Refining the classification of the irreps of the $1D$ N-Extended Supersymmetry

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Abstract

In hep-th/0511274 the classification of the fields content of the linear finite irreducible representations of the algebra of the $1D$ N-Extended Supersymmetric Quantum Mechanics was given. In hep-th/0611060 it was pointed out that certain irreps with the same fields content can be regarded as inequivalent. This result can be understood in terms of the "connectivity" properties of the graphs associated to the irreps. We present here a classification of the connectivity of the irreps, refining the hep-th/0511274 classification based on fields content. As a byproduct, we find a counterexample to the hep-th/0611060 claim that the connectivity is uniquely specified by the *sources* and *targets* of an irrep graph. We produce one pair of $N = 5$ irreps and three pairs of $N = 6$ irreps with the same number of sources and targets which, nevertheless, differ in connectivity.

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1 Introduction

The structure of the irreducible representations of the N-extended supersymmetric quantum mechanics has been elucidated only recently (see [1, 2, 3, 4, 5]). One is concerned with the problem of classifying the finite linear irreducible representations of the supersymmetry algebra

$$\begin{aligned} \{Q_i, Q_j\} &= \delta_{ij} H, \\ [Q_i, H] &= 0, \end{aligned} \tag{1.1}$$

where Q_i are N odd supercharges ($i = 1, \ldots, N$), while the bosonic central extension H can be regarded as a hamiltonian (therefore $H \equiv i\frac{d}{dt}$) of a supersymmetric quantum mechanical system. The finite linear irreps of (1.1) consist of an equal finite number n of bosonic and fermionic fields (depending on a single coordinate t, the time) upon which the supersymmetry operators act linearly.

In [1] it was proven that all (1.1) irreps fall into classes of equivalence determined by the irreps of an associated Clifford algebra. As one of the corollaries, a relation between n (the total number of bosonic, or fermionic, fields entering the irrep) and the value N of the extended supersymmetry was established.

A dimensionality $d_i = d_1 + \frac{i-1}{2}$ (d_1 is an arbitrary constant) can be assigned to the fields entering an irrep. The difference in dimensionality between a given bosonic and a given fermionic field is a half-integer number. The fields content of an irrep is the set of integers $(n_1, n_2, \ldots, n_l)$ specifying the number n_i of fields of dimension d_i entering the irrep. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is known as the length of the irrep. Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa. In both cases the equality $n_1+n_3+\ldots = n_2+n_4+\ldots = n$ is guaranteed. A multiplet is bosonic (fermionic) if its n_1 component fields of lower dimensions are bosonic (fermionic). The representation theory does not discriminate the overall bosonic or fermionic nature of the multiplet.

In [2] the allowed $(n_1, n_2, \ldots, n_l)$ fields contents of the N-extended (1.1) superalgebra were classified (the results were explicitly furnished for $N \leq 10$). In [5] it was further pointed out that an equivalence relation could be introduced in such a way that the fields content uniquely specifies the irreps in the given class. On physical grounds, irreps with different fields content produce quite different supersymmetric physical systems. For instance, the fields content determines the dimensionality of the target space of the one-dimensional N-extended supersymmetric sigma models, see e.g. [6]. Similarly, dimensional reductions of supersymmetric field theories produce extended supersymmetric one-dimensional quantum mechanical systems with specific field contents, see e.g. [7]. The classification of the (1.1) irreps fields contents has very obvious physical meaning. This part of the program of classifying irreps, due to [2], can now be considered completed.

In the last year, the (1.1) irreps were investigated in [3] in terms of filtered Clifford modules. In [3] and [4] it was pointed out that certain irreps admitting the same fields content can be regarded as inequivalent. These results were obtained by analyzing the "connectivity properties" (more on that later) of certain graphs associated to the irreps. A notion of equivalence class among irreps (spotting their difference in "connectivity") was introduced. In [4], two examples were explicitly presented. They involved a pair of $N = 6$ irreps with $(6, 8, 2)$ fields content and a pair of $N = 5$ irreps with $(6, 8, 2)$ fields content. In [4] the classification of the irreps which differ by connectivity was left as an open problem.

In this letter we point out that, using the approach of [2], we can easily classify the connectivity properties of the irreps of given fields contents. The explicit results will be presented for $N \leq 8$. Since the $N \leq 4$ cases are trivial, the connectivity being uniquely determined by the fields content, we explicitly present the results for $N = 5, 6, 7, 8$.

