# Cliffordized NAC Supersymmetry and PT-symmetric Hamiltonians*

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**Abstract**

It is shown that non-anticommutative supersymmetry can be described through a Cliffordization of the superspace fermionic coordinates. A NAC supersymmetric quantum mechanical model is shown to be a $PT$-symmetric hamiltonian.

## 1 Introduction

We discuss in this work the main results and approaches of [1] to the formulation of Non-Anticommutative (NAC) supersymmetry in terms of a Cliffordization of the fermionic superspace coordinates.

Most of the existing works in the literature concerning the formulation of Non-anticommutative supersymmetry adopt (see e.g. [2]) the viewpoint of the deformation theory, as discussed in [3]. It was shown, e.g., in [4] that a Moyal-star deformation of the Grassmann algebra produces a Clifford algebra. Needless to say, Grassmann generators can be regarded as fermionic parameters entering a superspace. The deformation results in an overall Cliffordization of the superspace. Several papers, either analyzing specific models [5, 6, 7], or the general properties of the deformed supersymmetric theories [8, 9, 10], were produced within this framework.

In [1], a different viewpoint was advocated. Instead of introducing non-anticommutativity from a starting undeformed algebra, a top-down approach, much in the spirit of the Inonü-Wigner contraction, was proposed. From the very beginning, the supergroup associated to a superLie algebra can be constructed through "exponentiation" of the super-Lie algebra generators, by relaxing the condition that their associated odd parameters are of Grassmann type. In the particular case of the supergroup associated to the one-dimensional supersymmetric quantum mechanics, the odd-parameters can be assumed to satisfy a Clifford-algebra, whose normalization depends on (the inverse of) a mass-scale parameter. In the limit for the mass-scale going to infinity one recovers, by contraction, the standard supersymmetry.

The construction of [1] is based on a further feature, the introduction of a an extension of the ordinary Berezin calculus (derivation and integration), which can be made consistent for odd Clifford variables (whose square is non-vanishing). A superspace, spanned by odd Clifford variables, can be defined and the fermionic covariant derivatives can be computed with a standard procedure. The fermionic covariant derivatives, in the odd-Clifford variables superspace, do not satisfy, in general, the graded Leibniz rule. While this feature is problematic in dealing with theories presenting chiral superfields (the product of chiral superfields is no longer a chiral superfield, if the graded Leibniz rule is violated),

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unconstrained superfields do not pose a problem. We show that the simplest non-trivial example where this framework can be applied consists of an $N = 2$ non-anticommutative supersymmetric quantum mechanical system for the real $(1, 2, 1)$ bosonic superfield $\Phi$ with constant kinetic term and a trilinear superpotential. The theory is derived in terms of the component fields. It is further shown that its bosonic sector, divided into physical and auxiliary fields (the latter satisfy purely algebraic equations of motion) is such that, depending on the type of non-anticommutative deformation and coupling constants, the purely real algebraic equations of motion of the auxiliary fields can admit complex (imaginary) solutions. Inserting these solution into the equations of motion of the bosonic dynamical fields, one is naturally led to Bender-Boettcher [11, 12, 13] $PT$-symmetric pseudohermitian hamiltonians.

This feature appears naturally in the Clifford approach to NAC supersymmetry and is due to a very basic property. The auxiliary fields entering the odd-Clifford valued superfields satisfy algebraic equations of motion which are no longer linear, as in the ordinary supersymmetry case, but of higher order. Even if the algebraic equation admits real coefficient, the explicit solution is not necessarily real.

# 2 Odd Clifford variables and fermionic covariant derivatives

Let us take $N = 2$ odd Clifford variables $\theta_i$ $(i = 1, 2)$ s.t.

$$\theta_i\theta_j + \theta_j\theta_i = 2\eta_{ij}, \tag{1}$$

where $\eta_{ij}$ is a diagonal matrix with diagonal entries $\frac{\pm 1}{M}$ ($M$ is a mass parameter). The above algebra is Euclidean whether $\eta_{11} = \eta_{22}$, Lorentzian if $\eta_{11} = -\eta_{22}$. A consistent calculus generalizing the Berezin calculus [14] can be introduced by assuming the basic relations for the odd derivatives $\partial_{\theta_i}$, $\partial_{\theta_i}1 = 0$, $\partial_{\theta_i}\theta_j = \delta_{ij}$, and extending it to arbitrary powers of the odd Clifford variables by taking into account the graded Leibniz rule.

