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AN ACTION PRINCIPLE FOR A HOT PLASMA IN CURVED SPACE-TIME

**H. Heintzmann* and

*M. Novello

* Centro Brasileiro de Pesquisas Físicas
Av. Wenceslau Bras nº 71 - fundos
20000 Rio de Janeiro - RJ - BRASIL

** Permanente adress: Institut für Theoretische Physik
D5000 Köln 41, W-germany

ABSTRACT

We provide a manifestly covariant extension of the Low-Sturrock Lagrangian and investigate its invariance properties. It is pointed out that the Boltzmann - Vlasov equation follows from a gauge invariance of the Lagrangian, which is a consequence of the lack of uniqueness to label particle trajectories. Whitham's wave-vector action is shown to be conserved in curved space-time and the equivalent dielectric tensor for a cold plasma is derived in lowest order eikonal approximation.

I. INTRODUCTION

Although recent years have seen a considerable progress[1] in the formulation of relativistic thermodynamics and plasma physic a selfconsistent, Poincaré-covariant statistical mechanics of particles interacting also by long-range forces has not been achieved (and is nowhere in sight). Pulsar magnetospheres, accreting binary stellar systems and relativistic jets of plasma such as possibly observed in Quasars and in SS433, however, all demand a fully relativistic treatment including curved space-time and provide thus a strong motivation to incorporate general relativity into plasma physics.
As we are not interested here in a description of the background plasma in curved space-time but rather in the propagation of disturbances (waves) through this plasma we circumvent the above mentioned difficulties[2] by assuming that (globally) a partition function for the plasma exists and that it obeys the Liouville equation in curved space-time[2]. We shall further assume that the background fields are known exactly and we shall ignore the back-reaction of the disturbance on space-time. We shall treat the extreme plasma limit, where the wavelength of the perturbation is much larger than interparticle distances. Our treatment is therefore complementary to the one given recently by Thorne[3] who considered the near vacuum case, which may be more appropriate for the study of the propagation of X-rays and γ-rays in concrete astrophysical situations, whereas ours applies more to the propagation of radio waves.

Our study will be based on a relativistic version of the eikonal method, which will be described in the first section and on an action principle, which allows one to derive many useful results in an economic and elegant way. The reader, who is interested in the mathematical subtilties of the problem is referred to the article by Brener and Ehlers (who consider however only the cold plasma case and who use a less elegant method. In the third section we introduce the averaged Lagrangian and in the fourth section we apply it to small amplitude waves. The final section is devoted to some applications of the method.

1. The relativistic eikonal method

Generally one cannot solve Maxwell's equations exactly for waves propagating through an inhomogeneous medium. The problem becomes even more complex if one wishes to do this for a medium in accelerated motion or in curved space-time, as is for example necessary if one wishes to describe wave propagation through a pulsar magnetosphere or through the accreting gas of a black hole. The standard treatments using plane wave solutions to describe wave propagation in a homogeneous plasma in a flat space time[5] are no longer applicable. In the geometrical optics approximation[6], where the wavelengths λ are small in comparison with the "inhomogeneity scale" ℓ, a refined version of Hamilton's theory of rays[7], first described by Weinberg[8] leads to radiative transfer equations which determine the change in wave vector, polarization state and intensity. In this eikonal method one ignores internal reflection. A proof does not exist, but the general conjecture is that the amplitudes of the reflected waves go to zero as $\exp(-\ell/\lambda)$. Thus they cannot be obtained by a method which is essentially a power series expansion in λ/ℓ. In the eikonal method it is assumed that the four-potential of the waves $\delta A^i(x)$ can be written as*

$$\delta A^k(x) = f(x)\; a^k(x)\; e^{i\psi(x)} \tag{1.1}$$

The scalar f is called the amplitude, the suitably normalized four-vector a^k (x) is called the polarization vector, and ψ (x) is the eikonal. f(x) and $a^k(x)$ are slowly varying functions of x, whereas the eikonal $\psi(x)$ describes the rapid oscillations of the waves, $\psi_{|i}:=k_i$, where k_i, is the wave four-vector. The four components of k_i are not independent, but for given medium they are related by a scalar relation which is characteristic for the medium and which is called the dispersion relation, $D = D(k_i, x_j) = 0$.

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* we use the notation of Landau and Lifshitz[7], i.e. latin indices run from zero to three, greek ones from one to three. The signature of the metric is (- +++). and g is the determinant of the metric g_{ab}.

In order to solve the eikonal equation $D(\psi_{|i}, x_j) = 0$ and to construct $\psi(x)$ one uses the method of ray tracing[4)8)], i. e. one solves for the ordinary differential equations

$$\dot{x}^i = \frac{dx^i}{d\ell} = \frac{\partial D}{\partial k_i} \tag{1.2}$$

$$\dot{k}^i := k^i{}_{;j}\, \dot{x}^j = -\frac{\partial D}{\partial x_i}$$

$$k_{i;j} = k_{j;i}$$

and $\psi(x)$ is then given by

$$\psi(x) = \int k_i\, \dot{x}^i\, d\ell \tag{1.3}$$

Locally the waves described by (1.1) are therefore plane waves, and the long range effects of the medium and the geometry are taken into account by propagation laws. The most general way of describing wave propagation would certainly be that of phenomenological electrodynamics[7)9)] where one uses the induction tensor δH^{ab} together with the field tensor $\delta F^{ab} = \delta A^{b|a} - \delta A^{a|b}$. δH^{ab} satisfies the field equation

$$\delta H^{ab}{}_{;b} = \frac{4\pi}{c}\, j^a_{ext} \tag{1.4}$$

and is related to δF^{ab} by the permeability four-tensor. To lowest order eikonal approximation this relation can be written[9)]

$$\delta H^{ab}(x) = \tilde{\varepsilon}^{ab}{}_{cd}(x,k)\,\delta F^{cd}(x) \tag{1.5}$$

and , in the absence of external currents, leads to

$$\delta F^{ab} = i f (k^a a^b - k^b a^a) e^{i\psi} \tag{1.6}$$

$$\delta H^{ab} k_b = i f \tilde{\varepsilon}^{ab}{}_{cd} k_b (k^c a^d - a^c k^d) e^{i\psi} = 0$$

