$) by ascribing appropriate values for ν, or (ii) to obtain one-sided solutions by choosing ν such that $n \geq 0$. At the end of the present section, we use the first alternative to rederive some Poole's solutions [14, 15] for the Mathieu equation, having period $2\pi m$, where m is any integer equal or greater than 2. In section 2.2, we use the second alternative for the general case. These latter solutions afford solutions with period π or 2π for the Mathieu equation, in contrast to the solutions obtained in the first alternative.

On the other hand, if there is no arbitrary parameter in the differential equation, the parameter ν takes the role of free parameter in the sense that it must be adjusted to ensure the validity of the characteristic equation and, consequently, the convergence of the series. By this reason, ν is also called characteristic index or parameter [16]. Examples of equations requiring a phase parameter are discussed in section 4.

From the above pair of solutions for the GSWE, by using the Ince limit (5), we readily find the solution $U^{0}_{1\nu}(z)$ written in the first pair below. To get the Ince limit of the solution $U^{\infty}_{1\nu}(z)$, we define c_n as

$$b_n^{(1)} = (i\eta)^{n+\nu}\Gamma(i\eta - n - \nu)c_n.$$

This imply that

$$U^{\infty}_{1\nu}(z) = e^{i\omega z} z^{1-(B_2/2)} \sum_{n=-\infty}^{\infty} c_n \Gamma(i\eta - n - \nu)(-qz)^{n+\nu}\Psi\left(n+\nu+1+i\eta, 2n+2\nu+2; -\frac{qz}{i\eta}\right),$$

where $q = -2\eta\omega$. The recurrence relations for c_n are

$$\bar{\alpha}_n c_{n+1} + \beta_n^{(1)} c_n + \bar{\gamma}_n c_{n-1} = 0, \ (-\infty < n < \infty)$$

with

$$\bar{\alpha}_n = \frac{i\eta}{i\eta - n - \nu - 1}\alpha_n^{(1)}, \ \bar{\gamma}_n = \frac{i\eta - \nu - 1}{i\eta}\gamma_n^{(1)}.$$

On the other hand, we have [12]

$$\lim_{a\to\infty}[\Gamma(a+1-c)\Psi(a, c; x/a)] = 2x^{(1-c)/2} K_{c-1}(2\sqrt{x}) \qquad (13)$$

where $K_\lambda(\xi)$ denotes the modified Bessel function of the second kind [17] whose definition in terms of irregular confluent hypergeometric functions is [12]

$$K_{\lambda}(\xi)=K_{-\lambda}(\xi)=\sqrt{\pi}\ e^{-\xi}(2\xi)^{\lambda}\Psi\left(\lambda+\frac{1}{2},2\lambda+1;2\xi\right). \tag{14}$$

Then, using (13) we find that for $i\eta\to\infty$ (n fixed and q =constant)

$$\Gamma(i\eta-n-\nu)(-qz)^{n+\nu}\Psi\left(n+\nu+1+i\eta,2n+2\nu+2;-\tfrac{qz}{i\eta}\right)\to 2(-qz)^{1/2}K_{2n+2\nu+1}(\pm 2i\sqrt{qz}),$$

$$\lim\bar{\alpha}_n\to\lim\alpha_n^{(1)},\quad \lim\bar{\gamma}_n\to\lim\gamma_n^{(1)}\Rightarrow\lim c_n\to\lim b_n^{(1)}.$$

Using these results, we find the Ince limit of $U_{1\nu}^{\infty}$, written in the first pair below. Although this is a formal derivation, the solution may be checked by inserting it into Eq. (6) (see Appendix B). In addition, from the relation [17]

$$\lim_{|\xi|\to\infty}K_{\lambda}(\xi)\sim\sqrt{\frac{\pi}{2\xi}}\ e^{-\xi},\quad -\frac{3\pi}{2}<\arg\xi<\frac{3\pi}{2} \tag{15}$$

we see that the expansions in series of Bessel functions have the behavior given by

$$\lim_{z\to\infty}U_{j\nu}^{\infty}(z)\sim e^{\pm 2i\sqrt{qz}}z^{(1/4)-(B_2/2)},\quad -\frac{3\pi}{2}<\arg(\pm 2i\sqrt{qz})<\frac{3\pi}{2},\quad (j=1,2)$$

in accordance with Eq. (9). The second pair of solutions follows from the first one through the rule T_2, as mentioned before. Moreover, solutions for the Mathieu equation are obtained by using Eqs. (7a) and by noting that in this case the hypergeometric functions can be rewritten in terms of trigonometric functions.

First pair

$$\begin{aligned}U_{1\nu}^{0}(z)&=\sum_{n=-\infty}^{\infty}b_n^{(1)}F\left(\frac{B_2}{2}-n-\nu-1,n+\nu+\frac{B_2}{2};B_2+\frac{B_1}{z_0};1-\frac{z}{z_0}\right),\\ U_{1\nu}^{\infty}(z)&=z^{(1-B_2)/2}\sum_{n=-\infty}^{\infty}b_n^{(1)}K_{2n+2\nu+1}\left(\pm 2i\sqrt{qz}\right),\end{aligned} \tag{16a}$$

where in the recurrence relations (11a)

$$\begin{aligned}\alpha_n^{(1)}&=qz_0\frac{\left(n+\nu+2-\frac{B_2}{2}\right)\left(n+\nu+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)}{(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)},\\ \beta_n^{(1)}&=4B_3-2qz_0+4\left(n+\nu+1-\tfrac{B_2}{2}\right)\left(n+\nu+\tfrac{B_2}{2}\right)-2qz_0\frac{\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{(n+\nu)(n+\nu+1)},\\ \gamma_n^{(1)}&=qz_0\frac{\left(n+\nu+\frac{B_2}{2}-1\right)\left(n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+\nu-\frac{1}{2}\right)(n+\nu)}.\end{aligned} \tag{16b}$$

If $B_2+(B_1/z_0)$ is zero or a negative integer we have the solution $U_{2\nu}^{0}$ instead of $U_{1\nu}^{0}$.

For the Mathieu equation we use Eqs. (7a) and the formula [11]

$$F\left[-a,a;(1/2);\sin^2(\sigma u)\right]=\cos(2a\sigma u).$$

Thence, we obtain even solutions with respect to u, namely,

$$\begin{aligned} W_{1\nu}^{0}(u)&=\sum_{n=-\infty}^{\infty}b_n^{(1)}\cos[(2n+2\nu+1)\sigma u], && |\cos(\sigma u)|<\infty,\\ W_{1\nu}^{\infty}(u)&=\sum_{n=-\infty}^{\infty}b_n^{(1)}K_{2n+2\nu+1}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], && |\cos(\sigma u)|>1, \end{aligned} \tag{17a}$$

with the simplified recurrence relations ($a=2q-4B_3$)

$$qb_{n+1}^{(1)}+\left[(2n+2\nu+1)^2-a\right]b_n^{(1)}+qb_{n-1}^{(1)}=0. \tag{17b}$$

Second pair

$$\begin{aligned} U_{2\nu}^{0}(z)&=(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\ z^{1+\frac{B_1}{z_0}}\sum_{n=-\infty}^{\infty}b_n^{(2)}\times\\ &\qquad F\left(-n-\nu-\tfrac{B_2}{2}+1,n+\nu+2-\tfrac{B_2}{2};2-B_2-\tfrac{B_1}{z_0};1-\tfrac{z}{z_0}\right),\\ U_{2\nu}^{\infty}(z)&=(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\ z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}}\sum_{n=-\infty}^{\infty}b_n^{(2)}K_{2n+2\nu+1}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{18a}$$

where

$$\begin{aligned} \alpha_n^{(2)}&=qz_0\frac{\left(n+\nu+\frac{B_2}{2}\right)\left(n+\nu+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)},\quad \beta_n^{(2)}=\beta_n^{(1)},\\ \gamma_n^{(2)}&=qz_0\frac{\left(n+\nu+1-\frac{B_2}{2}\right)\left(n+\nu-\frac{B_2}{2}-\frac{B_1}{z_0}\right)}{\left(n+\nu-\frac{1}{2}\right)(n+\nu)}, \end{aligned} \tag{18b}$$

in the recurrence relations (11a) for $b_n^{(2)}$. If $B_2+(B_1/z_0)$ is a positive integer equal or greater than 2 we have the solution $U_{1\nu}^0$ instead of $U_{2\nu}^0$. Note that, in writing the solution $U_{2\nu}^0$, we have used the relation

$$F(a,b;c;y)=(1-y)^{c-a-b}F(c-a,c-b;c;y). \tag{19}$$

For the Mathieu equation we use the relation [11]

$$F\left(a,1-a;\frac{3}{2};\sin^2(\sigma u)\right)=\frac{\sin[(2a-1)\sigma u]}{(2a-1)\sin(\sigma u)}$$

and, in addition, define c_n as $b_n^{(2)}=(2n+2\nu+1)c_n$. So, we find that the recurrence relations for c_n become identical to the ones for $b_n^{(1)}$, giving the odd solutions

$$\begin{aligned} W_{2\nu}^{0}(u)&=\sum_{n=-\infty}^{\infty}b_n^{(1)}\sin[(2n+2\nu+1)\sigma u],\\ W_{2\nu}^{\infty}(u)&=\tan(\sigma u)\sum_{n=-\infty}^{\infty}(2n+2\nu+1)\,b_n^{(1)}K_{2n+2\nu+1}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], \end{aligned} \tag{20}$$

where $|\cos(\sigma u)| < \infty$ and $|\cos(\sigma u)| > 1$, respectively.