The connectivity of the irreps (inspired by the graphical presentation of the irreps known as "Adinkras" [8]) can be understood as follows. For the class of irreducible representations under consideration, any given field of dimension d is mapped, under a supersymmetry transformation, either

a) to a field of dimension $d + \frac{1}{2}$ belonging to the multiplet* or,

b) to the time-derivative of a field of dimension $d - \frac{1}{2}$.

If the given field belongs to an irrep of the N-extended (1.1) supersymmetry algebra, therefore $k \leq N$ of its transformations are of type a), while the $N - k$ remaining ones are of type b). Let us now specialize our discussion to a length-3 irrep (the interesting case for us). Its fields content is given by $(n_1, n, n - n_1)$, while the set of its fields is expressed by $(x_i; \psi_j; g_k)$, with $i = 1, \ldots, n_1$, $j = 1, \ldots, n$, $k = 1, \ldots, n - n_1$. The x_i's are 0-dimensional fields (the ψ_j are $\frac{1}{2}$-dimensional and the g_k 1-dimensional fields, respectively). The connectivity associated to the given multiplet is defined in terms of the ψ_g symbol. It encodes the following information. The n $\frac{1}{2}$-dimensional fields ψ_j are partitioned in the subsets of m_r fields admitting k_r supersymmetry transformations of type a). We have $\sum_r m_r = n$. Please notice that k_r can take the 0 value. The ψ_g symbol is expressed as

$$\psi_g \equiv m_{1k_1} + m_{2k_2} + \ldots \tag{1.2}$$

Please notice that an analogous symbol, x_ψ, obtained from the previous one by replacing the ψ_j fields with the x_i fields and the g_k fields with the ψ_j fields is always trivial. An N-irrep with $(n_1, n, n - n_1)$ fields content always produce $x_\psi \equiv n_{1N}$. Using the methods of [2], we are able to classify here the admissible ψ_g connectivities of the irreps. The pair of $N = 6$ $(6, 8, 2)$ irreps and the pair of $N = 5$ $(6, 8, 2)$ irreps of [4] fall into the two admissible classes of ψ_g connectivity for the corresponding values of N and fields content.

In [4] the two sets of three ordered numbers (for length-3 multiplets), $S = [s_1, s_2, s_3]$ and $T = [t_1, t_2, t_3]$, the "sources" and "targets" respectively, have been introduced. The integer s_i gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which do not result as an a)-supersymmetry transformation of at least one field of dimension $d_i - \frac{1}{2}$. The integer t_i gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which only admit supersymmetry transformations of type b). For a multiplet of $(n_1, n, n - n_1)$ fields content, necessarily $s_1 = n_1$, $s_3 = 0$, together with $t_1 = 0$ and $t_3 = n - n_1$. S, and T are fully determined once

*or to its opposite, the sign of the transformation being irrelevant for our purposes.

s_2 and t_2, respectively, are known. The complete list of ψ_g connectivities for length-3 multiplets contains more information than S and T. As for the targets, it is obvious that t_2 can be recovered from ψ_g. As for the sources, using the $(n_1, n, n-n_1) \leftrightarrow (n-n_1, n, n_1)$ irreps duality discussed in [2], s_2 is recovered from the ψ_g connectivity of the associated dual multiplet. In Section **3** we produce the list of the allowed connectivities. In [4] it is claimed that inequivalent irreps can be discriminated by the sources and targets S, T numbers *alone*. On the other hand, from our list of allowed connectivities, we obtain several pairs of irreps differing in connectivities but admitting the same S and T numbers of sources and targets. In Section **4** we summarize the previous results, presenting the full list of $N \leq 8$ irreps differing by sources and targets, as well as the full list of $N \leq 8$ irreps with the same sources and targets and different ψ_g connectivity. We explicitly present the $N = 5$ supersymmetry transformations of one such a pair of irreps (the $N = 5$ $(4, 8, 4)$ multiplets). We postpone to the Conclusions a discussion of the possible interpretations of our finding.

This paper is structured as follows. In the next Section, the needed ingredients and [2] conventions are reviewed. The main results are presented in Section **3**. The irreps connectivities are furnished for all cases which can potentially produce inequivalent results (therefore, for the $N = 5, 6, 7$ length-3 and length-4 irreps). In Section **4** it is pointed out that the ψ_g connectivities computed in Section **3** can discriminate irreps which are not discriminated by the sets of "sources and targets" numbers employed in [4]. Further comments and open problems are discussed in the Conclusions. To make the paper self-consistent, an Appendix with our conventions of the $Cl(0, 7)$ Clifford generators (used to construct the $N = 5, 6, 7, 8$ supersymmetry operators) is added.