The trivial symmetry relations for $\tilde{\varepsilon}^{ab}{}_{cd}$ are

$$\tilde{\varepsilon}^{ab}{}_{cd} = - \tilde{\varepsilon}^{ba}{}_{cd} = - \tilde{\varepsilon}^{ab}{}_{dc}$$

and for a loss-free medium due to the Onsager relations

$$\tilde{\varepsilon}_{abcd} = \bar{\tilde{\varepsilon}}_{cdab} =: \tilde{\varepsilon}^*{}_{abcd} \qquad (1.7)$$

where the bar denotes the complex conjugate quantity.
Introducing $L^{ad} := \tilde{\varepsilon}^{abcd} k_b k_c$, equ. (1.6) can be written

$$L^{ad} a_d = 0 \qquad (1.8)$$

$$L^{ad} = (L^{ad})^* = \bar{L}^{da}$$

Due to the symmetry relations of $\tilde{\varepsilon}$ this equation has the trivial solution $a_d = k_d$. This implies that $\det \| L^{ad} \| \equiv 0$, i.e. $\det \| L^{ad} \| = 0$ is not the dispersion relation. It can be shown[5] that a covariant dispersion relation is obtained if one chooses a specific gauge for a_d, for example $a_d k^d = 0$, and adds this gauge condition $L^{ad} a_d \to (L^{ad} + k^a k^d) a_d$ to $L^{ad} a_d$. $\operatorname{Det} \| L^{ad} + k^d k^a \| = 0$ is then the dispersion relation (including the spurious mode $k_i k^i = 0$ corresponding to $k^i a_i = 0$). Equ. (1.8) determines further the polarization vector a^i (up to a gauge mode).
In order to determine the amplitude f, however, dynamical quantities are needed (and one has to proceed to first order approximation to obtain these). In geometric optics[7] Landau and Lifshitz use the physical argument that in a loss-free medium energy must be conserved on the average, leading to the requirement $\operatorname{div} \vec{S} = 0$ from which the amplitude can be obtained by simple ray tracing since $\vec{S}$ points in the direction of the group velocity. In a medium in arbitrary motion energy is not conserved but the number of "photons", as represented by the numbers of rays, still is. Multiplying equ. (1.8) by $\bar{a}^a$ we obtain $\bar{L} = \bar{a}_a L^{ad} a_d = 0$.

It is easy to show that $\bar{L}$ is proportional to the dispersion relation[5], and in geometrical optics the space part of

$$N^{i} := \frac{\partial}{\partial k_{i}} \bar{L} = \bar{a}_{a} a_{d} \frac{\partial}{\partial k_{i}} L^{ad} \tag{1.9}$$

coincides with $\omega^{-1} \vec{S}$, i.e. it satisfies $N^{\alpha}{}_{|\alpha} = 0$. The relativistic generalization of the conserved current therefore will be (1.9) if we can show that $N^{i}{}_{;i} = 0$ in general. N^{i} is sometimes called wave-vector action, and its conservation gives rise, in the standard manner, to the adiabatic invariant $I = N^{i} d\Sigma_{i} = N^{0} dV$, where dV is an arbitrary, infinitesimal volume element which is propagated along the rays of equ. (1.2). In linear theory the average energy-momentum tensor $\bar{T}^{ik}$ is related to N^{k} as follows[5) 6)]

$$\bar{T}^{ik} = k^{i} N^{k} \tag{1.10}$$

Now suppose the physical system under consideration (crystal, plasma, etc.) can be derived from a Lagrangian.
In this case equ. (1.10) is nothing but the averaged canonical energy momemtum tensor[6], and for the divergence of $\bar{T}^{ik}$ we would obtain

$$\bar{T}^{ik}{}_{;k} = - \bar{L}^{|i} \tag{1.11}$$

Using $\bar{L} = 0$ as the dispersion relation in (1.2) then immediately leads to ($\bar{L}^{|i}$ is the functional and not the total derivative with respect to x^{i})

$$N^{i}{}_{;i} = 0 \quad N^{i} := \frac{\partial}{\partial k_{i}} \bar{L} \tag{1.12}$$

The method of an averaged Lagrangian, which was first considered by Whitham[12] and subsequently investigated by many authors, for example[13) 14)], is therefore an elegant and powerful means to provide a manifestly covariant basis for relativistic geometrical optics. (Note that N^{α} was introduced for the first time by Sturrock[10] (1962), although his work did not have much influence on later development).

However, the price one pays is a loss of generality in that one does not always know how to construct a Lagrangian for a given physical system. Moreover, if one is interested in the phase of the amplitude one has to take recourse to the full system of differential equations[4) 8) 9)]. On the other hand, if one has a Lagrangian at hand, many results follow with considerable ease from its invariance properties via E. Noether's theorems[15) 16)]. In particular, equ. (1.12) is a direct consequence of the eikonal ansatz (1.1) and holds rigorously and not only to lowest order eikonal approximation. As mentioned above, to establish equ. (1.12) $\tilde{\varepsilon}^{ab}{}_{cd}$ must be known to first order eikonal approximation. This fact was overlooked by the authors of ref. 9, and their result concerning the divergence of N^i is wrong. A second gratifying aspect is that our treatment can immediately be extended to curved space-times.

2. The field Lagrangian for particle displacements

It is well know how to describe particle motion in given exterior fields by means of a variational principle and so is the representation of field equations for the electromagnetic field by similar principles[6)]. The synthesis of both into one single principle is complicated by the fact that particles and fields are described by different types of variables: "Lagrangian" for the particles and "Eulerian" for the fields[18)]. Sturrock[17)] and Low[18)] have devised a procedure to overcome this difficulty at least in part. They introduced displacements ξ^k which describe the motion of particles under the action of a perturbation field such that the unperturbed trajectory $x^k = x^k(s)$ goes over into $\tilde{x}^k = x^k + \xi^k$. ξ^k will be a "field-like" variable if we require that $\tilde{x}^k = x^k + \xi^k(x)$. It is well known that such a labelling is not unique since the transformation $(\tilde{\xi}^k(x) = \Delta x^k + \xi^k(x+\Delta x))$

$$\tilde{\xi}^k = \xi^k + ds(u^k + \xi^k{}_{|i}\, u^i) \qquad (2.1)$$

in which ds may be regarded as an arbitrary infinitesimal function that generates a set of representations of the same physical system[17)]. This gives rise to a further gauge invariance of the theory (apart from the indeterminacy of the potentials of the electromagnetic field[6] and leads to a "strong conservation law"[15][16], which, as we shall show, is just the conservation of particles for each species (whereas the gauge invariance of the electromagnetic potentials leads only to charge conservation of the *total* charge[6], which is weaker - only for a charge-separated plasma do the two coincide).

The result of the procedure, however, is rather formal, and only if the displacements are small so that the Lagrangian allows a power series expansion in the displacement amplitude does one obtain physically useful Lagrangians which can be treated in the standard manner[6]. However, the advantage of the more formal treatment is that it leads to manifestly covariant expressions, valid rigorously, and that it allows one easily to investigate the invariance properties of the Lagrangians to all orders of approximation.