With similar techniques derived in the ordinary Grassmann case (for $M \to \infty$), it is possible to introduce the fermionic covariant derivatives $D_j$ and the two supersymmetry generators $Q_j$, given by

$$\begin{aligned} D_j &= \partial_{\theta_j} - i\theta_j\partial_t + i\frac{\epsilon_j}{M}\partial_{\theta_j}\partial_t, \\ Q_j &= \partial_{\theta_j} + i\theta_j\partial_t - i\frac{\epsilon_j}{M}\partial_{\theta_j}\partial_t \end{aligned} \tag{2}$$

(where $\epsilon_j = \eta_{jj}$) and satisfying the algebra

$$\begin{aligned} \{D_i, D_j\} &= -\delta_{ij}H, \\ \{D_i, Q_j\} &= 0, \\ \{Q_i, Q_j\} &= \delta_{ij}H. \end{aligned} \tag{3}$$

The integration in the $N = 2$ superspace is given by

$$\int\int d\theta_1 d\theta_2 \;\; \theta_2\theta_1 = 1 \tag{4}$$

and zero otherwise (the even powers of $\theta_i$ are replaced by the corresponding powers in $\frac{1}{M}$).

# 3 The $N=2$ NAC Supersymmetric Quantum Mechanics

A real bosonic $N=2$ superfield $\Phi$ is given by

$$\Phi = \phi + i\psi_1\theta_1 + i\psi_2\theta_2 + if\theta_1\theta_2, \quad (5)$$

with real bosonic, $\phi$ and $f$, component fields of mass-dimension $d$ and $d+1$, respectively ($f$ is the auxiliary field). The real component fermionic fields $\psi_1$ and $\psi_2$ have mass-dimension $d+\frac{1}{2}$.

In the odd Clifford case the ordinary superfield multiplication must be replaced by the (anti)symmetrized $*$-multiplication defined as follows

$$A_1 * A_2 = \frac{1}{2}(A_1A_2 + (-1)^{deg(A_1)deg(A_2)}A_2A_1), \quad (6)$$

where $deg(A_i)$ denotes the grading (bosonic, $=0$, or fermionic, $=1$) of the superfields.

The fermionic covariant derivatives $D_1$, $D_2$ do not obey, in the odd Clifford case for Euclidean or Lorentzian deformations of the Grassmann algebra, the graded Leibniz rule.

# 4 The model

The $N=2$ free kinetic action of the real superfield $\Phi$ can be written as

$$S_{N=2,kin.} = \frac{1}{2m}\int dt \int d\theta_1 d\theta_2 (D_1\Phi * D_2\Phi). \quad (7)$$

It reads, in component fields,

$$S_{N=2,kin.} = \frac{1}{2m}\int dt(\dot{\phi}^2 + f^2 - i\dot{\psi}_1\psi_1 - i\dot{\psi}_2\psi_2). \quad (8)$$

The general $N=2$ action is

$$S_{N=2} = S_{N=2,kin.} + S_{N=2,pot.}, \quad (9)$$

where $S_{N=2,pot.}$ is the potential term.

For the $N=2$ harmonic oscillator the potential term is quadratic in $\Phi$,

$$S_{N=2,pot.} = i\frac{\omega}{2}\int dt \int d\theta_1 d\theta_2 (\Phi * \Phi), \quad (10)$$

with $\omega$ an adimensional constant.