As we have explained earlier, if there is a free parameter in the differential equation, it is possible to satisfy the characteristic equation for any noninteger or half-integer ν. We use this fact to rederive some Poole's solutions [14, 15] to the Mathieu equation. For this, in the previous $W_{1\nu}^{0}(u)$ and $W_{1\nu}^{0}(u)$ we take

$$2\nu + 1 = l/m, \quad \sigma = 1, \tag{21}$$

where l and m are integers prime to one another, $l < m$. Then, we find the two-sided Poole solutions $W_1^P(u)$ and $W_1^P(u)$ given by

$$W_1^P(u) = \sum_{n=-\infty}^{\infty} b_n^{(1)} \cos\left[\left(2n + \frac{l}{m}\right) u\right], \quad W_2^P(u) = \sum_{n=-\infty}^{\infty} b_n^{(1)} \sin\left[\left(2n + \frac{l}{m}\right) u\right]. \tag{22}$$

The first is even with respect to u and the second is odd, and both of them have period $2\pi m$, $m > 1$. Since they have the same series coefficients, we can combine them to find another Poole solution, that is,

$$W^P(u) = \sum_{n=-\infty}^{\infty} b_n^{(1)} \exp\left[i\left(2n + \frac{l}{m}\right) u\right]. \tag{23}$$

Furthermore, for an arbitrary ν we find

$$W(u) = \sum_{n=-\infty}^{\infty} b_n^{(1)} \exp\left[i\left(2n + 2\nu + 1\right) u\right], \tag{24}$$

which is also a solution already known in the literature [11, 15].

2.2. Solutions without phase parameter

Now we truncate the solutions obtained in section 2.1 by taking $n \geq 0$. This gives ν in terms of some parameters of the differential equation. The resulting solutions are convergent only if there is a free parameter to be determined from the characteristic equation.

This truncation reverses the procedure by which the solution $U_{1\nu}^{0}(z)$ for the GSWE, given in Eq. (12a), was obtained. Indeed, that solution was constructed [18] as a generalization of an one-sided series of Jacobi polynomials, constructed by Fackerell and Crossman [19] to solve the angular Teukolsky equations of the relativistic astrophysics. Despite this, the truncated solutions found in [7] are more general than the Fackerell-Crossman ones because no particular values are attached to the parameters of the GSWE and also because the truncation was extended to the Leaver expansion $U_{1\nu}^{\infty}(z)$. I addition, these one-sided series are suitable to get solutions in finite series, the so called quasi-polynomial solutions. In effect, a solution whose coefficients b_n obey recurrence relations as

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \quad n \geq 0, \quad b_{-1} = 0$$

becomes a quasi-polynomial solution with $0 \leq n \leq N-1$ whenever $\gamma_N = 0$ for some $n = N$ [20] .

For the truncated solutions – denoted by (U_i^0, U_i^∞), $i = 1, 2, 3, 4$ – the recurrence relations and the characteristic equations have one of the three forms written below. The first case ($\alpha_{-1} = 0$) is the general one and the others ($\alpha_{-1} \neq 0$) may occur only for special cases.

$$\left.\begin{array}{l} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1), \end{array}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{25}$$

$$\left.\begin{array}{l} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_1 b_2 + \beta_1 b_1 + [\alpha_{-1} + \gamma_1]\, b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 2), \end{array}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\,[\alpha_{-1} + \gamma_1]}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{26}$$

$$\left.\begin{array}{l} \alpha_0 b_1 + [\beta_0 + \alpha_{-1}]\, b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1), \end{array}\right\} \Rightarrow \beta_0 + \alpha_{-1} = \frac{\alpha_0\gamma_1}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -}\cdots. \tag{27}$$

Note that we have $n \geq -1$ in α_n, $n \geq 0$ in β_n and $n \geq 1$ in γ_n.

These forms for the recurrence relations are the same that appear in truncation of the expansions (12a) for the GSWE [7]. As a matter of fact, the solutions of the present section are the Ince limit of solutions for the GSWE given in section 3 of Ref. [7]. However, in order to illustrate how these recurrence relations are obtained, we insert the solution $U_{1\nu}^{\infty}$ given in (16a) into the Ince limit of the GSWE. Then, for $n \geq 0$, from Eq. (B4) we find that

$$\sum_{n=0}^{\infty} \alpha_{n-1}^{(1)} b_n^{(1)} K_{2n+2\nu-1}(\xi) + \sum_{n=0}^{\infty} \beta_n^{(1)} b_n^{(1)} K_{2n+2\nu+1}(\xi) + \sum_{n=0}^{\infty} \gamma_{n+1}^{(1)} b_n^{(1)} K_{2n+2\nu+3}(\xi) = 0.$$

Setting $m = n - 1$, $m = n$ and $m = n + 1$ in the first, second and third terms, respectively, this equation becomes

$$\begin{array}{l} \alpha_{-1} b_0 K_{2\nu-1}(\xi) + [\alpha_0 b_1 + \beta_0 b_0]\, K_{2\nu+1}(\xi) + [\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0]\, K_{2\nu+3}(\xi) + \\ \displaystyle\sum_{n=2}^{\infty} [\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1}]\, K_{2n+2\nu+1}(\xi) = 0, \end{array} \tag{28}$$

where we have dropped the upper suffixes. Therefore, if we can choose ν so that $\alpha_{-1} = 0$, we find the first set of recurrence relations. However, notice that

$$\alpha_{-1} = \frac{q z_0 \left(\nu + 1 - \frac{B_2}{2}\right)\left(\nu - \frac{B_1}{x_0} - \frac{B_2}{2}\right)}{2\nu(\nu + 1/2)} = 0, \ \text{if} \left\{ \begin{array}{l} \nu = \frac{B_2}{2} - 1 \ \text{for}\ B_2 \neq 1, 2; \\ \nu = \frac{B_1}{x_0} + \frac{B_2}{2} \ \ \text{for}\ \frac{B_1}{x_0} + \frac{B_2}{2} \neq 0, \frac{1}{2}. \end{array}\right.$$

Hence we see that there are two possible choices for ν and, for each of them we have two cases in which α_{-1} may differ from zero. Let us consider only the case $\nu = (B_2/2) - 1$. Then, for the exceptional case $B_2 = 1$ ($\nu = -1/2$), we find $K_{2\nu-1} = K_{2\nu+3} = K_2$ (since $K_\lambda = K_{-\lambda}$) and therefore the Bessel functions in the first and third terms of Eq. (28) are equal, giving the recurrence relations (26). Similarly, if $B_2 = 2$ ($\nu = 0$) we find $K_{2\nu-1} = K_{2\nu+1} = K_1$ and this leads to the recurrence relations (27). In this manner we obtain the first pair given below. The remaining can be derived from this by using the transformations rules T_1 and T_2 as

$$\left(U_1^0, U_1^\infty\right) \stackrel{T_1}{\longleftrightarrow} \left(U_2^0, U_2^\infty\right) \stackrel{T_2}{\longleftrightarrow} \left(U_3^0, U_3^\infty\right) \stackrel{T_1}{\longleftrightarrow} \left(U_4^0, U_4^\infty\right) \stackrel{T_2}{\longleftrightarrow} \left(U_1^0, U_1^\infty\right).$$

The condition on each pair is imposed in order to assure that the special functions are independent in both solutions; it guarantees either that there is no vanishing denominator in the recurrence relations. Furthermore, we have additional restrictions on the parameters of the solutions U_i^0. Thus, if $B_2+(B_1/z_0)$ is zero or a negative integer, the hypergeometric functions are not defined in U_1^0 and U_2^0 but are defined in U_3^0 and U_4^0, and vice-versa. The results for the Mathieu equations are already known [9], but note that the recurrence relations for this case come from the three Eqs. (25-27) above.