Sturrock starts from the particle action S: $= \sum_A S_{mA} + S_{mfA}$, where the sum over A is over the different charged species and where[6]

$$S_{mA} = -c \sum_{l=1}^{N_A} m_A \int ds_{Al} \tag{2.2}$$

$$S_{mfA} = \frac{1}{c} \sum_{l=1}^{N_A} e_A \int A_k(x_{AL})\, dx^k_{Al} = \frac{1}{c} \int \sqrt{-g}\, j_k(x) A^k(x)\, d^4x \tag{2.3}$$

Henceforth we shall drop the index A and consider each kind of particles separately; in the final result one can easily restore the sum over all species. The equations of motion for the background plasma are (for each species)

$$u^i_{;k} u^k = \frac{e}{mc^2} F^i{}_k u^k \tag{2.4}$$

The electromagnetic field equations follow from the total action $S_{tot} = S + S_f$

$$S_f = -\frac{1}{16\pi} \int \sqrt{-g}\, d^4x\, F^{ik} F_{ik} \qquad F_{ik} = A_{k;i} - A_{i;k} \tag{2.5}$$

and read[6])

$$F^{ik}{}_{;k} = \frac{4\pi}{c} j^{i} \tag{2.6}$$

If, under the action of a perturbation force, the particles at x go to $\tilde{x}$, the new equations will follow from the actions

$$\tilde{S} = - mc \sum_{l=1}^{N} \int d\tilde{s}_l + \frac{e}{c} \sum_{l=1} \int \tilde{A}_k \, d\tilde{x}_l^k \tag{2.7}$$

and

$$\tilde{S} = - \frac{1}{16\pi} \int \sqrt{-g}\, \tilde{F}^{ik} \, \tilde{F}_{ik} \, d^4 x + \frac{1}{c} \int \sqrt{-g}\, \tilde{j}^{i}(x) \tilde{A}_{i}(x) d^4 x \tag{2.8}$$

respectively, and give

$$\tilde{u}^{i}{}_{;k} \, \tilde{u}^{k} = \frac{e}{mc^2} \, \tilde{F}^{i}{}_{k} \tilde{u}^{k}$$

$$\tilde{u}^{i} = \tilde{u}^{i}(\tilde{s}) \qquad \tilde{u}^{i}\tilde{u}_{i} = -1 \tag{2.9}$$

$$\tilde{F}^{ik}{}_{;k} = \frac{4\pi}{c} \tilde{j}^{i}(x) \tag{2.10}$$

We will derive now the above equations from the following "field" Lagrangian:

$$S = - \frac{1}{16\pi} \int \sqrt{-g} \; \tilde{F}^{ik} \, d^4 x - \frac{e}{c} \int gN(x,u).(u^{i}+\dot{\xi}^{i}).\tilde{A}_{i}(x+\xi) d^4 u d^4 x + + mc \int gN(x,u) \sqrt{-(u^{i}+\dot{\xi}^{i})(u_{i}+\dot{\xi}_{i})} \; d^4 u d^4 x \tag{2.11}$$

Here $\xi^{i} = \xi^{i}(x,u)$ has been extended to cover the hot plasma case through each point in coordinate space particles pass with different four velocities u^{i} , $\xi^{i} = \xi^{i}(x,u)$. The invariant four volume element $d\Omega$ of the metric g_{ab} is given by

$\sqrt{-g}\; d^4 x = \sqrt{-g} dx^0 dx^1 dx^2 dx^3$. The metric g_{ab} induces in the tangent bundle T_p a metric the volume element of which is given by[2]) $d\pi = \sqrt{-g}\; d^4 n = \sqrt{-g}\; dn^0 \; dn^1 dn^2 dn^3$. Note that ξ^{i} is not a vector in general but $\dot{\xi}^{i}$ is. Consequently, the total derivative of ξ^{i} is now

$$\dot{\xi}^{i} := \frac{D}{ds}\xi^{i} := \xi^{i}_{;k}u^{k} + \xi^{i}_{\nabla k}b^{k} \qquad \xi^{i}_{\nabla k} := \frac{\partial}{\partial u^{k}}\xi^{i} \tag{2.12}$$

$N(x,u)$ is the distribution function of the background plasma. We shall show below that due to the gauge invariance under transformation (2.1) N must obey Boltzmann-Vlasov equation[2] in phase space

$$L(N) = N_{|i}u^{i} + N_{\nabla i}b^{i} = 0 \tag{2.13}$$

where L is the Liouville operator and where b^{i} is given by (2.4) so that $b^{i}_{\nabla i} = 0$. Instead of working in the physical seven-dimensional, phase space we prefer to work in a fictitious[19] eight-dimensional phase space by taking into account the identity $\tilde{u}^{i}\tilde{u}_{i} = -1$ by means of a δ function and the fact that $\tilde{u}^{0} \geq 1$ by means of a θ-function. Our N is related to the usual f_0 by

$$N(x,u) = 2\theta(u^{0})\,\delta(-1-u^{i}u_{i})\,f_{0}(x,u) \tag{2.14}$$

and the four-current j^{i} is defined

$$j^{i}(x) = \int\sqrt{-g}\,u^{i}\,N(x,u)\,d^{4}u \tag{2.15}$$

Replacing b^{i} in the Liouville equation (2.13) by means of equ. (2.4) which reads explicitely $\dot{b}^{i} + \Gamma^{i}_{ab}n^{a}u^{b} = \frac{e}{mc^{2}}F^{i}{}_{a}n^{a}$ we obtain the well-known form of the Boltzmann equation in curved space-time.

$$u^{a}N_{|a} + \left(\frac{e}{mc^{2}}F^{a}{}_{b}u^{b} + \Gamma^{a}{}_{bc}u^{b}u^{c}\right)N_{\nabla a} = 0 \tag{2.16}$$

As we are considering only electromagnetic forces, so that $(F^{a}{}_{b}u^{b})_{\nabla b} = 0$, we find using the identity $b^{a}{}_{\nabla a} = -2\Gamma^{a}_{ab}u^{b} = -\frac{g_{|a}}{g}u^{a}$ that the 8- "current" $j^{A} := (Nu^{a},\, Nb^{a})$ in phase space is conserved $J^{A}{}_{;A} = \frac{1}{g}((gNu^{a})_{|a} + (gNb^{a})_{\nabla a}) = 0$, which in turn guarantees particle conservation (in phase-space)

$$dN = g(x)Nu^{0}d^{3}x\,d^{4}u = g(\tilde{x})\tilde{N}\tilde{u}^{0}d^{3}\tilde{x}\,d^{4}\tilde{u} \tag{2.17}$$

under the action of a perturbation force. $\tilde{N} = \tilde{N}(\tilde{x},\tilde{u})$ is the new distribution function and obeys

$$\tilde{L}(\tilde{N}) = \frac{D}{d\tilde{s}} \tilde{N}(\tilde{x},\tilde{u}) = 0 \tag{2.18}$$

where $\frac{D}{d\tilde{s}}$ is to be taken along the perturbed orbit. From equ. (2.17) we obtain the transformation law for N

$$\tilde{N}(\tilde{x},\tilde{u}) = N(x,u) \frac{d\tilde{s}}{ds} \frac{\partial^4 x}{\partial^4 \tilde{x}} \frac{\partial^4 u}{\partial^4 \tilde{u}} \frac{g(\tilde{x})}{g(x)} \tag{2.19}$$

with (2.19) it is easy to show that (2.11) leads back to (2.7) and (2.8), respectively, which guarantees complete covariance of our "Lagrangian" and also its gauge invariance under transformations (2.1).

