

Central extensions, classical non-equivariant maps and residual symmetries*

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The arising of central extensions is discussed in two contexts. At first classical counterparts of quantum anomalies (deserving being named as “classical anomalies”) are associated with a peculiar subclass of the non-equivariant maps. Further, the notion of “residual symmetry” for theories formulated in given non-vanishing EM backgrounds is introduced. It is pointed out that this is a Lie-algebraic, model-independent, concept.

1. INTRODUCTION

We discuss here the contents of two papers, [1] and [2], where two definitions have been proposed for two different phenomenons which are both related with the arising of centrally extended symmetry algebras.

In [1] the notion of “classical anomalies” has been introduced to describe a classical counterpart for the well-known notion of quantum anomalies. It can be said that a classical anomaly is present whenever the Noether charges of a given theory, endowed with a classical Poisson brackets structure, no longer close the original symmetry algebra of the action, but only its centrally extended version. Classical Poisson brackets are already sufficient to produce such an effect (i.e., it is not necessary to introduce a full commutator algebra for quantum operators). Perhaps the best known example is the Liouville theory [3], whose stress-energy tensor, even classically [4], satisfy $Vir \oplus Vir$, while the original conformal symmetry algebra of the classical action is $Witt \oplus Witt$, the direct sum of two copies of centerless Virasoro algebras. Even simpler examples can be given [1]. It is worth to point out that a “classical anomaly” is a very peculiar type of classical non-equivariant map. Indeed, it is a non-equivariant map associ-

ated with the Noether charges, i.e. the symmetries, of a classical action.

The second topic here discussed is the notion of “residual symmetries” introduced in [2]. These ones correspond to the surviving symmetries once an external (for sake of clarity let’s take an electromagnetic, not necessarily constant) background is turned on.

Previous works such as [5] investigated this issue for very simplified field models (e.g., in [5] a $U(1)$ free massive bosonic field in $1 + 1$ dimensions, minimally coupled to the external EM background was considered). On the other hand, as shown in [2] and discussed in section 3, the notion of “residual symmetry” is purely Lie-algebraic and model-independent. Any original Lie algebra, or better a D-module realization of it, admits its associated residual symmetry. To give an example, for a generic constant EM background, the Poincaré algebra in $(2 + 1)$ dimensions admits as residual symmetry the 5-generators Lie algebra $\mathcal{P}_c(2) \oplus o(2)$, where $\mathcal{P}_c(2)$ is the two-dimensional centrally extended Poincaré algebra discussed in [6]. According to the relative strength of the external electric versus magnetic field it could be of Euclidean or Minkowskian type.

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2. CLASSICAL ANOMALIES AS PECULIAR NON-EQUIVARIANT MAPS

The class of systems under consideration here consists of the classical dynamical systems which admit both a lagrangian and a hamiltonian description. It will be further assumed that the action \mathcal{S} admits an invariance under a group of symmetries which can be continuous (Lie), infinite-dimensional and/or super. The conserved Noether charges are associated to each generator of the symmetries of the action. When the hamiltonian dynamics is considered, the phase space of the theory possesses an algebraic structure given by the Poisson brackets. The existence of such a structure makes it possible to compute the Poisson bracket between any two given Noether charges. In the standard situation, the Poisson brackets among Noether charges realize a closed algebraic structure which is isomorphic to the original algebra of the symmetries of the action. It turns out, however, that this is not always the case. Indeed, it can happen that the algebra of Noether charges under Poisson bracket structure close a centrally extended version of the original symmetry algebra. Mimicking the quantum case, the following definition can be proposed for a classical dynamical system. The system is said to possess an anomalously realized symmetry, or in short a “classical anomaly”, if the following condition is satisfied: the symmetry transformations of the action admit Noether generators whose Poisson brackets algebra is a centrally extended version of the algebra of symmetry transformations.

Therefore a classical anomaly is a very specific case of “non-equivariant map” (for a discussion in a finite-dimensional setting see [7]). Not all non-equivariant maps discussed in the literature are classical anomalies. For instance the one-dimensional free-particle conserved quantities p (the momentum) and $pt - mx$ generate a non-equivariant map (the Poisson bracket between p and $pt - mx$ is proportional to the mass m). However, despite being conserved, they do not generate a symmetry of the action and for that reason they are not Noether charges.

