Braided Structure in 4-dimensional conformal Quantum Field Theory

Dedicated to Gerhard Mack and Robert Schrader on the occasion of their 60th birthday

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Abstract

Higher dimensional conformal QFT possesses an interesting braided structure which, different from the d=1+1 models, is restricted to the timelike region and therefore easily escapes euclidean action methods. It lies behind the spectrum of anomalous dimensions which may be viewed as a kind of substitute for a missing particle interpretation in the presence of interactions.

1 Introductory remarks

Since the early 60ies the use of conformal field theory in particle physics has been beset by physical doubts concerning incompatibilities with the LSZ framework of interacting (Wigner) particles. Indeed, besides free zero mass particles there are strictly speaking no other conformal situations which are consistent with a particle interpretation [1]. The analytic simplifications of massless limits are (again apart from free situations) in some sense paid for by the conceptual complications within the particle setting; the latter allow at best to extract extremely inclusive scattering data from interacting conformal field theory using a scattering theory which is based on probabilities instead of amplitudes [2] i.e. conformal theory is not directly a theory of particles. Nevertheless, as it is already well known from chiral theories, conformal field theory with its arena of spacetime charge flows and their fusions is presently the most useful theoretical laboratory to explore nonperturbative aspects of QFT in a mathematically controllable way.

This mentioned particle physics weakness was more than compensated for by the profound application to the theory of critical phenomena in statistical mechanics. In fact there have been two approaches in that direction. The older and in its aims more ambitious and general way is *extrinsic* i.e. one tries to approach critical theories from a generic i.e. noncritical setting by the renormalization group action in the commutative Euclidean functional integral setting. This is the method of [1] either Wilson (momentum space) or Kadanoff (x-space). These methods work well in the standard Euclidean setting if the functional integral approach coupled with dimensional perturbative ideas; it gives surprisingly good numerical results in the Lagrangian Callan-Symanzik setting. Its main aim is to obtain a numerical approximation for the critical indices of a given Hamiltonian or action in a continuous or discretized system.

The advent of chiral conformal models exemplified (in a more restricted setting) another, but this time truely intrinsic approach. In this the critical theories where not constructed by finding renormalization group transformations in an ambient coupling constant space which drive the system towards the scale invariant fixed points, but rather by finding enough intrinsic properties of conformal models which eventually lead to their classification and explicit constructions. Since the detailed connection between the concrete condensed matter and its microscopic description is in many cases unknown anyhow, this second method, if combined with the universality class concept and precise measurements really explains the critical indices as anomalous dimensions in an associated real time conformal field theory. As we will see below, they are related to a classifiable and computable algebraic structure. A more detailed application to a classification of exact critical indices of systems related to higher dimensional conformal field theories will be presented elsewhere [3].

A prototype of such an intrinsic approach is the BPZ scheme [4] of conformal blocks[2] which was

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[1]A formulation in the real-time setting of Wightman (i.e. noncommutative in the old-fashioned sense) which probably requires the use of more subtle conditional expectations on operator algebras has never been achieved. On the other hand there are also several real-time properties whose statistical mechanics interpretation is unknown to this date, e.g. the issue of the timelike structure in this article.

[2]It was foreshadowed by Kadanoff's ideas on "Coulomb Gas Representations".

successfully applied towards a construction of minimal models. There one aims at the critical theories right away and worries about their noncritical deformations (which convert them into islands in a continuum of theories in one phase) afterwards. The mathematical basis are certain algebras (Virasoro-, current-,W-) and their positive energy representations.

Unfortunately these algebras are limited to two-dimensional QFT, but fortunately it turned out that one can understand a substantial part of the two-dimensional situation by adapting the general theory of superselection rules of Doplicher-Haag-Roberts to low dimensions [5], thus increasing the chances of generalizations away from two dimensions. Whereas the Wilson-Kadanoff method had no known counterpart on the noncommutative real time side, the real-time DHR method has no known Euclidean formulation. The power of this more noncommutative DHR method of superselection charges in chiral theories resembles somewhat the power of the more noncommutative transfer matrix method in Baxter's work on 2-dim. lattice models in statistical mechanics. As a result of the restrictive power of relativistic causality and locality and the ensuing vacuum structure which distinguishes QFT from quantum mechanics, the method of algebraic QFT (AQFT) is more generic than the transfer-matrix method. In particular its validity is not limited to subclasses of chiral theories, but promises to classify and construct all of them [6].

The main point is the classification of admissible braid-group statistics [7]. This is part of the classification of superselected charges and their fusion. It does not require the study of genuine space-time properties of the dynamics of charge flows and can be done in terms of combinatorial algebras (type II_1 von Neumann algebras, in particular braid group intertwiner algebras with Markov traces on them). This is analogous (but more difficult) to the internal symmetry data for free field theory. As in the latter case the spin/statistics+group representation data uniquely fix the free field, the spacetime aspects of chiral models are completely determined by the charge superselection data, albeit in a quite sophisticated way. This tight connection in chiral theories[3] perhaps canot be expected for higher dimensional conformal theories; the present investigation does not try to resolve the problem whether in addition to the time- and spacelike- superselection data there are also deformation parameters which locally do not change the superselection class ($\simeq$ "phases" separated by phase transitions) but modify the strength of interactions and cannot be absorbed into the braid group statistics data.