Let us next show that a variation of (2.11) with respect to the "field" $\xi^i(x,u)$ leads to equ. (2.9). The variation is to be performed in phase space and leads to

$$\frac{\delta \hat{L}}{\delta \xi^i} := \left(\frac{\partial \hat{L}}{\delta \dot{\xi}^i}\right)^{\bullet} - \frac{\partial \hat{L}}{\partial \xi^i} = \left(\frac{\partial \hat{L}}{\partial \xi^i{}_{|a}}\right)_{|a} + \left(\frac{\partial \hat{L}}{\partial \xi^i{}_{\nabla a}}\right)_{\nabla a} - \frac{\partial \hat{L}}{\partial \xi^i} = 0 \tag{2.20}$$

where

$$\hat{L} := + mc\, Ng\sqrt{-(u^i+\dot{\xi}^i)(u_i+\dot{\xi}_i)} - \frac{e}{c} Ng(u^i+\dot{\xi}^i)\tilde{A}_i(x+\xi) \tag{2.21}$$

$$= - N \hat{\hat{L}} g$$

and the definition for $\dot{\xi}^i$ equ.(2.12) should be remembered.

We find

$$\frac{\delta \hat{L}}{\delta \xi^i} = -Ng \frac{d\tilde{s}}{ds} \left(mc \frac{D}{d\tilde{s}} \tilde{u}^i - \frac{e}{c} \tilde{F}^{ik}\tilde{u}_k\right) - i(N)G^i = 0 \tag{2.22}$$

$$G^i := mc\, \tilde{u}^i + \frac{e}{c} \tilde{A}^i(\tilde{x})$$

which is easiest derived in Fermi coordinates.

Invariance under the gauge transformation (2.1) leads to (2.22) contracted by $\tilde{u}_i$ as a strong conservation law, i.e. for arbitrary $\tilde{u}$ and $\tilde{A}$, which implies

$$\frac{\delta \hat{\hat{L}}}{\delta \xi} = 0 \qquad L(N) = 0 \tag{2.23}$$

i.e. equs. (2.18) and (2.9). Equ. (2.22) agrees with that of Sturrock[17] derived for the cold plasma in flat space-time, and in the nonrelativistic limit our Lagrangian (2.11) goes over into Low's Lagrangian[18]. Note that variation with respect to $\tilde{A}$ (x) is trivial if one uses the equivalent form (2.8) which goes over into (2.11) by means of (2.19) and a relabelling of the coordinates

$$\tilde{F}(x)^{ik}{}_{;k} = \frac{4\pi}{c} \tilde{j}^{i}(x) = \frac{4\pi e}{c}\int \sqrt{-g}\tilde{N}(x,u)\; u^{i} d^4 u \tag{2.24}$$

Invariance of the total action under coordinate transformations

$$x^{i} \rightarrow x^{i} + \varepsilon^{i}$$

with constant ε^{i} leads to the pseudo energy momentum tensor

$$T^{ik} = A_{l}^{|i} \frac{\partial}{\partial A_{l|k}} (L_f + \int d^4 u\hat{L}) - g^{ik}(L_f + \int \hat{L} d^4 u) + \int \xi_{l}^{|i} \frac{\partial \hat{L}}{\partial \xi_{l|k}} d^4 u \tag{2.25}$$

which obeys

$$T^{ik}{}_{;k} = -L^{|i} - \int d^4 u \hat{L}^{\nabla i} \tag{2.26}$$

Here $\hat{L}^{\nabla i}$ and $L^{|i}$ are again the functional (not the total) derivatives with respect to u^{i} and x^{i}, i.e. only the explicit dependence of $\hat{L}$ and L on x and u is to be differentiated. The complicated form of our pseudo-energy-momentum tensor is a consequence of the hybrid nature of our Lagrangians; L_f is a density in the four dimensional position space whereas $\hat{L}_{fm}$ and $\hat{L}_m$ are densities in the eight-dimensional phase space. Using the field equations and performing some simple manipulations[6], one easily arrives at (2.22). In order to reduce (2.26) to the standard result (equ. (1.11) without average) we have to show that the integral vanishes. To this end we note - see (2.21) - that $\hat{L}$ can be written $\hat{L} =: N\hat{\hat{L}}$ and that N does not depend on the field ξ, so that the functional derivative of N is just the total derivative $\hat{L}^{\nabla i} = \hat{\hat{L}} \frac{d}{du_i} N + N\hat{L}^{\nabla i}$. Partial integration of the first term then gives (with the usual requirement that N(u) vanishes fast enough at the boundary of velocity space)

$$\hat{L}^{\nabla i} = N(\hat{\hat{L}}^{\nabla i} - \frac{d}{du_i} \hat{\hat{L}}) = N \frac{\delta \hat{\hat{L}}}{\delta \xi} . \delta \xi^{\nabla i} \qquad (2.27)$$

which in fact vanishes due to equ. (2.23). So far we have not made any approximations, which guarantees that our latter results will inherit the invariance properties to all orders of approximation.