2 Basic notions and conventions

In this Section we summarize the basic notions, results and conventions of [2] that will be needed in the following. Up to $N \leq 8$, inequivalent connectivities are excluded for $N = 1, 2, 3, 4$ and can only appear, in principle, for $N = 5, 6, 7, 8$. The irreps of the $N = 5, 6, 7, 8$ supersymmetric extensions can be obtained through a dressing of the $N = 8$ length-2 root multiplet (see [2] and the comment in [5]). For simplicity, we can therefore limit the discussion of the [2] construction starting from the $N = 8$ length-2 root multiplet. It involves 8 bosonic and 8 fermionic fields entering a column vector (the bosonic fields are accommodated in the upper part). The 8 supersymmetry operators $\widehat{Q}_i$ $(i = 1, \ldots, 8)$ in the $(8, 8)$ $N = 8$ irrep are given by the matrices

$$\widehat{Q}_j = \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \qquad \widehat{Q}_8 = \begin{pmatrix} 0 & \mathbf{1}_8 \\ \mathbf{1}_8 \cdot H & 0 \end{pmatrix} \tag{2.3}$$

where the γ_j matrices $(j = 1, \ldots, 7)$ are the 8×8 generators of the $Cl(0, 7)$ Clifford algebra and $H = i\frac{d}{dt}$ is the hamiltonian. The $Cl(0, 7)$ Clifford irrep is uniquely defined up to similarity transformations and an overall sign flipping [9]. Without loss of generality we can unambiguously fix the γ_j matrices to be given as in the Appendix. Please notice that each γ_j matrix (and the $\mathbf{1}_8$ identity) possesses 8 non-vanishing entries, one in each column and one in each row. The whole set of non-vanishing entries of the eight

(A.1) matrices fills the entire $8 \times 8 = 64$ squares of a "chessboard". The chessboard appears in the upper right block of (2.3).

The length-3 and length-4 $N = 5, 6, 7, 8$ irreps (no irrep with length $l > 4$ exists for $N \leq 9$, see [2]) are acted upon by the Q_i's supersymmetry transformations, obtained from the original $\widehat{Q}_i$ operators through a dressing,

$$\widehat{Q}_i \quad \rightarrow \quad Q_i = D\widehat{Q}_i D^{-1}, \tag{2.4}$$

realized by a diagonal dressing matrix D. It should be noticed that only the subset of "regular" dressed operators Q_i (i.e., having no $\frac{1}{H}$ or higher poles in its entries) act on the new irreducible multiplet. Without loss of generality, for our purpose of computing the irreps connectivities, the diagonal dressing matrix D which produces an irrep with $(n_1, n, n - n_1)$ fields content can be chosen to have its non-vanishing diagonal entries given by $\delta_{pq} d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = n + 1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, n$. Any permutation of the first n entries produces a dressing which is equivalent, for computing both the fields content and the ψ_g connectivity, to D .

Similarly, the $(n_1, n_2, n - n_1, n - n_2)$ length-4 multiplets are acted upon by the Q_i operators dressed by D, whose non-vanishing diagonal entries are now given by $\delta_{pq} d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = 2n - n_2 + 1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, 2n - n_2$.

The $N = 5, 6, 7, 8$ length-2 $(8, 8)$ irreps are unique (for the given value of N), see [5].

It is also easily recognized that all $N = 8$ length-3 irreps of given fields content produce the same value of ψ_g connectivity (1.2). For what concerns the length-3 $N = 5, 6, 7$ irreps the situation is as follows. Let us consider the irreps with $(k, 8, 8 - k)$ fields content. Its supersymmetry transformations are defined by picking an $N < 8$ subset from the complete set of 8 dressed Q_i operators. It is easily recognized that for $N = 7$, no matter which supersymmetry operator is discarded, any choice of the seven operators produces the same value for the ψ_g connectivity. Irreps with different connectivity can therefore only be found for $N = 5, 6$. The $\begin{pmatrix} 8 \\ 6 \end{pmatrix} = 28$ choices of $N = 6$ operators fall into two classes, denoted as A and B, which can, potentially, produce $(k, 8, 8 - k)$ irreps with different connectivity. Similarly, the $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = 56$ choices of $N = 5$ operators fall into two A and B classes which can, potentially, produce irreps of different connectivity. Please notice that, for some given $(k, 8, 8 - k)$ irrep, the value of ψ_g connectivity computed in both $N = 5$ (as well as $N = 6$) classes can actually coincide. In the next Section we will show when this feature indeed happens.