First pair: $B_2 \neq 0, -1, -2, \cdots$. This first pair corresponds to $\nu = (B_2/2) - 1$ in $(U_{1\nu}^0, U_{1\nu}^\infty)$.

$$\begin{aligned} U_1^0(z) &= \sum_{n=0}^{\infty} b_n^{(1)} F\left(-n, n+B_2-1; B_2+\frac{B_1}{z_0}; 1-\frac{z}{z_0}\right), \\ U_1^\infty(z) &= z^{(1-B_2)/2} \sum_{n=0}^{\infty} b_n^{(1)} K_{2n+B_2-1}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{29a}$$

with the following coefficients

$$\begin{aligned} \alpha_n^{(1)} &= \frac{qz_0(n+1)\left(n-\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}\right)\left(n+\frac{B_2}{2}+\frac{1}{2}\right)}, \\ \beta_n^{(1)} &= 4B_3 - 2qz_0 + 4n\left(n+B_2-1\right) - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}-1\right)\left(n+\frac{B_2}{2}\right)}, \\ \gamma_n^{(1)} &= \frac{qz_0(n+B_2-2)\left(n+B_2+\frac{B_1}{z_0}-1\right)}{\left(n+\frac{B_2}{2}-\frac{3}{2}\right)\left(n+\frac{B_2}{2}-1\right)}, \end{aligned} \tag{29b}$$

in the recurrence relations for the $b_n^{(1)}$, namely: Eqs. (25) if $B_2 \neq 1, 2$; Eqs. (26) if $B_2 = 1$; Eqs. (27) if $B_2 = 2$.

For the Mathieu equation we find solutions

$$\begin{aligned} W_1^0(u) &= \sum_{n=0}^{\infty} b_n^{(1)} \cos(2n\sigma u), & |\cos(\sigma u)| < \infty, \\ W_1^\infty(u) &= \sum_{n=0}^{\infty} b_n^{(1)} K_{2n}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], & |\cos(\sigma u)| > 1, \end{aligned} \tag{30a}$$

with the simplified recurrence relations

$$\begin{aligned} &qb_1^{(1)} - ab_0^{(1)} = 0, \quad qb_2^{(1)} + \left[4-a\right] b_1^{(1)} + 2qb_0^{(1)} = 0, \\ &qb_{n+1}^{(1)} + \left[4n^2 - a\right] b_n^{(1)} + qb_{n-1}^{(1)} = 0 \ (n \geq 2). \end{aligned} \tag{30b}$$

These solutions are even with respect to u and, for $\sigma = 1$, the solution $W_1^0(u)$ has period π.

Second pair: $(B_2/2) + (B_1/z_0) \neq -1, -3/2, -2, -5/2 \cdots$. This pair of solutions can also be obtained by taking $\nu = (B_2/2) + (B_1/z_0)$ in $(U_{1\nu}^0, U_{1\nu}^0)$.

$$\begin{aligned} U_2^0(z) &= z^{1+(B_1/z_0)} \sum_{n=0}^{\infty} b_n^{(2)} F\left(-n, n+1+B_2+\frac{2B_1}{z_0}; B_2+\frac{B_1}{z_0}; 1-\frac{z}{z_0}\right), \\ U_2^\infty(z) &= z^{(1-B_2)/2} \sum_{n=0}^{\infty} b_n^{(2)} K_{2n+1+B_2+(2B_1/z_0)}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{31a}$$

where

$$\begin{aligned} \alpha_n^{(2)} &= \frac{qz_0(n+1)\left(n+2+\frac{B_1}{x_0}\right)}{\left(n+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right)\left(n+\frac{3}{2}+\frac{B_2}{2}+\frac{B_1}{z_0}\right)}, \\ \beta_n^{(2)} &= 4B_3 - 2qz_0 + 4\left(n+1+\frac{B_1}{x_0}\right)\left(n+B_2+\frac{B_1}{x_0}\right) - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}+\frac{B_1}{x_0}\right)\left(n+1+\frac{B_2}{2}+\frac{B_1}{x_0}\right)}, \\ \gamma_n^{(2)} &= \frac{qz_0\left(n+B_2+\frac{B_1}{x_0}-1\right)\left(n+B_2+\frac{2B_1}{z_0}\right)}{\left(n-\frac{1}{2}+\frac{B_2}{2}+\frac{B_1}{z_0}\right)\left(n+\frac{B_2}{2}+\frac{B_1}{z_0}\right)}, \end{aligned} \tag{31b}$$

in the recurrence relations for $b_n^{(2)}$: Eqs. (25) if $(B_2/2) + (B_1/z_0) \neq 0, -1/2$; Eqs. (26) if $(B_2/2) + (B_1/z_0) = -1/2$; Eqs. (27) if $(B_2/2) + (B_1/z_0) = 0$.

For the Mathieu equation we again have even solutions

$$\begin{aligned} W_2^0(u) &= \sum_{n=0}^{\infty} b_n^{(2)} \cos[(2n+1)\sigma u], & |\cos(\sigma u)| < \infty, \\ W_2^\infty(u) &= \sum_{n=0}^{\infty} b_n^{(2)} K_{2n+1}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], & |\cos(\sigma u)| > 1, \end{aligned} \tag{32a}$$

with the recurrence relations

$$\begin{aligned} & qb_1^{(2)} + [q+1-a]\, b_0^{(2)} = 0, \\ & qb_{n+1}^{(2)} + \left[(2n+1)^2 - a\right] b_n^{(2)} + qb_{n-1}^{(2)} = 0 \ (n \geq 1). \end{aligned} \tag{32b}$$

If $\sigma = 1$ the solution $W_2^0(u)$ has period 2π.

Third pair: $B_2 \neq 4, 5, 6, \cdots$. This corresponds to $\nu = 1 - (B_2/2)$ in $(U_{2\nu}^0, U_{2\nu}^\infty)$.

$$\begin{aligned} U_3^0(z) &= (z-z_0)^{1-B_2-\frac{B_1}{z_0}} z^{1+\frac{B_1}{z_0}} \sum_{n=0}^{\infty} b_n^{(3)} F\left(-n, n+3-B_2; 2-B_2-\frac{B_1}{z_0}; 1-\frac{z}{z_0}\right), \\ U_3^\infty(z) &= (z-z_0)^{1-B_2-\frac{B_1}{z_0}} z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}} \sum_{n=0}^{\infty} b_n^{(3)} K_{2n+3-B_2}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{33a}$$

with the coefficients

$$\begin{aligned}
\alpha_n^{(3)} &= \frac{qz_0\ (n+1)\left(n+2+\frac{B_1}{z_0}\right)}{\left(n+2-\frac{B_2}{2}\right)\left(n+\frac{5}{2}-\frac{B_2}{2}\right)},\\
\beta_n^{(3)} &= 4B_3-2qz_0+4(n+1)(n+2-B_2)-\frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+1-\frac{B_2}{2}\right)\left(n+2-\frac{B_2}{2}\right)},\\
\gamma_n^{(3)} &= \frac{qz_0\ (n+2-B_2)\left(n+1-B_2-\frac{B_1}{z_0}\right)}{\left(n+\frac{1}{2}-\frac{B_2}{2}\right)\left(n+1-\frac{B_2}{2}\right)}.
\end{aligned} \tag{33b}$$

in the recurrence relations: Eqs. (25) if $B_2 \neq 2,3$; Eqs. (26) if $B_2 = 3$; Eqs. (27) if $B_2 = 2$.

For the Mathieu equation we redefine the coefficients $b_n^{(3)}$ as $b_n^{(3)} \to (2n+2)b_n^{(3)}$. Then we find the odd solutions

$$\begin{aligned}
W_3^0(u) &= \sum_{n=0}^{\infty} b_n^{(3)} \sin[(2n+2)\sigma u], & |\cos(\sigma u)| < \infty,\\
W_3^{\infty}(u) &= \tan(\sigma u) \sum_{n=0}^{\infty} (2n+2)\, b_n^{(3)} K_{2n+2}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], & |\cos(\sigma u)| > 1,
\end{aligned} \tag{34a}$$

with the recurrence relations

$$\begin{aligned}
&qb_1^{(3)} + [4-a]\, b_0^{(3)} = 0,\\
&qb_{n+1}^{(3)} + \left[4\,(n+1)^2 - a\right] b_n^{(3)} + qb_{n-1}^{(3)} = 0,\ \ (n \geq 1).
\end{aligned} \tag{34b}$$

For $\sigma = 1$ the solution $W_3^0(u)$ has period π.