3. The averaged Lagrangian

In order to proceed we make the further assumption that the solutions of (2.9) and (2.10) are strictly periodic. It is then possible to introduce ψ as a new, independent variable and to formulate a modified variational principle. To this end we define new fields

$$B^{i}(x,\psi) = A^{i}(x) \qquad \eta^{i}(x,u,\psi) = \xi^{i}(x,u) \tag{3.1}$$

For the partial derivations we then have

$$\begin{aligned} A^{i}{}_{|k} &= B^{i}_{|k} + B^{i}_{|\psi}\psi_{|k} \\ \xi^{i}{}_{|k} &= \eta^{i}_{|k} + \eta^{i}_{|\psi}\psi_{|k} \qquad \xi^{i}{}_{\nabla k} = \eta^{i}{}_{\nabla k} \end{aligned} \tag{3.2}$$

and the new Lagrangian will be

$$L = L(B,\eta) = L(B,\eta,B^{i}_{|k} + B^{i}_{|\psi}\psi_{|k} \,,\, \eta^{i}_{|k} + \eta^{i}_{|\psi}\,\psi_{|k},\, \eta^{i}_{\nabla k}) \tag{3.3}$$

Now supose we have a solution of equs. (2.9) and (2.10). $\psi_{|a} = k_a$ is then a known function of x, $k_a = k_a(x)$. We will insert this function into (3.2) and (3.3), so that [20) 21)] L is a function of ψ only:

$$L = L(x,u,B,\eta,B^{i}_{|l} + B^{i}_{|\psi}\, k_l,\, \eta^{i}_{|l} + \eta^{i}_{|\psi}k_l,\, \eta^{i}{}_{\nabla l}) \tag{3.4}$$

Since B and η are periodic, so is L. We now consider the following action

$$S = \int_0^{2\pi} d\psi \int d^4x \left(\int d^4u\,(\hat{L}) + L_f\right) \tag{3.5}$$

and vary S first with respect to B . We obtain

$$\delta S = \int_0^{2\pi} d\psi \int d^4x \left\{- \left(\frac{\partial L}{\partial B^{i}_{|l}}\right)_{|l} - \left(\frac{\partial L}{\partial B^{i}_{|\psi}}\right)_{|\psi} + \frac{\partial L}{\partial B^{i}}\right\} \delta B^{i} +$$

$$\int d^4x \left[\delta B^{i} \frac{\partial L}{\partial B^{i}_{|\psi}}\right]_0^{2\pi} \tag{3.6}$$

and an analogous equation for η^i. For periodic δB^i the "surface term" in (3.6) vanishes since L is periodic, and we obtain

$$\left(\frac{\partial L}{\partial B^i_{|l}}\right)_{|l} + \left(\frac{\partial L}{\partial B_{|\psi}}\right)_{|\psi} - \frac{\partial L}{\partial B^i} = 0 \qquad (3.7)$$

which is nothing but the original

$$\left(\frac{\partial L}{\partial A^i_{|l}}\right)_{|l} - \frac{\partial L}{\partial A^i} = 0 \qquad (3.8)$$

Therefore we can introduce the averaged Lagrangian

$$\bar{L} := \int_0^{2\pi} d\psi \, L \qquad (3.9)$$

This Lagrangian is invariant under the gauge transformation $\psi \to \psi + \alpha$ for constant α and therefore leads to a conserved current (E. Noether, part one of her theorem [23])

$$N^i_{\ |i} = 0 \qquad N^i := \frac{\partial \bar{L}}{\partial k_i} \qquad (3.10)$$

For a direct proof one integrates the identity $0 = \int_0^{2\pi} (\frac{d}{d\psi} L) d\psi$ and uses the modified field equations (3.7). One then obtains

$$0 = \int_0^{2\pi} \left\{ \frac{\partial}{\partial x^i} \left(B^l_{|\psi} \frac{\partial L}{\partial B^l_{|i}}\right) + \int d^4 u \left[\frac{\partial}{\partial x^i} \eta^l_{|\psi} \frac{\partial \hat{L}}{\partial \eta^l_{|i}} + \frac{\partial}{\partial u^i} \left(\eta_{|\psi} \frac{\partial \hat{L}}{\partial \eta_{\nabla i}}\right) \right] \right\} d\psi$$

and inspection of (3.4) shows that in fact

$$\frac{\partial L}{\partial B^i_{|l}} B^i_{|\psi} + \frac{\partial L}{\partial \eta^i_{|l}} \eta^i_{|\psi} = \frac{\partial L}{\partial k_l} \qquad (3.12)$$

We have therefore shown that even for such a complicated nonlinear Lagrangian there still exists a conserved current if the Lagrangian allows for periodic solutions.

This is certainly true for those small amplitude oscillations where the dispersion relation allows for real solutions so that the eikonal ansatz (1.1) is justified.

4. Small amplitude waves: the linearized Vlasov equation

We now expand our Lagrangians (2.11) into a power series of the amplitude. We write $\tilde{A}^i = A^i + \delta A^i$, where A^i is the four potential of the background plasma. Note that our definition is slightly inconsistent with (1.1) and that we are considering now only

real quantities and introduce the Lagrange density $L = \sqrt{-g}L$.
L_f leads to only two terms: $L_f^1 = -\frac{1}{4\pi} a_{i|k}F^{ik}$ and $L_f^2 = -\frac{1}{16\pi} f_{ik}f^{ik}$, $f^{ik} = a_{k|i} - a_{i|k}$. All higher order terms vanish. In Low's norelativistic treatment the same would be true for the matter Lagrangian, but the relativistic Lagrangian gives contributions to all orders. We define the projection operatior into the local rest frame $h^a_b := \delta^a{}_b + u^a u_b$ and write

$$\sqrt{-(u^i+\dot\xi^i)(u_i+\dot\xi_i)} = (1-(u^a\dot\xi_a))\left[1-\frac{h_{ab}\dot\xi^a\dot\xi^b}{(1-(u^a\dot\xi_a))^2}\right]^{1/2} . \qquad (4.1)$$

The power series expansion of (2.11) is then straightforward. Adding to the first order Lagrangian the divergences (which do not change the field equations)

$\frac{1}{4\pi}(a_i F^{ik})_{|k}$ and $\dot C_1$ with $C_1 = N(mcu^i - \frac{e}{c}A^i)\xi_i$,

the first order Lagrangian is seen to vanish identically because of the field equations of the background plasma.
To second order approximation we obtain after adding $\dot C_2$ with $C_2 = -\frac{e}{c} N\,(a^i + \frac{1}{2}A^i{}_{|k}\xi^k)\xi_i$

$$\hat L_m^2 = -mc\,\frac{N}{2}\,h_{ab}\dot\xi^a\dot\xi^b + \frac{e}{2c}N(F_{ab}\xi^a\dot\xi^b + u_c F^c{}_{a|b}\xi^a\xi^b) \qquad (4.2)$$

$$L_{fm}^2 = -\frac{eN}{c}\,f^{ab}\xi_a u_b \quad , \quad L_f^2 = -\frac{1}{16\pi}\,f^{ik}f_{ik}$$

Variation with respect to a^i gives Maxwell's equation (4.3)

$$(\sqrt{-g}f^{ik})_{|k} = \frac{4\pi}{c}\,e\int\sqrt{-g}d^4u\,\frac{[N(\xi^i u^k - u^i\xi^k)\sqrt{-g}]_{|k}}{\sqrt{-g}} = \frac{4\pi}{c}\,\delta j^i$$

and with respect to ξ gives the equation of motion

$$L^a := (h^a{}_b\dot\xi^b)^{\cdot} - \frac{e}{mc^2}(F^a{}_b\dot\xi^b + F^a_{b|c}u^b\xi^c) = \frac{e}{mc^2}\,f^a_b u^b \qquad (4.4)$$