To be specific, we present a list of representatives of the supersymmetry operators for each N and in each $N = 5, 6$ A, B class. We have

$$\begin{array}{rcl} N = 8 & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8 \\ N = 7 & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 6 \ (case\ A) & \equiv & Q_1, Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 6 \ (case\ B) & \equiv & Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \\ N = 5 \ (case\ A) & \equiv & Q_3, Q_4, Q_5, Q_6, Q_7 \\ N = 5 \ (case\ B) & \equiv & Q_2, Q_3, Q_4, Q_5, Q_6 \end{array} \tag{2.5}$$

We are now in the position to compute the connectivities of the irreps (the results are furnished in the next Section). Quite literally, the computations can be performed by filling a chessboard with pawns representing the allowed configurations.

3 Classification of the irreps connectivities

In this Section we report the results of the computation of the allowed connectivities for the $N = 5, 6, 7$ length-3 and length-4 irreps. As discussed in the previous Section, the only values of $N \leq 8$ which allow the existence of multiplets with the same fields content but inequivalent connectivities are $N = 5$ and $N = 6$. We also produce the S and T allowed sources and targets numbers for the irreps. As recalled in the Introduction, the S sources can be recovered from a symbol, denoted as " ${}_x\psi$", expressing the partitions of the n $\frac{1}{2}$-dimensional fields ψ_j in terms of the $h_r \leq N$ number of supersymmetry transformations of a) type which map the x_i fields on a given $\frac{1}{2}$-dimensional field. Due to the irrep $(n_1, n, n-n_1) \leftrightarrow (n-n_1, n, n_1)$ duality discussed in [2], ${}_x\psi$ is recovered from the ψ_g connectivity of its dual irrep. Indeed

$$
{}_x\psi[(k, n, n-k)_*] \quad = \quad \psi_g[(n-k, n, k)_*] \tag{3.6}
$$

(the suffix $* \equiv A, B$ has been introduced in order to discriminate, when needed, the A and B subcases of $N = 5, 6$).

Our results concerning the allowed ψ_g connectivities of the length-3 irreps are reported in the following table

$length-3$	$\underline{N=7}$	$\underline{N=6}$ $\swarrow\ \searrow$ $\underline{N=6_A}$		$\underline{N=6_B}$	$\underline{N=5}$ $\swarrow\ \searrow$ $\underline{N=5_A}$		$\underline{N=5_B}$
$(7,8,1)$	7_1+1_0		6_1+2_0			5_1+3_0	
$(6,8,2)$	6_2+2_1	6_2+2_0	$-$	4_2+4_1	$4_2+2_1+2_0$	$-$	2_2+6_1
$(5,8,3)$	5_3+3_2	$4_3+2_2+2_1$	$-$	2_3+6_2	$4_3+3_1+1_0$	$-$	$1_3+5_2+2_1$
$(4,8,4)$	4_4+4_3	4_4+4_2	$-$	8_3	4_4+4_1	$-$	4_3+4_2
$(3,8,5)$	3_5+5_4	$2_5+2_4+4_3$	$-$	6_4+2_3	$1_5+3_4+4_2$	$-$	$2_4+5_3+1_2$
$(2,8,6)$	2_6+6_5	2_6+6_4	$-$	4_5+4_4	$2_5+2_4+4_3$	$-$	6_4+2_3
$(1,8,7)$	1_7+7_6		2_6+6_5			3_5+5_4	

(3.7)

The ψ_g connectivities of the $N = 5$ (and $N = 6$) A and B subcases collapse to the same value for the $(1, 8, 7)$ and $(7, 8, 1)$ irreps, proving that these multiplets do not admit inequivalent connectivities.