Fourth pair: $(B_2/2) + (B_1/z_0) \neq 1, 3/2, 2, 5/2 \cdots$. This can also be obtained by setting $\nu = -(B_2/2) - (B_1/z_0)$ in $(U_{2\nu}^0, U_{2\nu}^{\infty})$

$$\begin{aligned}
U_4^0 &= (z-z_0)^{1-B_2-\frac{B_1}{z_0}} \sum_{n=0}^{\infty} b_n^{(4)} F\left(-n, n+1-B_2-\frac{2B_1}{z_0}; 2-B_2-\frac{B_1}{z_0}; 1-\frac{z}{z_0}\right),\\
U_4^{\infty} &= (z-z_0)^{1-B_2-\frac{B_1}{z_0}} z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}} \sum_{n=0}^{\infty} b_n^{(4)} K_{2n+1-B_2-(2B_1/z_0)}\left(\pm 2i\sqrt{qz}\right),
\end{aligned} \tag{35a}$$

with coefficients

$$\begin{aligned}
\alpha_n^{(4)} &= \frac{qz_0\ (n+1)\left(n-\frac{B_1}{z_0}\right)}{\left(n+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n+\frac{3}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},\\
\beta_n^{(4)} &= 4B_3-2qz_0+4\left(n-\frac{B_1}{z_0}\right)\left(n-B_2+1-\frac{B_1}{z_0}\right)-\frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},\\
\gamma_n^{(4)} &= \frac{qz_0\ \left(n+1-B_2-\frac{B_1}{z_0}\right)\left(n-B_2-\frac{2B_1}{z_0}\right)}{\left(n-\frac{1}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},
\end{aligned} \tag{35b}$$

in the recurrence relations: Eqs. (25) if $(B_2/2)+(B_1/z_0)\neq 0,1/2$; Eqs. (26) if $(B_2/2)+(B_1/z_0)=1/2$; Eqs. (27) if $(B_2/2)+(B_1/z_0)=0$.

For the Mathieu equation we redefine $b_n(4)$ according to $b_n^{(4)}\to(2n+1)b_n^{(4)}$ and find the odd solutions

$$\begin{aligned} W_4^0(u) &= \sum_{n=0}^{\infty} b_n^{(4)} \sin[(2n+1)\sigma u], && |\cos(\sigma u)|<\infty, \\ W_4^{\infty}(u) &= \tan(\sigma u) \sum_{n=0}^{\infty} (2n+1)\, b_n^{(4)} K_{2n+1}\left[\pm 2i\sqrt{q}\cos(\sigma u)\right], && |\cos(\sigma u)|>1, \end{aligned} \tag{36a}$$

with the recurrence relations

$$\begin{aligned} & q b_4^{(4)} + [1-q-a]\, b_0^{(4)} = 0, \\ & q b_{n+1}^{(4)} + \left[(2n+1)^2 - a\right] b_n^{(4)} + q b_{n-1}^{(4)} = 0 \ (n\geq 1). \end{aligned} \tag{36b}$$

Now, for $\sigma=1$, $W_4^0(u)$ has period 2π.

3. Ince's limits for the double-confluent Heun equation

As in the case of the Ince limit of the GSWE, we have found no solution in the literature for the Ince limit of the DCHE. The solutions below are obtained by taking the limit $z_0\to 0$ (Leaver limit) of the solutions given in section 2 for the Ince limit of the GSWE. For this we use the formulas [12]

$$\lim_{c\to\infty} F\left(a,b;c;1-\frac{c}{y}\right) = y^a \Psi(a,a+1-b;y), \tag{37a}$$

$$\lim_{\alpha\to\infty}\left(1+\frac{y}{\alpha}\right)^{\alpha} = e^y \Rightarrow \lim_{z_0\to 0}\left(1-\frac{z_0}{z}\right)^{-B_1/z_0} = e^{B_1/z}. \tag{37b}$$

Actually, it is not necessary to use the second equation above, since we can get one pair of solutions as the limit of the first pair of section 2.1 and, then, generate the other pair by means of the transformation rule

$$\tau U(z) = e^{B_1/z} z^{2-B_2} U(-B_1, 4-B_2, B_3+2-B_2; q; z), \tag{38}$$

where $U(z)=U(B_1,B_2,B_3;q;z)$ denotes known solutions of Eq. (8). On the other hand, to check that the solutions $U_i^0(z)$ exhibit the behavior given in Eq. (4) when $z\to 0$, we may use the relation [12]

$$\lim_{|y|\to\infty}\Psi(a,b;y)\sim y^{-a}[1+O(|y|^{-1}],\quad -\frac{3\pi}{2}<\arg y<\frac{3\pi}{2}. \tag{39}$$

3.1. Solutions with a phase parameter

For the solution $U^0_{1\nu}$ of section 2.1, we find that the limit of the hypergeometric functions when z_0 tends to zero, B_2 and B_1 being fixed ($c = B_2 + B_1/z_0 \to \infty$), is given by

$$\lim_{z_0\to 0} F\left(n+\nu+\frac{B_2}{2}, -n-\nu-1+\frac{B_2}{2}; B_2+\frac{B_1}{z_0}; 1-\frac{z}{z_0}\right)$$
$$\propto z^{-\nu-\frac{B_2}{2}}\left(\frac{B_1}{z}\right)^n \Psi\left(n+\nu+\frac{B_2}{2}, 2n+2\nu+2; \frac{B_1}{z}\right).$$

Then, considering also the solution $U^\infty_{1\nu}$ and the limits for the coefficients in the recurrence relations, we get the first pair of solutions with a phase parameter ν (different of integer or half-integer). The rule τ leads to the second pair.

First pair

$$\begin{aligned} U^0_{1\nu}(z) &= z^{-\nu-\frac{B_2}{2}} \sum_{n=-\infty}^{\infty} b^{(1)}_n \left(\frac{B_1}{z}\right)^n \Psi\left(n+\nu+\frac{B_2}{2}, 2n+2\nu+2; \frac{B_1}{z}\right), \\ U^\infty_{1\nu}(z) &= z^{(1-B_2)/2} \sum_{n=-\infty}^{\infty} b^{(1)}_n K_{2n+2\nu+1}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{40a}$$

where in the recurrence relations (11a)

$$\begin{aligned} \alpha^{(1)}_n &= -\frac{qB_1\left(n+\nu+2-\frac{B_2}{2}\right)}{(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)}, \\ \beta^{(1)}_n &= 4B_3 + 4\left(n+\nu+1-\frac{B_2}{2}\right)\left(n+\nu+\frac{B_2}{2}\right) - \frac{qB_1(B_2-2)}{(n+\nu)(n+\nu+1)}, \\ \gamma^{(1)}_n &= \frac{qB_1\left(n+\nu+\frac{B_2}{2}-1\right)}{\left(n+\nu-\frac{1}{2}\right)(n+\nu)}. \end{aligned} \tag{40b}$$

Second pair

$$\begin{aligned} U^0_{2\nu}(z) &= e^{B_1/z} z^{-\nu-\frac{B_2}{2}} \sum_{n=-\infty}^{\infty} b^{(2)}_n \left(-\frac{B_1}{z}\right)^n \Psi\left(n+\nu+2-\frac{B_2}{2}, 2n+2\nu+2; -\frac{B_1}{z}\right), \\ U^\infty_{2\nu}(z) &= e^{B_1/z} z^{(1-B_2)/2} \sum_{n=-\infty}^{\infty} b^{(2)}_n K_{2n+2\nu+1}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{41a}$$

where

$$\alpha^{(2)}_n = \frac{qB_1\left(n+\nu+\frac{B_2}{2}\right)}{(n+\nu+1)\left(n+\nu+\frac{3}{2}\right)},\ \beta^{(2)}_n = \beta^{(1)}_n,\ \gamma^{(2)}_n = -\frac{qB_1\left(n+\nu+1-\frac{B_2}{2}\right)}{\left(n+\nu-\frac{1}{2}\right)(n+\nu)}, \tag{41b}$$

in the recurrence relations (11a) for $b^{(2)}_n$.

3.2. Solutions without phase parameter

These solutions may be derived by truncating the solutions of section 3.1. In this case, we see that there is only one choice for ν in each pair. Alternatively, the solutions can be found by applying the Leaver procedure to the first and third pairs of section 2.2.