We shall now show that δj^i in equ. (4.3) is given by the usual definition for the current equ. (2.15)

$$\delta j^i = \int\sqrt{-g}\,\delta N u^i\,d^4u \qquad (4.5)$$

and that δN obeys the linearized Vlasov equation. To this end we define the perturbed distribution function by (2.19). To linear order we have in Fermi coordinates in which along a particle's trajectory $\sqrt{-g} = 1$, $\Gamma^a_{bc} = 0$

$$\frac{d\tilde{s}}{ds} = 1 - u^a \dot{\xi}_a$$

$$\tilde{u}^a = u^a + h^a{}_b \dot{\xi}^b \qquad \frac{\partial \tilde{u}^a}{\partial u^b} = \delta^a_b + (h^a{}_c \dot{\xi}^c)_{\nabla b} \tag{4.6}$$

$$\tilde{x}^a = x^a + \xi^a \qquad \frac{\partial \tilde{x}^a}{\partial x^b} = \delta^a_b + \xi^a{}_{|b}$$

$$\Delta := \left(\frac{d\tilde{s}}{ds}\right)\left(\frac{\partial^4 x}{\partial^4 \tilde{x}}\right)\left(\frac{\partial^4 u}{\partial^4 \tilde{u}}\right) = 1 - u^a \dot{\xi}_a - \xi^a{}_{|a} - (h^a_b \dot{\xi}^b)_{\nabla a}$$

and from (2.19) together with (4.6) we obtain

$$\delta N = -(N\xi^a)_{|a} - (Nh^a{}_b \dot{\xi}^b)_{\nabla a} - (u\dot{\xi})N \tag{4.7}$$

Inserting (4.7) into (4.5), we obtain after a partial integration

$$\delta j^i = e \int \left[u^i - (N\xi^k)_{|k} - (Nh^a_b \dot{\xi}^b)_{\nabla a} - (u\dot{\xi})N \right] d^4u =$$

$$\int \left[(u^i \xi^k N)_{|k} + \dot{\xi}^i N \right] d^4u$$

Using $\dot{N} = 0$, the last term can be written $(\xi^i N)^{\cdot}$, and another partial integration using $b^i_{\nabla i} = 0$ in fact gives (4.3). It remains to be shown that N of equ. (4.7) obeys the linearized Vlasov equation:

$$\frac{D}{ds} \delta N = \delta \dot{N} = -\frac{e}{mc^2} N_{\nabla i} f^{ik} u_k \tag{4.8}$$

Equs. (4.6) and (4.7) are invariant only under the restricted gauge transformation $\xi^a \to \xi^a + \sigma u^a$ with constant σ whereas (4.3) and (4.4) are invariant under the gauge transformation with arbitrary σ. Therefore we can choose the gauge $u^a \dot{\xi}_a = 0$ without a loss of generality and still have the freedom of the restricted gauge transformation.

Using the identities $\frac{D}{ds}(A^i{}_{|i}) = (\frac{D}{ds} A^i)_{|i} - A^i{}_{\nabla j} b^j{}_{|i}$ and

$\frac{D}{ds} A^i_{\nabla i} = (\frac{D}{ds} A^i)_{\nabla i} - A^i_{|i} - A^i{}_{\nabla j} b^j{}_{\nabla i}$ we obtain with $\dot{N} = 0$

$$\delta \dot{N} = -(NL^a)_{\nabla a} \tag{4.9}$$

where L^a was defined in equ. (4.4) and is now

$$L^a = \ddot{\xi}^a - \frac{e}{mc^2}(F^a{}_b \dot{\xi}^b + F^a{}_{b|c} u^b \xi^c) = \frac{e}{mc^2} f^a{}_b u^b \tag{4.10}$$

according to which $L^{a}{}_{\nabla a} = 0$, so that in fact the linearized Vlasov equation follows. Equ. (4.9) was derived (nonrelativiscally) in Low's paper. Low noted that this does not yet finish the proof since the class of solutions of the equation of motion (4.4) can actually be larger than the class for which (4.9) holds. We shall show that we can use the remaining gauge freedom to establish a one-to-one correspondence between the two classes of solutions (without changing the current (4.3) or (4.5), respectively). Multiplying equ. (4.10) with u_a, we obtain by means of (4.4) $L^a u_a = 0$. This may be written as

$$u_a \ddot{\xi}^a (u^a \dot{\xi}_a)^{\cdot} - (\dot{u}^a \dot{\xi}_a) = \frac{e}{mc^2} (u_a F^a{}_b \dot{\xi}^b + u_a F^a{}_{b|c} u^b \xi^c) = - (\dot{u}_a \dot{\xi}^a)$$

$$\text{i.e. } (u^a \dot{\xi}_a)^{\cdot} = 0 \qquad (4.11)$$

so that $u^a \dot{\xi}_a = C$ for all solutions of (4.10). If this constant C happens to be different from zero, we can gauge it to zero with a restricted gauge transformation $\xi^a \to \xi^a + C \cdot u^a$ (without affecting the physical components ξ^a, as is easily seen in the local rest frame). The proof of (4.9) is somewhat tedious but can be simplified if one proceeds as follows. One starts from $\delta N = N\Delta - N(\tilde{x},\tilde{u})$ and uses $\dot{N}(x,u) = 0$. This gives $-\dot{N}(\tilde{x},\tilde{u}) = -N_{\nabla i} (\dot{\tilde{b}}^i - b^i)$, which is just $-N_{\nabla i} \dot{L}^i$ according to (4.4), (2.4) and (2.9)

$$\dot{\Delta} = -L^a{}_{\nabla a} \qquad (4.12)$$

and

$$-\dot{N}_{|a} \xi^a - \dot{N}_{\nabla a} h^a{}_b \dot{\xi}^b = \dot{N}(\tilde{x},\tilde{u}) = 0 \qquad (4.13)$$

The remaining terms are then just $-N_{\nabla a} L^a$, which proves (4.9). We have therefore shown that although the Vlasov equation does not follow as a variational equation from the Lagrangian, it is contained implicitly in the field equations and may replace equ. (4.4).