It is helpful to produce tables with the values of the ψ_g connectivity, the S sources and the T targets for the irreps admitting inequivalent connectivities. For $N = 6$ we get

$N=6:$	*connectivities*	*sources*	*targets*
$(6,8,2)_A$	6_2+2_0	$S=[6,0,0]$	$T=[0,2,2]$
$(6,8,2)_B$	4_2+4_1	$S=[6,0,0]$	$T=[0,0,2]$
$(5,8,3)_A$	$4_3+2_2+2_1$	$S=[5,0,0]$	$T=[0,0,3]$
$(5,8,3)_B$	2_3+6_2	$S=[5,0,0]$	$T=[0,0,3]$
$(4,8,4)_A$	6_2+2_0	$S=[4,0,0]$	$T=[0,0,4]$
$(4,8,4)_B$	4_2+4_1	$S=[4,0,0]$	$T=[0,0,4]$
$(3,8,5)_A$	$4_3+2_2+2_1$	$S=[3,0,0]$	$T=[0,0,5]$
$(3,8,5)_B$	2_3+6_2	$S=[3,0,0]$	$T=[0,0,5]$
$(2,8,6)_A$	2_6+6_4	$S=[2,2,0]$	$T=[0,0,6]$
$(2,8,6)_B$	4_5+4_4	$S=[2,0,0]$	$T=[0,0,6]$

(3.8)

For $N=5$ we obtain

$N=5:$	*connectivities*	*sources*	*targets*
$(6,8,2)_A$	$4_2+2_1+2_0$	$S=[6,0,0]$	$T=[0,2,2]$
$(6,8,2)_B$	2_2+6_1	$S=[6,0,0]$	$T=[0,0,2]$
$(5,8,3)_A$	$4_3+3_1+1_0$	$S=[5,0,0]$	$T=[0,1,3]$
$(5,8,3)_B$	$1_3+5_2+2_1$	$S=[5,0,0]$	$T=[0,0,3]$
$(4,8,4)_A$	4_4+4_1	$S=[4,0,0]$	$T=[0,0,4]$
$(4,8,4)_B$	4_3+4_2	$S=[4,0,0]$	$T=[0,0,4]$
$(3,8,5)_A$	$1_5+3_4+4_2$	$S=[3,1,0]$	$T=[0,0,5]$
$(3,8,5)_B$	$2_4+5_3+1_2$	$S=[3,0,0]$	$T=[0,0,5]$
$(2,8,6)_A$	$2_5+2_4+4_3$	$S=[2,2,0]$	$T=[0,0,6]$
$(2,8,6)_B$	6_4+2_3	$S=[2,0,0]$	$T=[0,0,6]$

(3.9)

We postpone to Section **4** the discussion of our results.

3.1 Connectivities of the length-4 multiplets

Up to $N \leq 8$, the only admissible $(n_1, n_2, n-n_1, n-n_2)$ length-4 fields contents for the $(x_i; \psi_j; g_k; \omega_l)$ irreps are given below (see [2]). Here x_i ($i=1,\ldots,n_1$) denote the 0-dimensional fields, ψ_j ($j=1,\ldots,n_2$) denote the $\frac{1}{2}$-dimensional fields, g_k ($k=1,\ldots,n-n_1$) denote the 1-dimensional fields and, finally, ω_l ($l=1,\ldots n-n_2$) denote the $\frac{3}{2}$-dimensional auxiliary fields.

The analysis of the connectivities of the length-4 irreps is done as in the case of the length-3 irreducible multiplets. Since we have an extra set of fields w.r.t. the length-3 multiplets, the results can be expressed in terms of one more non-trivial symbol. Besides ψ_g, we introduce the g_ω symbol as well. The definition of g_ω follows the definition of ψ_g in (1.2). The difference of g_ω w.r.t. ψ_g is that the g_k fields enter now in the place of the ψ_j fields, while the ω_l fields enter in the place of the g_k fields.

Contrary to the case of the length-3 irreps, the connectivity of the length-4 irreps is uniquely specified in terms of N and the length-4 fields content. The complete list of results is presented in the following table

$length-4$	$su.sies$	ψ_g	g_ω
$(1,7,7,1):$	$N=7$	7_6	7_1
	$N=6$	1_6+6_5	6_1+1_0
	$N=5$	2_5+5_4	5_1+2_0
$(2,7,6,1):$	$N=6$	1_5+6_4	6_1
	$N=5$	$1_5+2_4+4_3$	5_1+1_0
$(2,6,6,2):$	$N=6$	6_4	6_2
	$N=5$	2_4+4_3	4_2+2_1
$(1,6,7,2):$	$N=6$	6_5	6_2+1_0
	$N=5$	1_5+5_4	$4_2+2_1+1_0$
$(1,5,7,3):$	$N=5$	5_4	4_3+3_1
$(3,7,5,1):$	$N=5$	3_4+4_2	5_1
$(1,3,3,1):$	$N=3$	3_2	3_1

(3.10)