First pair: $B_2 \neq 0, -1, -2, \cdots$. This corresponds to $\nu = (B_2/2) - 1$ in $(U^0_{1\nu}, U^\infty_{1\nu})$.

$$\begin{aligned} U_1^0(z) &= z^{1-B_2} \sum_{n=0}^{\infty} b_n^{(1)} \left(\frac{B_1}{z}\right)^n \Psi\left(n + B_2 - 1, 2n + B_2; \frac{B_1}{z}\right), \\ U_1^\infty(z) &= z^{(1-B_2)/2} \sum_{n=0}^{\infty} b_n^{(1)} K_{2n+B_2-1}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{42a}$$

with the following coefficients

$$\begin{aligned} \alpha_n^{(1)} &= -\frac{qB_1(n+1)}{\left(n+\frac{B_2}{2}\right)\left(n+\frac{B_2}{2}+\frac{1}{2}\right)}, \\ \beta_n^{(1)} &= 4B_3 + 4n\left(n + B_2 - 1\right) - \frac{qB_1(B_2-2)}{\left(n+\frac{B_2}{2}-1\right)\left(n+\frac{B_2}{2}\right)}, \\ \gamma_n^{(1)} &= \frac{qB_1(n+B_2-2)}{\left(n+\frac{B_2}{2}-\frac{3}{2}\right)\left(n+\frac{B_2}{2}-1\right)}. \end{aligned} \tag{42b}$$

in the recurrence relations for the $b_n^{(1)}$: Eqs. (25) if $B_2 \neq 1, 2$; Eqs. (26) if $B_2 = 1$; Eqs. (27) if $B_2 = 2$.

Second pair: $B_2 \neq 4, 5, 6, \cdots$. It corresponds to $\nu = 1 - (B_2/2)$ in $(U^0_{2\nu}, U^\infty_{2\nu})$ but can also be obtained from the first pair via the rule τ.

$$\begin{aligned} U_2^0(z) &= e^{B_1/z} z^{-1} \sum_{n=0}^{\infty} b_n^{(2)} \left(-\frac{B_1}{z}\right)^n \Psi\left(n + 3 - B_2, 2n + 4 - B_2; -\frac{B_1}{z}\right), \\ U_2^\infty(z) &= e^{B_1/z} z^{(1-B_2)/2} \sum_{n=0}^{\infty} b_n^{(2)} K_{2n+3-B_2}\left(\pm 2i\sqrt{qz}\right), \end{aligned} \tag{43a}$$

where

$$\begin{aligned} \alpha_n^{(3)} &= \frac{qB_1\ (n+1)}{\left(n+2-\frac{B_2}{2}\right)\left(n+\frac{5}{2}-\frac{B_2}{2}\right)}, \\ \beta_n^{(3)} &= 4B_3 + 4(n+1)(n+2-B_2) - \frac{qB_1(B_2-2)}{\left(n+1-\frac{B_2}{2}\right)\left(n+2-\frac{B_2}{2}\right)}, \\ \gamma_n^{(3)} &= -\frac{qB_1\ (n+2-B_2)}{\left(n+\frac{1}{2}-\frac{B_2}{2}\right)\left(n+1-\frac{B_2}{2}\right)}. \end{aligned} \tag{43b}$$

in the recurrence relations for $b_n^{(2)}$: Eqs. (25) if $B_2 \neq 2, 3$; Eqs. (26) if $B_2 = 3$; Eqs. (27) if $B_2 = 2$.

4. Potential applications

As we have mentioned, the Schrödinger equation with inverse fourth and sixth-power potentials can be reduced, respectively, to the double-confluent Heun equation (3) and its Ince limit (8). Singular potentials like these have appeared in the description of intermolecular forces [23] and in the scattering of ions by polarizable atoms. For the sake of illustration, we consider the last problem.

Before discussing these examples, let us present the so called normal forms of the DCHE, that is, the forms in which there is no first-order derivative terms in the differential equations. The general procedure for this, consists in writing the equation as

$$\frac{d^2U}{dz^2}+p(z)\frac{dU}{dz}+q(z)U=0.$$

Then, the substitution

$$U(z)=F(z)\exp\left(-\frac{1}{2}\int p(z)dz\right)$$

gives a first normal form, namely,

$$\frac{d^2F}{dz^2}+I(z)F=0,\quad I(z)=q(z)-\frac{1}{2}\frac{dp(z)}{dz}-\frac{1}{4}[p(z)]^2.$$

From this, other normal forms are obtained by the transformations

$$z=h(\vartheta),\quad F(z)=\sqrt{\frac{dh}{d\vartheta}}\,G(\vartheta)$$

which yield

$$\frac{d^2G}{d\vartheta^2}+J(\vartheta)G=0,\quad J(\vartheta)=I[h(\vartheta)]\left(\frac{dh}{d\vartheta}\right)^2+\frac{1}{2}\frac{d^3h}{d\vartheta^3}/\frac{dh}{d\vartheta}-\frac{3}{4}\left(\frac{d^2h}{d\vartheta^2}/\frac{dh}{d\vartheta}\right)^2.$$

By employing this procedure, Lemieux and Bose [21] have derived several normal forms for the general Heun equation and its confluent cases, excepting the triconfluent equation. These forms are useful to recognize whether a given equation belongs to the Heun class. Nevertheless, to find the solutions for the equation, we have to come back to the form for which the solutions were established, as below. The three Lemieux-Bose normal forms for the DCHE, together with the transformations of variables, are the following:

$$\begin{aligned}&U(z)=z^{-B_2/2}e^{B_1/(2z)}F(z),\\&\tfrac{d^2F}{dz^2}+\left[\omega^2-\tfrac{2\eta\omega}{z}+\tfrac{1}{z^2}\left(B_3-\tfrac{B_2^2}{4}+\tfrac{B_2}{2}\right)+\tfrac{B_1}{z^3}\left(1-\tfrac{B_2}{2}\right)-\tfrac{B_1^2}{4z^4}\right]F=0;\end{aligned}\tag{44}$$

$$\begin{aligned}&z=\rho^2,\quad U(z)=\rho^{(1-2B_2)/2}e^{B_1/(2\rho^2)}G(\rho)\Leftrightarrow G(\rho)=z^{(2B_2-1)/4}e^{-B_1/(2z)}U(z),\\&\tfrac{d^2G}{d\rho^2}+\left[4\omega^2\rho^2-8\eta\omega+\tfrac{4}{\rho^2}\left(B_3-\tfrac{B_2^2}{4}+\tfrac{B_2}{2}-\tfrac{3}{16}\right)+\tfrac{4B_1}{\rho^4}\left(1-\tfrac{B_2}{2}\right)-\tfrac{B_1^2}{\rho^6}\right]G=0;\end{aligned}\tag{45}$$

$$\begin{aligned}&z=e^{\lambda u},\quad U(z)=H(u)\exp\left[\tfrac{1}{2}\lambda(1-B_2)u+\tfrac{B_1}{2}e^{-\lambda u}\right]\ \Leftrightarrow\ H(u)=z^{(B_2-1)/2}e^{-B_1/(2z)}U(z),\\&\tfrac{d^2H}{du^2}+\lambda^2\left[B_3-\left(\tfrac{1-B_2}{2}\right)^2-\tfrac{B_1^2}{4}e^{-2\lambda u}-B_1\left(\tfrac{B_2}{2}-1\right)e^{-\lambda u}-2\eta\omega e^{\lambda u}+\omega^2e^{2\lambda u}\right]H=0,\end{aligned}\tag{46}$$

where λ is a constant at our disposal, for example, $\lambda = 1$ or $\lambda = i$. Note that, since these transformations involve neither η nor ω, their Ince limits are obtained by putting $\omega^2 = 0$ and $2\eta\omega = -q$.