The restrictive role in terms of fusion and statistics of superselected charges is already visible in the analysis of 4-point functions from monodromy and short distance behavior in chiral conformal theories, but a systematic construction without relying on educated guesses and consistency checks still awaits elaboration. In any case the intrinsic approach which aims first at the explicit constructions of the conformal "islands", and then tests the infinitesimal surrounding by some kind of Zamolodchikov perturbation idea, has been quite successful and promises more to come.

The drawback of the exact intrinsic method is that it appears limited to chiral theories. Higher

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[3]Whenever there is a continuously varying coupling parameter as in the massless Thirring model, this can be absorbed into the chiral statistics parameters (= chiral anomalous dimensions modulo integers).

dimensional conformal field theories do not have the simplifying feature of chiral factorization, but they also seem to be formally close to free field theories in the aforementioned sense. As far back as the middle 70ies there were two ways of thinking about general conformal fields, either as Wightman fields on the covering of compactified Minkowski space with a globalized notion of causality and the field-state-vector relation (absence of local annihilators, i.e. the Reeh-Schlieder property) as it appeared in the work of Luescher and Mack [8], or as (controllable) nonlocal projection back into the compactified Minkowski spacetime as a kind of quasiperiodic operator-valued sections. Different from (smeared) Wightman fields, the projected fields have a source and range projector attached to them[4] and arise as the components of the L-M Wightman fields in a decomposition with respect to the center of the conformal covering group. The resulting decomposition theory of the covering fields appeared in joint work involving the present author [9] and was done independent of the Luescher-Mack work. It is best seen as a timelike analogue of the spin-statistics theorem in that it relates the phases under a full timelike sweep through the Dirac-Weyl compactified Minkowski world to the spectrum of anomalous scale dimensions modulo (half)integers (in the spin case the spatial 2π rotational sweep). This raises the question whether there could be also some timelike "statistics" (exchange structure) behind the anomalous dimension spectrum. It is the purpose of this short note to argue that this is indeed the case.

2   The Timelike Plektonic Structure

In a situation without controllable models one has to proceed in a structural way and at the end use the obtained structure to classify and construct models. As mentioned in the introduction there are two ways to explore the structure of conformal theories, the Luescher-Mack approach which uses Wightman fields on the conformal covering space and the approach of Swieca and myself which works with nonlocal quasiperiodic component fields on the compactified ordinary Minkowski spacetime. The connection is given by the decomposition formula for conformal fields

$$F(x) = \sum_{\alpha,\beta} F_{\alpha,\beta}(x), \ F_{\alpha,\beta}(x) \equiv P_\alpha F(x) P_\beta \qquad (1)$$

$$Z = \sum_\alpha e^{2\pi i \theta_\alpha} P_\alpha$$

Here Z is the generator of the center of the conformal covering group $\widetilde{SO(4,2)}$, P_α are its spectral projectors and the phases are the mod1 reductions of the anomalous dimensions associated with the family of fields sharing the same phase (conformal blocks). Whereas the $F(x)$ are Lüscher-Mack fields i.e. globally causal Wightman fields which live on the covering space (and therefore one should use the appropriate more complicated coordinates or charts), the $F_{\alpha\beta}(x)$ are (operator-valued) trivializing section

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[4]In chiral theories these operators obey exchange algebra relations and are often referred to as vertex operators as opposed to chiral observables which are described by Wightman fields.

on the Dirac-Weyl compactification $\bar{M}$ are apart from fulfilling a timelike numerical quasiperiodicity condition conventional objects of the "laboratory-world" (without requiring the heaven and hells of the covering) [10][11]. In chiral theories they become the "vertex operators" appearing in the conformal block analysis of [4].

Although in higher dimension one cannot have anything else than spacelike Bose/Fermi commutation relation of the standard field/particle statistics, the possibility of defining conformal observables as local fields which live on the compactification $\bar{M}$ prepares the ground for becoming aware of timelike commutativity (Huygens principle). Together with the ordering structure inside the timelike lightcone this allows for a consistent timelike braid group commutation structure. In terms of the above double indexed fields this means the validity of the following exchange algebra

$$F_{\alpha,\beta}(x)G_{\beta,\gamma}(y) = \sum_{\beta'} R_{\beta,\beta'}^{(\alpha,\gamma)} G_{\alpha,\beta'}(y) F_{\beta',\gamma}(x), \ x > y \tag{2}$$

$$F_{a,\beta}G_{\beta,\gamma} = \sum_{\beta'} R_{\beta,\beta'}^{(\alpha,\gamma)} G_{\alpha,\beta'} F_{\beta',\gamma}, \ locF > locG \tag{3}$$

where the second line refers to the bounded operators of the local algebras. The arguments that the R-matrices must be representers of the Artin braid group is the same as in d=1+1 where the spacelike region has the same topology as timelike light cones. These component fields have no physical role in the spacelike region in fact they are local and one has to sum them according to (1) in order to formulate Einstein causality. This timelike braid structure is of course consistent with the structure of the conformal 2- and 3-point functions. For example from

$$\langle F(x)F(y)^* \rangle \simeq lim_{\varepsilon \to 0} \frac{1}{\left[-(x-y)_\varepsilon^2 \right]^{\delta_A}} \tag{4}$$

$$(x-y)_\varepsilon^2 = (x-y)^2 + i\varepsilon(x_0 - y_0)$$

one reads off the timelike relation

$$\langle F(x)F(y)^* \rangle = e^{2i\delta_F} \langle F(y)^* F(x) \rangle , \ x > y$$