5. The equivalent permeability tensor

Having established that our basic equations (2.9) and (2.10) follow from the Lagrangians (4.1) and (4.2), we can now apply the result of section 3, equ. (3.10), if we make the eikonal ansatz (1.1). As the next step we use the virial theorem to eliminate ξ^i from the total averaged Lagrangian. Since L^2_m is quadratic in ξ and L_{mf} is linear, we obtain

$$2\bar{L}_m = -\bar{L}_{mf}$$

so that $\bar{L} = \bar{L}_m + \bar{L}_{mf} + \bar{L}_f = \bar{L}_f + \frac{1}{2}\bar{L}_{mf}$. Defining

$$\delta H^{ab} := -4\pi \frac{e}{c} \int \sqrt{-g} N \,(\xi^a u^b - u^a \xi^b) d^4 u + f^{ab} \qquad (5.1)$$

we obtain finally

$$\bar{L} = -\frac{1}{16\pi} \overline{f_{ab}\,\delta H^{ab}} \qquad (5.2)$$

which agrees with ref. 5. Applying (3.10), we set that the wave action $N^i = \frac{\partial \bar{L}}{\partial k_i}$ is conserved. We only have to determine the linear response tensor which relates ξ^i to f^{ik}. For a cold plasma this is straightforward. With the eikonal ansazt (1.1) we have to invert (4.10), which now reads to lowest order eikonal approximation

$$-(ku)^2 \xi^a + i(ku)\Omega^a{}_b \xi^b = + \frac{e}{mc^2} f^a{}_b u^b \qquad (5.3)$$

in order to obtain

$$\xi^a = \sigma^{ab} \omega_{bc} u^c \qquad (5.4)$$

with

$$\omega_{bc} := + \frac{e}{mc^2} f_{bc}$$

and

$$\Omega_{bc} := + \frac{e}{mc^2} F_{bc}\,, \qquad {}^*\Omega^a{}_b = e^a{}_{bcd}\Omega^{cd} \quad \text{the dual to } \Omega^a{}_b$$

The result is

$$\sigma^a{}_b = \frac{1}{(ku)^2} \cdot \frac{(ku)^2 \delta^a{}_b + i(ku)\,{}^*\Omega^a{}_b - {}^*\Omega^a_c\,{}^*\Omega^c{}_b}{-(ku)^2 + \frac{1}{2}(\Omega_{ij}\Omega^{ij})} \qquad (5.5)$$

The result (5.5) can be easily derived from $\Omega^a{}_b u^b = 0$, which holds to lowest order eikonal approximation, so that as a consequence we have ${}^*\Omega^a_c \Omega^c{}_b = 0$. Using further the identity

$$\Omega^{a}{}_{b}\Omega^{b}{}_{c} = {}^*\Omega^{a}{}_{b}{}^*\Omega^{b}{}_{c} - \frac{1}{2}(\Omega_{ab}\Omega^{ab})\delta^{a}{}_{c}$$

one easily shows that (5.5) is the inverse of $(ku)^2\,\delta^{a}{}_{b} - i(ku)\Omega^{a}{}_{b}$. In the local rest frame of the plasma the space part of equ.(5.5) reduces to the well-known conductivity tensor $\sigma^{\alpha}{}_{\beta}$ if we identify $(ku)^2 = +\omega$ and $\frac{1}{2}(\Omega_{ij}\Omega^{ij}) = \Omega_L^2 = \Omega^2$ where Ω_L is the Larmor frequency in the particle's rest frame. From (5.4) and (5.5) we read off the permeability tensor, using (5.1) and (1.5), (1.6)

$$\tilde{\varepsilon}^{ab}{}_{cd} = \delta^{[a}_{[c}\,\delta^{b]}_{d]} + \frac{\omega_p^2}{c^2}\,\sigma^{[a}_{[c}\,u^{b]}\,u_{d]} \tag{5.6}$$

where $A_{[a}B_{b]} := \frac{1}{2}(A_aB_b - B_aA_b)$.

Instead of solving equ.(4.10) for the hot plasma case (where u is an independent variable), we use the linearized Vlasov equation (4.8), which, as we have shown, is equivalent. To this end we employ a technique due to Sagdeev and Shafranov [22] (as described by Stix [23] and Clemnow and Dougherty [24]. To put the method in covariant form we borrow the formalism from Buneman [25]. One first solves equ. (4.8) in *Lagrangian coordinates* for the perturbation δN so that it is sufficient to insert the zero-order orbits into the right-hand side of equ. (4.8) since f^{ik} is already a quantity of first order. These orbits are parametrized so that at proper time $s' = s$ they pass through x^i and u^i, respectively. For a magnetized plasma the orbits are to lowest order in a locally flat coordinate system using Fermi coordinates

$$\bar{x}^i = x^i + \tau^i{}_k u^k + u^i(s'-s)$$

$$\tau^i{}_k = a\Omega^i{}_k + b\Omega^i{}_e\Omega^e{}_k + (s'-s)\,\delta^i{}_k \tag{5.7}$$

$$a := \frac{c}{\Omega^2}(\cos\Omega(s'-s)-1) \qquad b := \frac{c}{\Omega^3}(\Omega(s'-s) - \sin\Omega(s'-s))$$

$$\bar{u}^i = \dot{\tau}^i{}_k u^k \qquad \dot{\tau} = \frac{d}{ds}\tau$$

Equs. (5.7) are the appropriate generalization of the Lorentz rotators given by Buneman [25] (the notation is that used in equ. (5.5) and below). If one wishes to take into account particle drifts, higher order harmonics and the change in the amplitude of gyration via an adiabatically conserved magnetic moment covariant formulations are also available [26]. Replacing the Lagrangian by Eulerian coordinates, one obtains [13]

$$\delta N(x,u) = -\frac{e}{mc^2}\int_{s0}^{s} \bar{u}_l(s')\ f^{kl}(\bar{x}(s'))\ N_0\,(\bar{x}(s'),\ \bar{u}(s'))_{\bar{\nabla}k}\,ds' + \delta N(x_0, u_0) \tag{5.8}$$

To make the integral independent on initial data we imagine the perturbation to be switched on adiabatically in the infinite past and introduce the integration variable t= s' - s to obtain

$$\delta N(x,u) = \frac{e}{mc^2}\int_{-\infty}^{0} \bar{u}_l(t)\ f^{kl}(\bar{x}(t))N_0\,(\bar{x}(t),\ \bar{u}(t))_{\bar{\nabla}k}\,dt \tag{5.9}$$

By this trick the integral has become independent on the proper times. With our eikonal ansatz (1.1) we obtain finally

$$\delta N(x,u) = -\frac{e}{mc^2}\ f^{kl}(x)\ s_{kl}(u) \tag{5.10}$$

$$s_{kl} = \int_{-\infty}^{0} \bar{u}_l\ N_{\bar{\nabla}k}\ e^{i(k_a\tau^a{}_b u^b + k_a u^a t)}\,dt$$

Instead of keeping the discussion as general as possible, we shall illustrate the procedure by a simple example: we consider the one-dimensional relativistic gas. For a treatment of a magnetized plasma with an isotropic relativistic Boltzmann distribution function compare Buneman[25)].