4 On "irreps connectivities" versus "sources and targets"

¿From the results presented in (3.8) and (3.9) we obtain two corollaries. At first we notice that, besides the $N=6$ $(6,8,2)$ and $N=5$ $(6,8,2)$ pairs of cases presented in [4], there exists four extra pairs, for $N\leq 8$, of inequivalent irreps with the same fields content which differ by the values of the sources and targets. The whole list of such pairs is given by

$$\begin{array}{ll} N=6: & (6,8,2)_A \leftrightarrow (6,8,2)_B \\ N=6: & (2,8,6)_A \leftrightarrow (2,8,6)_B \\ N=5: & (6,8,2)_A \leftrightarrow (6,8,2)_B \\ N=5: & (5,8,3)_A \leftrightarrow (5,8,3)_B \\ N=5: & (3,8,5)_A \leftrightarrow (3,8,5)_B \\ N=5: & (2,8,6)_A \leftrightarrow (2,8,6)_B \end{array} \tag{4.11}$$

The above list produces the complete classification of inequivalent $N\leq 8$ irreps that, according to [4], are discriminated by different values of S and T *alone*.

On the other hand, a second corollary of the (3.8) and (3.9) results shows the existence of extra pairs of irreps (a single pair of irreps for $N=5$ and three pairs for $N=6$) sharing the same fields content $(n_1,n,n-n_1)$, the same sources $S=[s_1,s_2,s_3]$ and the same targets $T=[t_1,t_2,t_3]$ which, nevertheless, admit different ψ_g connectivity. They are given by

$$\begin{array}{ll} N=6: & (3,8,5)_A \leftrightarrow (3,8,5)_B \\ N=6: & (4,8,4)_A \leftrightarrow (4,8,4)_B \\ N=6: & (5,8,3)_A \leftrightarrow (5,8,3)_B \\ N=5: & (4,8,4)_A \leftrightarrow (4,8,4)_B \end{array} \tag{4.12}$$

In order to convince the reader of the existence of such irreps with same sources and targets but different connectivity it is useful to explicitly present the supersymmetry transformations (depending on the ε_i global fermionic parameters) in at least one case. We write below the unique pair of $N = 5$ irreps (the $(4,8,4)_A$ and the $(4,8,4)_B$ multiplets) differing by connectivity, while admitting the same number of sources and the same number of targets. We have

i) The $N = 5$ $(4,8,4)_A$ transformations:

$$\begin{array}{rcl}
\delta x_1 &=& \varepsilon_2\psi_3+\varepsilon_4\psi_5+\varepsilon_3\psi_6+\varepsilon_1\psi_7+\varepsilon_5\psi_8\\
\delta x_2 &=& \varepsilon_2\psi_4+\varepsilon_3\psi_5-\varepsilon_4\psi_6-\varepsilon_5\psi_7+\varepsilon_1\psi_8\\
\delta x_3 &=& -\varepsilon_2\psi_1-\varepsilon_1\psi_5-\varepsilon_5\psi_6+\varepsilon_4\psi_7+\varepsilon_3\psi_8\\
\delta x_4 &=& -\varepsilon_2\psi_2+\varepsilon_5\psi_5-\varepsilon_1\psi_6+\varepsilon_3\psi_7-\varepsilon_4\psi_8\\
\delta\psi_1 &=& -i\varepsilon_2\dot{x}_3-\varepsilon_4 g_1-\varepsilon_3 g_2-\varepsilon_1 g_3-\varepsilon_5 g_4\\
\delta\psi_2 &=& -i\varepsilon_2\dot{x}_4-\varepsilon_3 g_1+\varepsilon_4 g_2+\varepsilon_5 g_3-\varepsilon_1 g_4\\
\delta\psi_3 &=& i\varepsilon_2\dot{x}_1+\varepsilon_1 g_1+\varepsilon_5 g_2-\varepsilon_4 g_3-\varepsilon_3 g_4\\
\delta\psi_4 &=& i\varepsilon_2\dot{x}_2-\varepsilon_5 g_1+\varepsilon_1 g_2-\varepsilon_3 g_3+\varepsilon_4 g_4\\
\delta\psi_5 &=& i\varepsilon_4\dot{x}_1+i\varepsilon_3\dot{x}_2-i\varepsilon_1\dot{x}_3+i\varepsilon_5\dot{x}_4+\varepsilon_2 g_3\\
\delta\psi_6 &=& i\varepsilon_3\dot{x}_1-i\varepsilon_4\dot{x}_2-i\varepsilon_5\dot{x}_3-i\varepsilon_1\dot{x}_4+\varepsilon_2 g_4\\
\delta\psi_7 &=& i\varepsilon_1\dot{x}_1-i\varepsilon_5\dot{x}_2+i\varepsilon_4\dot{x}_3+i\varepsilon_3\dot{x}_4-\varepsilon_2 g_1\\
\delta\psi_8 &=& i\varepsilon_5\dot{x}_1+i\varepsilon_1\dot{x}_2+i\varepsilon_3\dot{x}_3-i\varepsilon_4\dot{x}_4-\varepsilon_2 g_2\\
\delta g_1 &=& -i\varepsilon_4\dot{\psi}_1-i\varepsilon_3\dot{\psi}_2+i\varepsilon_1\dot{\psi}_3-i\varepsilon_5\dot{\psi}_4-i\varepsilon_2\dot{\psi}_7\\
\delta g_2 &=& -i\varepsilon_3\dot{\psi}_1+i\varepsilon_4\dot{\psi}_2+i\varepsilon_5\dot{\psi}_3+i\varepsilon_1\dot{\psi}_4-i\varepsilon_2\dot{\psi}_8\\
\delta g_3 &=& -i\varepsilon_1\dot{\psi}_1+i\varepsilon_5\dot{\psi}_2-i\varepsilon_4\dot{\psi}_3-i\varepsilon_3\dot{\psi}_4+i\varepsilon_2\dot{\psi}_5\\
\delta g_4 &=& -i\varepsilon_5\dot{\psi}_1-i\varepsilon_1\dot{\psi}_2-i\varepsilon_3\dot{\psi}_3+i\varepsilon_4\dot{\psi}_4+i\varepsilon_2\dot{\psi}_6
\end{array} \qquad (4.13)$$