It is required at least a trilinear potential to spot the difference between the Grassmann and the odd Clifford realization of the $N = 2$ supersymmetry. The most general trilinear potential can be written as

$$S_{N=2,pot.} = i\int dt \int d\theta_1 d\theta_2 (c_1\Phi * \Phi * \Phi + c_2\Phi * \Phi + c_3\Phi). \tag{11}$$

In (11) the coefficients $c_i$'s are real and the $i$ normalizing factor is introduced to ensure the reality of the $N = 2$ potential.

Without loss of generality the $c_2$ constant can be set equal to zero ($c_2 = 0$) through a shift $\Phi \mapsto \Phi' = \Phi + c$, for a suitable value $c$. The constant $c_1$ can be normalized s.t.

$$c_1 = \frac{1}{6}, \tag{12}$$

leaving the trilinear potential depending on a single real parameter $\alpha = c_3$.

# 5 The $N = 2$ trilinear potential for $\epsilon = \epsilon_1\epsilon_2 = 0, -1, 1$.

The trilinear potential, for $\epsilon = \pm 1$, induces an action whose kinetic term is given as before, while the potential $V$ is given by

$$V = -\left(\frac{1}{2m}f^2 + \frac{1}{2}\phi^2 f - i\phi\psi_1\psi_2 + \frac{1}{6}\frac{\epsilon}{M^2}f^3 + \alpha f\right). \tag{13}$$

In the $\epsilon = 0$ Grassmann supersymmetry, for generic potentials, the equation of motion of the auxiliary field $f$ is a linear equation. For $\epsilon \neq 0$, the equation of motion for $f$ is an algebraic equation (a second order equation for the above example of the trilinear potential), with several branches of solutions. The prescription to correctly pick up a branch is discussed in the following.

For simplicity, and without loss of generality, it is convenient to identify the mass-scale $M$ with the mass-scale $m$ entering (7). We can therefore set $M = m$. The main features of the potential can be understood by taking its purely bosonic sector, consistently setting all fermionic fields to zero ($\psi_1 = \psi_2 = 0$). In the $\epsilon = 0$ Grassmann case, solving the equation of motion for $f$ and inserting back into $V$, we obtain

$$\frac{V}{m} = \frac{1}{8}(\phi^2 + 2\alpha)^2. \tag{14}$$

The corresponding theory admits two invariances: supersymmetry and $\mathbf{Z}_2$-invariance $\phi \mapsto -\phi$. We can distinguish three cases according to the value of $\alpha$. We have
*i)* $\alpha > 0$: the $\mathbf{Z}_2$-invariance is exact, while the supersymmetry is spontaneously broken,
*ii)* $\alpha = 0$: both the $\mathbf{Z}_2$-invariance and the supersymmetry are exact and, finally,
*iii)* $\alpha < 0$: the supersymmetry is exact, while the $\mathbf{Z}_2$-invariance is spontaneously broken (the "mexican hat"-shape potential).

This analysis can be repeated for $\epsilon = \pm 1$. We obtain the following equation of motion for the auxiliary field $f$:

$$f_\pm = m(-\epsilon \pm \sqrt{1 - \epsilon(2\alpha + \phi^2)}). \tag{15}$$

By specializing to $\epsilon = -1$ (the "Lorentzian" case) and setting

$$x = \sqrt{1+2\alpha+\phi^2}, \tag{16}$$

we obtain two branches for the potential $V$:

$$\frac{V_\pm}{m} = \pm\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}. \tag{17}$$

For $\alpha \geq -\frac{1}{2}$, $x$ is always real. The branches have to be chosen s.t. $V$ is bounded below. Therefore

$$\frac{V}{m} = \frac{1}{3}|x^3| - \frac{1}{2}x^2 + \frac{1}{6}. \tag{18}$$

Three cases have to be distinguished according to the value $\alpha \geq -\frac{1}{2}$. We have
*i)* $\alpha > 0$: the $\mathbf{Z}_2$-invariance $\phi \mapsto -\phi$ is exact, while the supersymmetry is spontaneously broken,
*ii)* $\alpha = 0$: both the $\mathbf{Z}_2$-invariance and the supersymmetry are exact and,
*iii)* $-\frac{1}{2} \leq \alpha < 0$: the supersymmetry is exact, while the $\mathbf{Z}_2$ invariance is spontaneously broken (this case corresponds to a deformed version of the "mexican hat" potential).