In the absence of a magnetic field the particles' trajectories are simply $\bar{u}^i = u^i$, $\bar{x}^i = x^i + u^i(s'-s)$ and s_{kl} of equ. (5.10) is

$$i.u_l N_{\nabla k}\ .\ (ku)^{-1} \tag{5.11}$$

and for the current we obtain

$$\delta j^a = -i\,\frac{e^2}{mc^2}\int d^4\ u\ u^a\ \frac{u_l N_{\nabla k}}{k_b u^b}\ f^{kl} \tag{5.12}$$

In evaluating this integral the Landau prescription must be used. Putting $f^{ab} = i(k^a\delta A^b - \delta A^a k^b)$ and using Maxwell's equation (4.3) we obtain

$$(k^a\delta A^b - k^b\delta A^a)k_b = -\frac{4\pi e^2}{mc^2}\int \frac{d^4u\ u^a N_{\nabla b}u_c}{k_r u^r}\ (k^b\delta A^c - \delta A^b k^c)$$

or

$$(k^a k_b - k^2\delta^a{}_b)\delta A^b = \sigma^a{}_b\delta A^b \tag{5.14}$$

$$\sigma^a_b = +\frac{4\pi e^2}{mc^2}\int \frac{d^4u}{k_r u^r}\ (u^a((N_{\nabla c}k^c)u_b - (u^s k_s)N_{\nabla b}))$$

$$= -\frac{4\pi e^2}{mc^2}\int \frac{d^4u.N}{(k_r u^r)^2}\left[k^a u_b + k_b u^a)(k_r u^r) - (k_r k^r)u^a u_b - (k^r u_r)^2\delta^a{}_b\right]$$

To obtain the last line a partial integration was perfomed. It makes explicit the symmetry of $\sigma^a{}_b$ and $\sigma^a{}_b k^b = 0$. Note that (5.14) coincides with (5.5) inserted into (5.1) for no magnetic field. However, in general this is not true, as can be easily verified since (5.5) has only a resonance at the fundamental Larmor frequency Ω_L.We specialize now N to be $N = 2n_0(x)f(u_1).\delta(u_2)\delta(u_3)\delta(-1-u^r u_r)\theta(u^0)$ where we have chosen x^1 for the direction of the anisotropy. Such a plasma may exist for example around a pulsar beyond the velocity of light cylinder. We find

$$\sigma^2_2 = \sigma^3_3 = \omega^2_p/c^2 = \frac{4\pi e^2 n_0}{mc^2}$$

and

$$\begin{aligned}\sigma^1_1 &= -\frac{\omega^2_p}{c^2}\int \frac{\omega u^1(N_{|0}u_1 + N_{|1}u^0)}{u^0 - cku^1}\, d^4u \\ &= -\frac{\omega^2_p}{c^2}\int\frac{uf'(u)\, du}{\sqrt{1+u^2}-nu} \qquad n := \frac{ck^1}{\omega}\end{aligned} \tag{5.15}$$

$$\sigma^0_0 = -c\omega^{-1}\sigma^0{}_\alpha k^\alpha$$

For longitudinal oscillations we obtain from $\delta A^\alpha = 0$ $(\alpha = 1,2,3)$ the dispersion relation

$$k^\alpha k_\alpha = \omega^{-2}\sigma^{\alpha\beta}k_\alpha k_\beta \tag{5.16}$$

whereas transverse waves in the Landau gauge $\delta A^0 = 0$, with the help of (5.14), lead to

$$(k^\alpha k_\beta + (\frac{\omega^2}{c^2} - k^\gamma k_\gamma)\delta^\alpha{}_\beta)\delta A^\beta = \sigma^\alpha{}_\beta\delta A^\beta \tag{5.17}$$

and for the principal modes (along $||$ or orthogonal $\perp$ to x^1) for which $k_\alpha\delta A^\alpha = 0$ is possible, we have the dispersion relation

$$1 - n^2 = \frac{c^2}{\omega^2}\sigma \tag{5.18}$$

where $\sigma = \sigma^1_1 = \sigma_{||}$ or $\sigma = \sigma^2_2 = \sigma^3_3 = \sigma_\perp = \frac{\omega^2_p}{c^2}$
To estimate the influence of a large velocity dispersion on the propagation of the mode parallel to the x^1 we approximate $f(u)$ by $f(u) = \frac{1}{2}\theta e^{-\theta|u|}$ where $2\theta^{-1} = kT = \langle\gamma\rangle$ is an effective tempe_rature and measures the velocity dispersion. We obtain

$$\sigma_{||} = \theta \frac{\omega_p^2}{c^2} \int_0^\infty \frac{ue^{-\theta u} du}{\sqrt{1+u^2}-nu} \tag{5.19}$$

In the limit $\theta^{-1} \to 0$ we obtain $\sigma_{||} = \omega_p^2/c^2$, as we must, and for $\theta \to 0$ we get $\sigma_{||} \to \theta^2 \frac{\omega_p^2}{c^2} (1+n) (\frac{1}{1-n^2})$ from which we read off that $1-n^2 \to \frac{\omega_p^2}{\omega^2 \langle\gamma\rangle}$, i.e the plasma frequency is reduced by $\langle\gamma\rangle^{-1/2}$ a well-known result. Finally we give the result for an isotropic distribution function. We choose x^3 for our references axis and obtain[24])

$$\sigma^3_3 = \sigma_{||} = \frac{\omega_p^2}{c^2} \int_0^{2\pi} d\psi \int_0^\pi \sin\theta d\theta \int_0^\infty \frac{u^3 du}{u^0} \left[- \frac{\cos^2 \theta f'}{1-z \cos\theta} \right] \tag{5.20}$$

$$\sigma^1_1 = \sigma^2_2 = \sigma_\perp = \frac{\omega_p^2}{c} \int_0^\pi d\psi \int_0^\pi \sin\theta \, d\theta \int_0^\infty \frac{u^3 du}{n^0} \left[-f' \frac{\sin^2\theta \cos^2\psi}{1-z \cos\theta} \right] \tag{5.21}$$

where the following convention is used

$$u^\alpha = (u \sin\theta \cos\psi, u \sin\theta \sin\psi, u \cos\theta)$$
$$\delta A^i = (o, \delta A, o, o) \qquad k^i = (\frac{\omega}{c}, o, o, k)$$

The integral over angles is elementary, and it is convenient to consider first $\sigma_{||} + 2\sigma_\perp$, which shows that the integrals behave like $\frac{1}{\langle\gamma\rangle} \ln \langle\gamma\rangle$, again a known result.

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