ii) The $N = 5$ $(4,8,4)_B$ transformations:

$$\begin{array}{rcl}
\delta x_1 &=& \varepsilon_5\psi_2+\varepsilon_2\psi_3+\varepsilon_4\psi_5+\varepsilon_3\psi_6+\varepsilon_1\psi_7\\
\delta x_2 &=& -\varepsilon_5\psi_1+\varepsilon_2\psi_4+\varepsilon_3\psi_5-\varepsilon_4\psi_6+\varepsilon_1\psi_8\\
\delta x_3 &=& -\varepsilon_2\psi_1-\varepsilon_5\psi_4-\varepsilon_1\psi_5+\varepsilon_4\psi_7+\varepsilon_3\psi_8\\
\delta x_4 &=& -\varepsilon_2\psi_2+\varepsilon_5\psi_3-\varepsilon_1\psi_6+\varepsilon_3\psi_7-\varepsilon_4\psi_8\\
\delta\psi_1 &=& -i\varepsilon_5\dot{x}_2-i\varepsilon_2\dot{x}_3-\varepsilon_4 g_1-\varepsilon_3 g_2-\varepsilon_1 g_3\\
\delta\psi_2 &=& i\varepsilon_5\dot{x}_1-i\varepsilon_2\dot{x}_4-\varepsilon_3 g_1+\varepsilon_4 g_2-\varepsilon_1 g_4\\
\delta\psi_3 &=& i\varepsilon_2\dot{x}_1+i\varepsilon_5\dot{x}_4+\varepsilon_1 g_1-\varepsilon_4 g_3-\varepsilon_3 g_4\\
\delta\psi_4 &=& i\varepsilon_2\dot{x}_2-i\varepsilon_5\dot{x}_3+\varepsilon_1 g_2-\varepsilon_3 g_3+\varepsilon_4 g_4\\
\delta\psi_5 &=& i\varepsilon_4\dot{x}_1+i\varepsilon_3\dot{x}_2-i\varepsilon_1\dot{x}_3-\varepsilon_5 g_2+\varepsilon_2 g_3\\
\delta\psi_6 &=& i\varepsilon_3\dot{x}_1-i\varepsilon_4\dot{x}_2-i\varepsilon_1\dot{x}_4+\varepsilon_5 g_1+\varepsilon_2 g_4\\
\delta\psi_7 &=& i\varepsilon_1\dot{x}_1+i\varepsilon_4\dot{x}_3+i\varepsilon_3\dot{x}_4-\varepsilon_2 g_1+\varepsilon_5 g_4\\
\delta\psi_8 &=& i\varepsilon_1\dot{x}_2+i\varepsilon_3\dot{x}_3-i\varepsilon_4\dot{x}_4-\varepsilon_2 g_2-\varepsilon_5 g_3\\
\delta g_1 &=& -i\varepsilon_4\dot{\psi}_1-i\varepsilon_3\dot{\psi}_2+i\varepsilon_1\dot{\psi}_3+i\varepsilon_5\dot{\psi}_6-i\varepsilon_2\dot{\psi}_7
\end{array}$$