In the three cases above, $x$ belongs to the real axis. On the other hand $x$ is constrained to satisfy $|x| \geq \sqrt{1+2\alpha}$ (the whole real axis is recovered for the special value $\alpha = -\frac{1}{2}$).

In the Lorentzian $\epsilon = -1$ case, for $\alpha \geq -\frac{1}{2}$, we obtained real potentials which are deformations of the "Grassmann" potential (14). On the other hand, the reality condition (for the classical theory, the hermiticity condition is understood for its quantum version) for the $N = 2$ odd Clifford action written in terms of the $N = 2$ superfield requires $\alpha$ to be an unconstrained real parameter. In particular, the values $\alpha < -\frac{1}{2}$ are allowed. In the Lorentzian case, such values correspond to a potential expressed in terms of $x$, where now $x$ takes value on the whole real axis and on the part of the imaginary axis constrained to $|x| \leq \sqrt{-2\alpha-1}$.

In the Euclidean $\epsilon = 1$ case, the two branches of the potential are still furnished by equation (17). On the other hand, the $x$ variable is now expressed in terms of the real field $\phi$ as

$$x = \sqrt{1-2\alpha-\phi^2}. \tag{19}$$

In the Euclidean odd Clifford supersymmetry the $x$ variable always takes some of its values on (part of) the imaginary axis. We can indeed distinguish three separate cases according to the value of the $\alpha$ parameter. We have
*i)* for $\alpha > \frac{1}{2}$, $x$ takes values on the part of the imaginary axis satisfying the constraint $|x| \geq \sqrt{2\alpha-1}$;
*ii)* for $\alpha = \frac{1}{2}$, $x$ takes value on the whole imaginary axis;
*iii)* for $\alpha < \frac{1}{2}$, $x$ takes value on the whole imaginary axis *and* the part of the real axis satisfying the constraint $|x| \leq \sqrt{1-2\alpha}$.

# 6 On the connection between $N=2$ NAC supersymmetry and $\mathcal{PT}$ - hamiltonians

Let us specialize now our discussion to the Euclidean-deformed $\alpha=\frac{1}{2}$ case. For this special choice of $\alpha$, we have $x=i\phi$, s.t. the purely bosonic effective action $S$ for $\phi$ is given by

$$S = \int dt\left(\frac{1}{2}\dot{\phi}^2+\frac{i}{3}\phi^3-\frac{1}{2}\phi^2-\frac{1}{6}\right) \tag{20}$$

(we set $m=1$ for simplicity).

This action induces a Bender-Boettcher [11, 12, 13] $\mathcal{PT}$-symmetric hamiltonian $H$ (with $p=\dot{\phi}$)

$$H = \frac{1}{2}p^2-\frac{i}{3}\phi^3+\frac{1}{2}\phi^2+\frac{1}{6}, \tag{21}$$

invariant under the coupled transformations (see [12])

$$\begin{aligned} \mathcal{P} &: \phi\mapsto-\phi, \quad p\mapsto -p,\\ \mathcal{T} &: \phi\mapsto\phi, \quad\quad p\mapsto -p, \quad i\mapsto -i. \end{aligned} \tag{22}$$

It is worth stressing the fact that our original odd-Clifford $N=2$ supersymmetric action for $\phi,\psi_1,\psi_2,f$ (no matter which Clifford deformation and which real value of the $\alpha$ parameter are taken) satisfies the reality condition. It's only after solving the equation of motion for the auxiliary field $f$ that the imaginary unit $i$ appears (for the above-discussed cases) in the reduced action. What we succeeded here is to directly link a $\mathcal{PT}$-symmetric hamiltonian with a Non-anticommutative $N=2$ supersymmetric quantum mechanical system.

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