Now we proceed with the scattering problem. The radial part $R(r) = \chi(r)/r$ of the wave function for the Schrödinger equation in three dimensions, for a particle with mass μ and energy E, is

$$\frac{d^2\chi(r)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r)\right]\chi(r) = 0, \tag{47}$$

where $k^2 = 2\mu E/\hbar^2$, l is the angular momentum and $V(r)$ is the potential. Now, according to Kleinman, Hahn and Spruch [22], for the interaction of a light particle of charge e' with a fixed atom of charge $Z\overline{e}$ containing z' electrons, we have

$$V(r) = \frac{(Z - z')\overline{e}e'}{r} - \frac{\alpha_1'(e')^2}{2r^4} - (\alpha_2' - 6a_0\beta_1')\frac{(e')^2}{2r^6}, \tag{48}$$

where r is the distance from the incident ion to the atom, $a_0 = \hbar^2/(\mu\overline{e}^2)$ is the Bohr radius, α_1' and α_2' are, respectively, the electric dipole and quadrupole polarizabilities of the atom and β_1' is a parameter resulting from a nonadiabatic correction (α_1', α_2' and β_1' are constants which describe the properties of the target only). For this potential, the Schrödinger equation becomes

$$\frac{d^2\chi}{dr^2} + \left[k^2 - \frac{2\mu(Z-z')\overline{e}e'}{\hbar^2 r} - \frac{l(l+1)}{r^2} + \frac{\mu\alpha_1'(e')^2}{\hbar^2 r^4} + \frac{\mu\left(\alpha_2' - 6a_0\beta_1'\right)(e')^2}{\hbar^2 r^6}\right]\chi = 0. \tag{49}$$

Therefore, for neutral targets ($Z = z'$) this is a particular case of the Ince limit of the DCHE, as we see from Eq. (45) with $\omega^2 = 0$, $2\eta\omega = -q$ and $z = \rho^2 = r^2$. On the hand, if the inverse sixth-power term vanishes ($\alpha_2' = 6a_0\beta_1'$), this radial Schrödinger equation is a particular case of the DCHE as seen from Eq. (44) with $B_2 = 2$, for neutral or ionized targets. In both cases the energy of the incident particle (k^2) is given and, consequently, there is no free parameter in these equations since the other constants are also fixed. Then, convergent solutions require a phase parameter ν, analogously to the scattering by the field of an electric dipole [2]. To obtain the radial dependence $R(r)$ we must convert Eq. (49) into the DCHE (3) and its limit (8). Below we discuss only the asymptotic behaviors of the solutions for the each case. By this reason, we do not write the recurrence relations for the coefficients.

Potential with inverse fourth and sixth-power terms. Eq. (45) suggests the substitutions

$$z = r^2, \quad \chi(r) = e^{-B_1/(2r^2)} r^{B_2-(1/2)} U(z = r^2) \quad \text{with}$$
$$B_1 = \pm\tfrac{e'}{\hbar}\sqrt{\mu(6a_0\beta_1' - \alpha_2')}, \quad B_2 = 2 - \tfrac{\alpha_1'(e')^2}{2\hbar^2 B_1}, \quad (6a_0\beta_1' \neq \alpha_2')$$

which transform the Schrödinger equation (49) into

$$z^2\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left[\left(\frac{B_2}{2} - \frac{1}{4}\right)\left(\frac{B_2}{2} - \frac{3}{4}\right) - \frac{l(l+1)}{4} + \frac{k^2}{4}z - \frac{\mu}{2}(Z - z')\sqrt{z}\right]U = 0.$$

Then, for $Z \neq z'$, the Schrödinger equation is more general than the Ince limit of DCHE. However, assuming a neutral target, we may form two pairs of solutions according to

$$R_{i\nu}(r) = \frac{1}{r}\chi_{i\nu}(r) = e^{-B_1/(2r^2)} r^{B_2-(3/2)} U_{i\nu}(z = r^2) \ (i = 1, 2) \tag{50}$$

where on the right-hand side the $U_{i\nu}$ represent the solutions with a phase parameter for the Ince limit of the DCHE, given in section 3.1. Then, taking into account that for this case $q = k^2/4 = \mu E/(2\hbar^2)$ and $z = r^2$, we find

$$\begin{aligned} R^0_{1\nu}(r) &= e^{-B_1/(2r^2)} r^{-2\nu-\frac{3}{2}} \sum_{n=-\infty}^{\infty} b_n^{(1)} \left(\frac{B_1}{r^2}\right)^n \Psi\left(n+\nu+\frac{B_2}{2}, 2n+2\nu+2; \frac{B_1}{r^2}\right), \\ R^\infty_{1\nu}(r) &= e^{-B_1/(2r^2)} r^{-1/2} \sum_{n=-\infty}^{\infty} b_n^{(1)} K_{2n+2\nu+1}\left(\pm ikr\right); \end{aligned} \tag{51}$$

$$\begin{aligned} R^0_{2\nu}(r) &= e^{B_1/(2r^2)} r^{-2\nu-\frac{3}{2}} \sum_{n=-\infty}^{\infty} b_n^{(2)} \left(-\frac{B_1}{r^2}\right)^n \Psi\left(n+\nu+2-\frac{B_2}{2}, 2n+2\nu+2; -\frac{B_1}{r^2}\right), \\ R^\infty_{2\nu}(r) &= e^{B_1/(2r^2)} r^{-1/2} \sum_{n=-\infty}^{\infty} b_n^{(2)} K_{2n+2\nu+1}\left(\pm ikr\right). \end{aligned} \tag{52}$$

From these expressions we obtain

$$\lim_{r\to\infty} R^\infty_{1\nu}(r) \propto \lim_{r\to\infty} R^\infty_{2\nu}(r) \sim \frac{e^{\mp ikr}}{r}, \quad -\frac{3\pi}{2} < \arg\left(\pm ikr\right) < \frac{3\pi}{2} \tag{53}$$

where we have employed the limit (15) for the modified Bessel functions. Thus, when $r \to \infty$, the solutions $R^\infty_{i\nu}$ are bounded even if k is a pure imaginary, since in this case $\exp(ikr)$ or $\exp(-ikr)$ goes to zero. At $r = 0$, Eq. (39) implies that

$$\begin{aligned} &\lim_{r\to 0} R^0_{1\nu}(r) \sim e^{-B_1/r^2} r^{B_2-(3/2)}, \quad -\frac{3\pi}{2} < \arg\frac{B_1}{r^2} < \frac{3\pi}{2}, \\ &\lim_{r\to 0} R^0_{2\nu}(r) \sim e^{B_1/r^2} r^{(5/2)-B_2}, \quad -\frac{3\pi}{2} < \arg\left(-\frac{B_1}{r^2}\right) < \frac{3\pi}{2}. \end{aligned} \tag{54}$$

Thence, if B_1 is a positive real number, the first limit goes to zero; if B_1 is a negative real number, the second limit goes to zero. However, if B_1 is a pure imaginary, we write

$$B_1 = iC, \quad B_2 = 2 + \frac{i\alpha_1'(e')^2}{2\hbar^2 C}$$

where C is real. Thus we find

$$|R^0_{1\nu}(r)| \propto |R^0_{2\nu}(r)| \sim \sqrt{r} \to 0.$$

Therefore, it is possible to find at least one pair of solutions for which both the solutions are bounded at the singularities.

Potential without inverse sixth-power term. From Eq. (44) we find that the substitutions

$$z=r,\quad \chi(r)=e^{-B_1/(2r)}r^{B_2/2}U(z=r)\ \text{ with }\ \hbar^2B_1^2=-4\mu(e')^2,\quad B_2=2$$

transform the Schrödinger equation (49) into

$$r^2\frac{d^2U}{dr^2}+(B_1+2r)\frac{dU}{dr}+\left[-l(l+1)-\frac{2\mu}{\hbar^2}(Z-z')\bar{e}e'r+k^2r^2+\frac{\mu(\alpha_2'-6a_0\beta_1')(e')^2}{\hbar^2r^4}\right]U=0.(55)$$

Thus, in absence of the inverse sixth-power term, the radial Schrödinger equation, even if we have a Coulomb term in the potential, may be solved by

$$R_{i\nu}(r)=e^{-B_1/(2r)}U_{i\nu}(z=r) \tag{56}$$

where $U_{i\nu}(z=r)$ are solutions with a phase parameter for the DCHE with $z=r$ and $B_2=2$ (see Appendix B). As

$$\omega=\pm k\leftrightarrow\pm\eta=\pm\frac{\mu}{k\hbar^2}(Z-z')\bar{e}e',\quad k=\frac{\sqrt{2\mu E}}{\hbar}, \tag{57}$$

those solutions give

$$\begin{aligned}
R^0_{1\nu}(r)&=e^{\pm ikr-\frac{B_1}{2r}}\sum_{n=-\infty}^{\infty}b_n\left(\frac{B_1}{r}\right)^{n+\nu+1}\Psi\left(n+\nu+1,2n+2\nu+2;\frac{B_1}{r}\right),\\
R^\infty_{1\nu}(r)&=e^{\pm ikr-\frac{B_1}{2r}}\sum_{n=-\infty}^{\infty}b_n(\mp 2ikr)^{n+\nu}\Psi(n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2ikr);
\end{aligned} \tag{58}$$

$$\begin{aligned}
R^0_{2\nu}(r)&=e^{\pm ikr+\frac{B_1}{2r}}\sum_{n=-\infty}^{\infty}b_n\left(-\frac{B_1}{r}\right)^{n+\nu+1}\Psi\left(n+\nu+1,2n+2\nu+2;-\frac{B_1}{r}\right),\\
R^\infty_{2\nu}(r)&=e^{\pm ikr+\frac{B_1}{2r}}\sum_{n=-\infty}^{\infty}b_n(\mp 2ikr)^{n+\nu}\Psi(n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2ikr).
\end{aligned} \tag{59}$$