On the other hand, infinite-dimensional non-

equivariant moment maps were constructed in [8]. In those papers the only explicit application concerned the dynamical systems of KdV type (classical integrable hierarchies). Such systems, in contrast with the examples discussed here, admits a hamiltonian description, but not a lagrangian formulation. Even if conserved quantities can be constructed, they can not be interpreted as Noether charges.

The possibility for a classical anomaly to occur is based on very simple and nice mathematical consistency conditions, implemented by the Jacobi-identity property of the given symmetry algebra. Let us illustrate this point by considering some generic (but not the most general) scheme. Let us suppose that the (bosonic) generators δ_a 's of a symmetry invariance of the action satisfy a linear algebra whose structure constants satisfy the Jacobi identity, i.e. $[\delta_a, \delta_b] = f_{ab}^c \delta_c$, while $[\delta_a, [\delta_b, \delta_c]] + [\delta_b, [\delta_c, \delta_a]] + [\delta_c, [\delta_a, \delta_b]] = 0$. The associated Noether charges Q_a 's are further assumed to be the generators of the algebra, i.e., applied on a given field ϕ they produce

$$\delta_a \phi = \{Q_a, \phi\}, \quad (1)$$

where the brackets obviously denote the Poisson-brackets.

The condition

$$[\delta_a, \delta_b] \phi = f_{ab}^c \delta_c \phi, \quad (2)$$

puts restriction on the possible Poisson brackets algebra satisfied by the Noether charges. It is certainly true that $\{Q_a, Q_b\} = f_{ab}^c Q_c$, (which corresponds to the standard case) is consistent with both (1) and (2). However, in a generic case, it is not at all a necessary condition since more general solutions can be found. Indeed, the presence of a central extension, expressed through the relation $\{Q_a, Q_b\} = f_{ab}^c Q_c + k * \Delta_{ab}$, (where k is a central charge and the function Δ_{ab} is antisymmetric in the exchange of a and b), is allowed.

Indeed, since the relation

$$\{Q_a, \{Q_b, \phi\}\} - \{Q_b, \{Q_a, \phi\}\} = \{\{Q_a, Q_b\}, \phi\} \quad (3)$$

holds due to the Jacobi property of the Poisson bracket structure (which is assumed to be satisfied), no contradiction can be found with (2);

the right hand side of (3) in fact is given by $\{f_{ab}{}^c Q_c + k * \Delta_{ab}, \phi\} = \{f_{ab}{}^c Q_c, \phi\} = f_{ab}{}^c \delta_c \phi$, due to the fact that k is a central term and has vanishing Poisson brackets with any field.

This observation on one hand puts restrictions on the possible symmetries for which a classical anomaly can be detected; the symmetries in question, on a purely algebraic ground, must admit a central extension. This is not the case, e.g., for the Lie groups of symmetry based on finite simple Lie algebras. On the other hand one is warned that, whenever a symmetry *does* admit an algebraically consistent central extension, it should be carefully checked, for any specific dynamical model which concretely realizes it, whether it is satisfied exactly or anomalously. This remark already holds at the classical level, not just for purely quantum theories.

Some further points deserve to be mentioned. The first one concerns the fact that the quantization procedure (which, for the cases we are here considering, can be understood as an explicit realization of an abstract Poisson brackets algebra as an algebra of commutators between operators acting on a given Hilbert space) can induce anomalous terms for theories which, in their classical version, are not anomalous in the sense previously specified. It therefore turns out that the occurrence of classical anomalies is a phenomenon which is “more difficult to observe” than the corresponding appearance of quantum anomalies since it occurs more seldom.

A second point concerns the fact that the algebra of Poisson brackets, as an abstract algebra, is assumed to satisfy the Leibniz property. This is no longer the case for its concrete realization given by the algebra of commutators. The Noether charges are in general non-linearly constructed with the original fields ϕ_i (which collectively denote the basic fields and their conjugate momenta) of a given theory. For such a reason it is only true in the classical case that, whenever an anomalous central charge in an infinitesimal linear algebra of symmetries is detected, it can be normalized at will by a simultaneous rescaling of all the fields ϕ_i involved ($\phi_i \mapsto \alpha \cdot \phi_i$) and of the Poisson brackets normalization ($\{.,.\} \mapsto \frac{1}{\alpha} \{.,.\}$), for an arbitrary real constant α . In the classical

case any central charge different from zero can therefore be consistently set equal to 1. However in the quantum case a specific value of the central charge is fixed by the type of representation of the symmetry algebra associated with the given model and is a genuine physical parameter (the role of the Virasoro central charge in labeling the conformal minimal models is an example). The above argument is not, however, (at least directly) applicable to non-linear symmetries, such as those leading to the classical counterparts of the Fateev–Zamolodchikov W -algebras.