$$\begin{aligned}\delta g_2 &= -i\varepsilon_3\dot{\psi}_1 + i\varepsilon_4\dot{\psi}_2 + i\varepsilon_1\dot{\psi}_4 - i\varepsilon_5\dot{\psi}_5 - i\varepsilon_2\dot{\psi}_8 \\ \delta g_3 &= -i\varepsilon_1\dot{\psi}_1 - i\varepsilon_4\dot{\psi}_3 - i\varepsilon_3\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_5 - i\varepsilon_5\dot{\psi}_8 \\ \delta g_4 &= -i\varepsilon_1\dot{\psi}_2 - i\varepsilon_3\dot{\psi}_3 + i\varepsilon_4\dot{\psi}_4 + i\varepsilon_2\dot{\psi}_6 + i\varepsilon_5\dot{\psi}_7 \end{aligned} \quad (4.14)$$

5 Conclusions

In this paper we computed the allowed connectivities of the finite linear irreducible representations of the (1.1) supersymmetry algebra. For length-3 irreps the connectivity is encoded in the ψ_g symbol (1.2) which specifies how the fields in an irrep are linked together by supersymmetry transformations. For $N \leq 8$ we classified which irreps with the same fields content admit different connectivities (they only exist for $N = 5, 6$). As a corollary, we classified the irreps with inequivalent "sources and targets" (see [4]). We found counterexamples to the [4] claim that the connectivity of the irreps is uniquely specified by their "sources and targets". Irreps sharing the same fields content and same sources and targets, but differing in connectivity were also classified. A possible interpretation of our result is that the class of equivalence among irreps discussed in [4] is not fully adequate to spot differences in irreps connectivities.

The approach here discussed can be straighforwardly generalized to compute the connectivities of the $N \geq 9$ irreps of [2].

Concerning physical applications, irreps were classified according to their fields content in [2]. The differences in fields content have obvious physical meanings (as already recalled, irreps with different fields content produce, e.g., one-dimensional supersymmetric sigma models which are embedded in target manifolds of different dimensionality, see [6]). In order to understand the physical implications of the several pairs of irreps with same fields content but different connectivity, it would be quite important to construct off-shell invariant actions for both irreps in the pair. As far as we know, the construction of such off-shell invariant actions has not been accomplished yet. For $N = 8$ a large class of off-shell invariant actions, for each given irrep, has been constructed in [6]. The list in [6] is not exhaustive (see, e.g., [2], where an extra off-shell invariant action was produced). It is possible, but unlikely, that the problem of constructing off-shell invariant actions for multiplets with different connectivities could be solved with the [6] formalism of constrained superfields (since we are dealing with $N > 4$ systems). It is unclear in fact how to constrain the superfields in the cases under consideration. On the other hand, the linear supersymmetry transformations of the irreps are already given. It therefore looks promising to use the "linear" approach developed in [2]. We are planning to address this problem in the future. Another issue deserving investigation concerns the puzzling similarities shared by both linear and non-linear representations of the (1.1) supersymmetry algebra, see e.g. [10] for a recent discussion. One of the main motivations of the present work concerns the understanding of the features of the large-N supersymmetric quantum mechanical systems, due to their implications in the formulation of the M-theory, see the considerations in [11] and [7]. The dimensional reduction of the 11-dimensional maximal supergravity (thought as the low-energy limit of the M-theory) produces an $N = 32$ supersymmetric one-dimensional quantum mechanical system.

Appendix

We present here for completeness the set (unique up to similarity transformations and an overall sign flipping) of the seven 8×8 gamma matrices γ_i which generate the $Cl(0,7)$ Clifford algebra. The seven gamma matrices, together with the 8-dimensional identity $\mathbf{1}_8$, are used in the construction of the $N = 5, 6, 7, 8$ supersymmetry irreps, as explained in the main text.

$$
\gamma_1 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
\qquad
\gamma_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

$$
\gamma_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\gamma_4 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

$$
\gamma_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\gamma_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\gamma_7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\mathbf{1}_8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\qquad (A.1)
$$

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