Using Eq. (39), we find

$$\lim_{r\to\infty}R^\infty_{1\nu}(r)\propto\lim_{r\to\infty}R^\infty_{2\nu}(r)\sim r^{\mp i\eta}\,\frac{e^{\pm ikr}}{r},\quad -\frac{3\pi}{2}<\arg\left(\mp ikr\right)<\frac{3\pi}{2} \tag{60}$$

Thus, when $r\to\infty$, the solutions $R^\infty_{i\nu}$ are bounded even if k is a pure imaginary number, since in this case the behavior of $\exp(ikr)$ or $\exp(-ikr)$ predominates over the other factor. At $r=0$, by using Eq. (39) we get

$$\begin{aligned}
&\lim_{r\to 0}R^0_{1\nu}(r)\sim e^{-B_1/(2r)},\quad -\frac{3\pi}{2}<\arg\frac{B_1}{r}<\frac{3\pi}{2}\\
&\lim_{r\to 0}R^0_{2\nu}(r)\sim e^{B_1/(2r)},\quad -\frac{3\pi}{2}<\arg\left(-\frac{B_1}{r}\right)<\frac{3\pi}{2}.
\end{aligned} \tag{61}$$

As B_1 is a pure imaginary number, we find that

$$|R^0_{1\nu}(r)| \propto |R^0_{2\nu}(r)| \sim 1$$

Therefore, in this case we can form two pairs of solutions which are regular at the singular points, both pairs having the same series coefficients. For neutral targets ($\eta = 0$) the previous results have already been found by Bühring who has treated the Schrödinger equation as a DCHE [16, 24]. Before this author, the Schrödinger equation (for neutral targets and an inverse fourth-power polarization potential) had been transformed into a Mathieu equation [25, 26]. Thus, the Bühring approach is profitable since it works for ionized targets, too. In addition, as we have seen, for inverse sixth-power polarization potential, the Schrödinger equation may be transformed to the Ince limit of the DCHE, provided that the target is neutral.

5. Final remarks

We have constructed the differential equation (6) by applying the Ince limit, defined in Eq. (5), to a generalized spheroidal wave equation (GSWE). The Leaver limit ($z_0 \to 0$) of that equation has afforded Eq. (8) that turns out to be the Ince limit of a double-confluent Heun equation (DCHE) as well. The subnormal Thomé behavior at $z = \infty$, for the solutions of the these Ince limits of the GSWE and DCHE, distinguishes such equations from the original GSWE and DCHE hitherto considered in the literature.

In section 2, a pair of solutions (with a phase parameter) for the Ince limit of the GSWE has been found as the Ince limit of a pair of solutions for the original GSWE. One solution is given by a series of hypergeometric functions and the other by a series of modified Bessel functions of the second kind. Both solutions in that pair have the same series coefficients but different regions of convergence, as in solutions for the Mathieu equations. Other pair has followed from the first one by means of a transformation rule. Hence, four pairs of solutions without phase parameter have resulted from the truncation of the series with a phase parameter, that is, by restricting the summation index of the series to $n \geq 0$.

In section 3, solutions for the Ince limit of the DCHE have been established by taking the Leaver limit of solutions for the Ince limit of the GSWE. These solutions, given by series of irregular confluent hypergeometric functions and modified Bessel functions, present the appropriate behavior at the irregular singularities $z = 0$ and $z = \infty$. Note, nonetheless, that in sections 3 and 4 we have dealt with expansions in series of modified Bessel functions only. Other possibilities may be investigated, specially solutions in series of Bessel function products, as these could have important properties as regards the convergence of the series.

In the solutions without phase parameter for the Ince limits of the GSWE and DCHE, there are three possible forms to the recurrence relations for the series coefficients. This fact is relevant in itself and, in particular, is essential to recover solutions for the Mathieu equation from the ones for the Ince limit of the GSWE.

The solutions we have obtained for the Mathieu equation are already known and exhibit the usual parity and periodicity properties. This includes also the solutions found by Poole, given by two-sided series ($-\infty < n < \infty$) and having period $2\pi m$, where m is any integer greater than 1. However, we note that other types of solutions for the Mathieu equations (and also for the Whittaker-Hill equations) are possible, since these equations may be considered as particular cases of both the GSWE and double-confluent Heun equations as well [27].

At last, notice that we have point out no application for Ince limit of the GSWE. Nevertheless, in section 4 we have seen that the Schrödinger equation (49) for the scattering of low-energy particles by polarizable targets leads to an DCHE and its Ince limit. The exception is the Schrödinger equation with Coulomb and inverse sixth-power terms which requires solutions for a more general equation, possibly similar to an equation considered by Kurth and Schmidt in [28].

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Appendix A: Degenerate DCHEs

Let us show that DCHE

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left(B_3-2\eta\omega z+\omega^2z^2\right)U=0,\quad (B_1\neq 0,\quad \omega\neq 0),$$

for $B_1=0$ and/or $\omega=0$ degenerates into a confluent hypergeometric equation or an equation with constant coefficients. Thus, if $B_1=0$ and $\omega\neq 0$, the substitutions

$$y=-2i\omega z,\quad U(z)=e^{-y/2}y^{\alpha}f(y),\quad \alpha^2-(1-B_2)\alpha+B_3=0$$

give the confluent hypergeometric equation

$$y\frac{d^2f}{dy^2}+[(2\alpha+B_2)-y]\frac{df}{dy}-\left(i\eta+\alpha+\frac{B_2}{2}\right)f=0.$$

If $B_1\neq 0$ and $\omega=0$, the change of variables

$$y=B_1/z,\quad U(z)=y^{\beta}g(y),\quad \beta^2-(B_2-1)\beta+B_3=0$$

leads to

$$y\frac{d^2g}{dy^2}+[(2\beta+2-B_2)-y]\frac{dg}{dy}-\beta g=0.$$

If $B_1=\omega=0$, we find an equation with constant coefficients by taking $z=\exp y$.

Now let us show that the Ince limit of the DCHE

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+(B_3+qz)\,U=0,\ (q\neq 0,\ B_1\neq 0)$$

also gives degenerate cases if $q\neq 0$ and/or $B_1\neq 0$. In fact, if $q=0$ and $B_1\neq 0$, this equation is equivalent to the DCHE with $\omega=0$ and $B_1\neq 0$. If $q\neq 0$ and $B_1=0$, the substitutions

$$\xi=\pm 2i\sqrt{qz},\quad U(z)=\xi^{1-B_2}T(\xi)$$

reduces the equation to the modified Bessel equation

$$\xi^2 \frac{d^2 T}{d\xi^2} + \xi \frac{dT}{d\xi} - \left[(1-B_2)^2 - 4B_3 + \xi^2 \right] T = 0.$$

Finally, for $q = B_1 = 0$, we find again an equation with constant coefficients by taking $z = \exp y$.

Appendix B: The solutions in series of Bessel functions

The solution $U_{1\nu}^{\infty}(z)$ in series of Bessel functions can also be constructed as follows. We perform the substitutions

$$\xi = \pm 2i\sqrt{qz}, \quad U(z) = \xi^{1-B_2} Y(\xi) \tag{B1}$$

in the Ince limit of the GSWE (6). This yields

$$\begin{aligned} \xi^2 \frac{d^2 Y}{d\xi^2} + \xi \frac{dY}{d\xi} - \xi^2 Y = & -4qz_0 \frac{d^2 Y}{d\xi^2} - \frac{4q(z_0 - 2B_1 - 2B_2 z_0)}{\xi} \frac{dY}{d\xi} \\ & + \left[4q(1-B_2) \frac{2B_1 + B_2 z_0 + z_0}{\xi^2} + (1-B_2)^2 + 4qz_0 - 4B_3 \right] Y. \end{aligned} \tag{B2}$$

Now we expand $Y(\xi)$ according to

$$Y(\xi) = \sum_{n=-\infty}^{\infty} b_n^{(1)} K_\lambda(\xi), \quad \lambda = 2n + 2\nu + 1, \tag{B3}$$

where $K_\lambda(\xi)$ denotes the modified Bessel function of the second kind [17]. The last equation and (B1) afford the solution $U_{1\nu}^{\infty}(z)$.