3. RESIDUAL SYMMETRIES IN THE PRESENCE OF AN EM BACKGROUND

Let us discuss in detail for the sake of simplicity the case of the residual symmetry for generic Poincaré-invariant field theories in (2+1)-dimension, coupled with an external constant EM background. The generalization of this procedure to higher-dimensional theories and non-constant EM backgrounds is straightforward and immediate.

In the absence of the external electric and magnetic field, the action \mathcal{S} is assumed to be invariant under a 7-parameter symmetry, given by the six generators of the (2 + 1)-Poincaré symmetry which, when acting on scalar fields (the following discussion however is valid no matter which is the spin of the fields) are represented by

$$\begin{aligned} P_\mu &= -i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \end{aligned} \quad (4)$$

(the metric is chosen to be + --), plus a remaining symmetry generator corresponding to the internal global $U(1)$ charge that will be denoted as Z .

It is further assumed that in the action \mathcal{S} the dependence on the classical background field is expressed in terms of the covariant gauge-derivatives

$$D_\mu = \partial_\mu - ieA_\mu,$$

with e the electric charge.

In the presence of constant external electric and magnetic fields, the $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ field-

strength is constrained to satisfy

$$F^{0i} = E^i, \quad F^{ij} = \epsilon^{ij} B, \quad (5)$$

where $\mu, \nu = 0, 1, 2$ and $i, j = 1, 2$. The fields E^i and B are constant. Without loss of generality the x^1, x^2 spatial axis can be rotated so that $E^1 \equiv E$, $E^2 = 0$. Throughout the text this convention is respected.

In order to recover (5), the gauge field A_μ must depend at most linearly on the coordinates $x^0 \equiv t$, $x^1 \equiv x$ and $x^2 \equiv y$.

The gauge-transformation

$$A_\mu \mapsto A_\mu' = A_\mu + \frac{1}{e} \partial_\mu \alpha(x^\nu) \quad (6)$$

allows to conveniently choose for A_μ the gauge-fixing

$$\begin{aligned} A_0 &= 0, \\ A_i &= E_i t - \frac{B}{2} \epsilon_{ij} x^j. \end{aligned} \quad (7)$$

The above choice is a good gauge-fixing since it completely fixes the gauge (no gauge-freedom is left). It will be soon evident that the residual symmetry is a truly physical symmetry, independent of the chosen gauge-fixing.

Due to (7), the action \mathcal{S} explicitly depends on the x^μ coordinates entering A_μ . The simplest way to compute the symmetry property of an action such as \mathcal{S} which explicitly depends on the coordinates consists in performing the following trick. At first A_μ is regarded on the same foot as the other fields entering \mathcal{S} and assumed to transform as standard vector field under the global Poincaré transformations, namely

$$A_\mu'(x^{\rho'}) = \Lambda_\mu{}^\nu A_\nu(x^\rho) \quad (8)$$

for $x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + a^\mu$.

For a generic infinitesimal Poincaré transformation, however, the transformed A_μ gauge-field no longer respects the gauge-fixing condition (7). In the active transformation viewpoint only fields are entitled to transform, not the space-time coordinates themselves. A_μ plays the role of a fictitious field, inserted to take into account the dependence of the action \mathcal{S} on the space-time coordinates caused by the non-trivial background.

Therefore, the overall infinitesimal transformation δA_μ should be vanishing. This result can be reached if an infinitesimal gauge transformation (6) $\delta_g(A_\mu)$ can be found in order to compensate for the infinitesimal Poincaré transformation $\delta_P(A_\mu)$, i.e. if the following condition is satisfied

$$\delta(A_\mu) = \delta_P(A_\mu) + \delta_g(A_\mu) = 0. \quad (9)$$

Only those Poincaré generators which admit a compensating gauge-transformation satisfying (9) provide a symmetry of the \mathcal{S} action (and therefore enter the residual symmetry algebra). This is a plain consequence of the original assumption of the Poincaré and manifest gauge invariance for the action \mathcal{S} coupled to the gauge-field A_μ .

Notice that the original Poincaré generators are deformed by the presence of extra-terms associated to the compensating gauge transformation. Let p denote a generator of (4) which “survives” as a symmetry in the presence of the external background. The effective generator of the residual symmetry is

$$\hat{p} = p + (\dots),$$

where (\dots) denotes the extra terms arising from the compensating gauge transformation associated to p . Such (\dots) extra terms are gauge-fixing dependent. The “residual symmetry generator” \hat{p} can only be expressed in a gauge-dependent manner. However, two gauge-fixing choices are related by a gauge transformation \mathbf{g} . The residual symmetry generator in the new gauge-fixing, denoted as \tilde{p} , is related to the previous one by an Adjoint transformation

$$\tilde{p} = \mathbf{g} \hat{p} \mathbf{g}^{-1}. \quad (10)$$

Therefore the residual symmetry algebra does not depend on the choice of the gauge fixing and is a truly physical characterization of the action \mathcal{S} .

The extra-terms (\dots) are necessarily linear in the space-time coordinates when associated with a translation generator, and bilinear when associated to a surviving Lorentz generator (for a constant EM background). Their presence implies the arising of the central term in the commutator of the deformed translation generators.

The residual symmetry algebra of the $(2 + 1)$ -Poincaré theory involves, besides the global $U(1)$ generator Z , the three deformed translations and just one deformed Lorentz generator (the remaining two Lorentz generators are broken).

Within the (7) gauge-fixing choice the deformed translations are explicitly given by

$$\begin{aligned} P_0 &= -i\partial_t - eEx, \\ P_1 &= -i\partial_x - \frac{e}{2}By, \\ P_2 &= -i\partial_y + \frac{e}{2}Bx. \end{aligned} \quad (11)$$

The deformed generator of the residual Lorentz symmetry is explicitly given, in the same gauge-fixing and for $E \neq 0$, by

$$\begin{aligned} M &= i(x\partial_t + t\partial_x) - i\frac{B}{E}(y\partial_x - x\partial_y) + \\ &\quad \frac{e}{2}(Et^2 + Ex^2 - Bty). \end{aligned} \quad (12)$$

The residual symmetry algebra can be easily computed. The $U(1)$ charge Z is no longer decoupled from the other symmetry generators. It appears instead as a central charge.

The 5-generator solvable, non-simple Lie algebra of residual symmetries admits a convenient presentation. The generator

$$\tilde{Z} \equiv BP_0 + EP_2 \quad (13)$$

not only commutes with all the other $*$ generators

$$[\tilde{Z}, *] = 0, \quad (14)$$

for $E \neq B$ it is not even present in the r.h.s., so that the residual symmetry algebra is given by a direct sum of $u(1)$ and a 4-generator algebra. The latter algebra is isomorphic to the centrally extended two-dimensional Poincaré algebra. Such an algebra is of Minkowskian or Euclidean type according to whether $E > B$ or respectively $E < B$. This point can be intuitively understood due to the predominance of the electric or magnetic effect (in the absence of the electric field the theory is manifestly rotational invariant, so that the Lorentz generator is associated with the Euclidean symmetry). We have explicitly, for $B > E$, that the algebra

$$\begin{aligned} [\overline{M}, S_1] &= iS_2, \\ [\overline{M}, S_2] &= -iS_1 \end{aligned} \quad (15)$$

is reproduced by

$$\begin{aligned} \overline{M} &= \frac{E}{\sqrt{B^2 - E^2}}M, \\ S_1 &= P_0 + \frac{B}{E}P_2, \\ S_2 &= \frac{\sqrt{B^2 - E^2}}{E}P_1, \end{aligned} \quad (16)$$

while for $E > B$ the algebra

$$\begin{aligned} [\tilde{M}, T_1] &= iT_2, \\ [\tilde{M}, T_2] &= iT_1, \end{aligned} \quad (17)$$

is reproduced by

$$\begin{aligned} \tilde{M} &= \frac{E}{\sqrt{E^2 - B^2}}M, \\ T_1 &= P_0 + \frac{B}{E}P_2, \\ T_2 &= -\frac{\sqrt{E^2 - B^2}}{E}P_1. \end{aligned} \quad (18)$$

In both cases the commutator between the translation generators S_1, S_2 , and respectively T_1, T_2 , develops the central term proportional to Z which can be conveniently normalized.

The residual symmetry algebra of the $(2 + 1)$ case for generic values of E and B (the special case $E = B$ is degenerate) is therefore given by the direct sum

$$u(1) \oplus \mathcal{P}_c(2). \quad (19)$$

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