When we insert (B3) into (B2), we use some difference-differential relations derived from the properties of K_λ [17]. Thus, we have

$$\xi^2 \frac{d^2 K_\lambda}{d\xi^2} + \xi \frac{dK_\lambda}{d\xi} - \xi^2 K_\lambda = \lambda^2 K_\lambda$$

on the left-hand side and

$$4\frac{d^2 K_\lambda}{d\xi^2} = K_{\lambda+2} + 2K_\lambda + K_{\lambda-2}, \quad \frac{4}{\xi}\frac{dK_\lambda}{d\xi} = -\frac{4\lambda}{\xi^2} K_\lambda + \frac{2}{\lambda - 1}\left(K_{\lambda-2} - K_\lambda \right)$$

on the right-hand side. This gives

$$\begin{aligned} & qz_0 \sum_{n=-\infty}^{\infty} \left[1 + \frac{2[1 - 2B_2 - (2B_1/z_0)]}{\lambda - 1} \right] b_n^{(1)} K_{\lambda-2}(\xi) \\ & + \sum_{n=-\infty}^{\infty} \left[\lambda^2 + 4B_3 - 2qz_0 - (1-B_2)^2 - \frac{2qz_0[1 - 2B_2 - (2B_1/z_0)]}{\lambda - 1} \right] b_n^{(1)} K_\lambda(\xi) \\ & + qz_0 \sum_{n=-\infty}^{\infty} b_n^{(1)} K_{\lambda+2}(\xi) \\ & = qz_0 \sum_{n=-\infty}^{\infty} \left[\left(1 - 2B_2 - \frac{2B_1}{z_0} \right) \lambda + (1-B_2)\left(1 + B_2 + \frac{2B_1}{z_0} \right) \right] b_n^{(1)} \frac{4K_\lambda(\xi)}{\xi^2}. \end{aligned}$$

To remove the term $4K_\lambda(\xi)/\xi^2$ on the right-hand side we use the relation

$$\frac{4K_\lambda}{\xi^2} = \frac{K_{\lambda-2}}{\lambda(\lambda-1)} - \frac{2K_\lambda}{(\lambda-1)(\lambda+1)} + \frac{K_{\lambda+2}}{\lambda(\lambda+1)}.$$

Then, reminding that $\lambda = 2n + 2\nu + 1$, we find

$$\sum_{n=-\infty}^{\infty} \alpha_{n-1}^{(1)} b_n^{(1)} K_{2n+2\nu-1} + \sum_{n=-\infty}^{\infty} \beta_n^{(1)} b_n^{(1)} K_{2n+2\nu+1} + \sum_{n=-\infty}^{\infty} \gamma_{n+1}^{(1)} b_n^{(1)} K_{2n+2\nu+3} = 0, \tag{B4}$$

where the coefficients $\alpha_n^{(1)}$, $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ are just the ones given in equations (16b). To get the recurrence relations with the form given in (11a), we change $n \to m+1$ and $n \to m-1$ in the first and third terms, respectively. After this, we equate to zero the coefficients of each independent $K_{2m+2\nu+1}(\xi)$.

On the other hand, to study the convergence of the series, we apply a Perron-Kreuser theorem [13] for the minimal solutions of the recurrence relations for $b_n^{(1)}$ and obtain (if $z_0 \neq 0$)

$$\lim_{n\to\infty} \frac{b_{n+1}^{(1)}}{b_n^{(1)}} = \lim_{n\to-\infty} \frac{b_{n-1}^{(1)}}{b_n^{(1)}} = -\frac{qz_0}{4n^2}. \tag{B5}$$

Using also the relation [17]

$$\lim_{\lambda\to\infty} K_\lambda(\xi) = \frac{1}{2}\Gamma(\lambda)\left(\frac{\xi}{2}\right)^{-\lambda}$$

and $K_{-\lambda}(\xi) = K_\lambda(\xi)$, we get

$$\lim_{n\to\infty} \frac{K_{2n+2\nu+3}(\xi)}{K_{2n+2\nu+1}(\xi)} = \lim_{n\to-\infty} \frac{K_{2n+2\nu-1}(\xi)}{K_{2n+2\nu+1}(\xi)} = -\frac{4n^2}{qz}.$$

Hence, we have

$$\lim_{n\to\infty} \frac{b_{n+1}^{(1)} K_{2n+2\nu+3}(\xi)}{b_n^{(1)} K_{2n+2\nu+1}(\xi)} = \lim_{n\to-\infty} \frac{b_{n-1}^{(1)} K_{2n+2\nu-1}(\xi)}{b_n^{(1)} K_{2n+2\nu+1}(\xi)} = \frac{z_0}{z}.$$

Therefore, by the ratio test the series converges for $|z| > |z_0|$. In (B5) we have supposed that $z_0 \neq 0$ but, if $z_0 = 0$, we find

$$\lim_{n\to\infty} \frac{b_{n+1}^{(1)}}{b_n^{(1)}} = \lim_{n\to-\infty} \frac{b_{n-1}^{(1)}}{b_n^{(1)}} = -\frac{B_1}{4n^3} \;\Rightarrow$$
$$\lim_{n\to\infty} \frac{b_{n+1}^{(1)} K_{2n+2\nu+3}(\xi)}{b_n^{(1)} K_{2n+2\nu+1}(\xi)} = \lim_{n\to-\infty} \frac{b_{n-1}^{(1)} K_{2n+2\nu-1}(\xi)}{b_n^{(1)} K_{2n+2\nu+1}(\xi)} = \frac{B_1}{nz}.$$

Thus, in this limit the series converges for $|z| > 0$ and per se this result is already included in $|z| > |z_0|$.

Appendix C: Solutions for the DCHE of section 4

The Leaver-type solutions for the DCHE (3) present some simplifications for $B_2 = 2$. The solutions given in Ref. [7] are expansions in series of regular and irregular confluent hypergeometric functions. However, to obtain the expected behavior at the singular points $z = 0$ and $z = \infty$, we have to choose the irregular functions. Then, by using the same notation of sections 2.1 and 3.1, we find that for $B_2 = 2$ the first pair of solutions with a phase parameter is given by

$$\begin{aligned} U_{1\nu}^{0}(z) &= e^{i\omega z} \sum_{n=-\infty}^{\infty} b_n \left(\frac{B_1}{z}\right)^{n+\nu+1} \Psi\left(n+\nu+1, 2n+2\nu+2; \frac{B_1}{z}\right), \\ U_{1\nu}^{\infty}(z) &= e^{i\omega z} \sum_{n=-\infty}^{\infty} b_n (-2i\omega z)^{n+\nu} \Psi(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega z), \end{aligned} \tag{C1}$$

and the second pair takes the form

$$\begin{aligned} U_{2\nu}^{0}(z) &= e^{i\omega z+\frac{B_1}{z}} \sum_{n=-\infty}^{\infty} b_n \left(-\frac{B_1}{z}\right)^{n+\nu+1} \Psi\left(n+\nu+1, 2n+2\nu+2; -\frac{B_1}{z}\right), \\ U_{2\nu}^{\infty}(z) &= e^{i\omega z+\frac{B_1}{z}} \sum_{n=-\infty}^{\infty} b_n (-2i\omega z)^{n+\nu} \Psi(n+\nu+1+i\eta, 2n+2\nu+2; -2i\omega z). \end{aligned} \tag{C2}$$

Then, we see that the two pairs have the same series coefficients b_n and the coefficients in the recurrence relations (11a) are simply

$$\alpha_n = i\omega B_1 \left(\tfrac{n+\nu+1-i\eta}{2n+2\nu+3}\right), \quad \beta_n = B_3 + (n+\nu)(n+\nu+1), \gamma_n = i\omega B_1 \left(\tfrac{n+\nu+i\eta}{2n+2\nu-1}\right). \tag{C3}$$

In these solutions ν cannot be integer or half-integer and the $U_{i\nu}^{0}$ converge for any finite z, whereas the $U_{i\nu}^{\infty}$ converge for $|z| > 0$. Note, moreover, that the irregular confluent hypergeometric functions that appear in $U_{i\nu}^{0}$ could be rewritten in terms of modified Bessel of the second kind by using the definition (14). In the solutions $U_{i\nu}^{\infty}$ the confluent hypergeometric functions could be rewritten in terms of the Hankel functions $H_{\rho}^{(1)}$ but only if $\eta = 0$ (neutral target, in the problem of section 4). For this we have to use the relation [12]

$$\Psi\left(\rho + \frac{1}{2}, 2\rho+1; -2ix\right) = \frac{i}{2\sqrt{\pi}} e^{i(\rho\pi - x)} H_{\rho}^{(1)}(x), \ \ \rho = n+\nu+\frac{1}{2}.$$

The asymptotic behaviors of the solutions given in (C1-2) may be found by using Eq. (39).

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