Let us now look at this situation from a more general analytic viewpoint of vacuum correlation functions. It is well known that Wightman functions allow for analytic continuations into the so-called BWH domain. As was observed first by Jost, this complex domain has real spacelike points which he was able to describe explicitly (Jost points). This fact is useful if one wants to describe the various operator orderings inside an n-point function in terms of one analytic master function by using spacelike local commutativity. What is less well known is that the use of conformal invariance allows for an additional analytic extension a la BHW which leads to timelike Jost points. Again the role of algebraic relations is to bind together the various timelike orderings into one analytic master function. But now, as a consequence

of the more general braid structure one will loose the uniqueness in that continuation i.e. one will end up with ramified coverings which are the analytic manifestations of Huygens-localizable fields (relative to the observables) which are timelike nonlocal (plektonic) among themselves. Whereas the intermediate projectors nearest to the vacuum are uniquely determined by the anomalous dimension of the field, there are the well-known branchings in other projectors. For the 4-pointfunction we obtain a sum of several contributions which are interrelated by crossing transformations which correspond to the application of the plektonic commutation relations.

$$W(x_4, x_3, x_2, x_1) := \sum_\gamma \langle F(x_4)F(x_3)P_\gamma F(x_2)F(x_1) \rangle \tag{5}$$

$$= \left[\frac{x_{42}^2 x_{31}^2}{(x_{43})_\varepsilon^2 (x_{32})_\varepsilon^2 (x_{21})_\varepsilon^2 (x_{14})_\varepsilon^2} \right]^{\delta_A} \sum_\gamma w_\gamma(u, v),$$

$$u = \frac{x_{43}^2 x_{21}^2}{(x_{42})_\varepsilon^2 (x_{31})_\varepsilon^2}, \quad v = \frac{x_{32}^2 x_{41}^2}{(x_{42})_\varepsilon^2 (x_{31})_\varepsilon^2}$$

This is all very similar to the well studied chiral situation except that the analytic consequences of monodromy relations are somewhat more involved because there is no factorization which reduces the number of cross ratios. The new aspect as compared to the exchange structure of chiral theories is the issue of compatibility between the plektonic timelike- anf the spacelike bosonic/fermionic exchange algebra. This issue was discussed in the above analytic setting of correlation functions in [13]. From these consistency relations between correlations of pointlike fields, Rehren [14] abstracted the following group structure which amalgamates the Artin braid group B_∞ with the symmetry group S_∞ ("mixed group" G_∞)

$$b_i t_j = t_j b_i, \ |i - j| \geq 2 \tag{6}$$

$$b_i t_j t_i = t_j t_i b_j, \ |i - j| = 1$$

$$b_i b_j t_i = t_j b_i b_j, \ |i - j| = 1$$

Here the b's are the generators of the braid group, i.e. they fulfill the well-known Artin relations among themselves, whereas the t's are the transpositions which generate the permutation group.

The question of how these consistency problems can be taken care of within the conceptually more complete and mathematically more rigorous DHR setting [5] still awaits solution; the problem here is that the spacelike Haag dualization generates a different net structure from the Haag dual net on $\bar{M}$. Whereas this is welcome since one wants to arrive at two different localized endomorphisms and this is only possible on two different nets, the exact positions of these nets remain an open problem for further research.

One important aspect of the endomorphism formalism would be the understanding of the central charges which are expected to arise from the global selfintertwiners related to the timelike charge transport

around $\bar{M}$ and their associated S-T $SL(2,\mathbb{Z})$-structure [6]. Whereas one expects the physical realization in the sense of Rehren's "statistics characters" [15] to continue to hold in higher dimensions, the derivation of modular identities based on Verlinde's geometric argument which led to the modular "duality" (dual temperature $\beta \to \frac{2\pi}{\beta}$) breaks down. This can be seen by explicitly calculating the partition function of the conformal Hamiltonian R_0 [13] which is equal to the true Hamiltonian of the associated anti- deSitter model. In the case of AdS_4 for which Fronsdal [16] computed its spectrum (i.e. R_0 in 3-dim. conformal theory), one obtains

$$tre^{-\beta R_0} = e^{-E_0\beta} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{N_n}}, \quad N_n = 2n+1 \tag{7}$$

$$R_0 = E_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{(2n+1)} n a_n^{(k)*} a_n^{(k)}$$

and it is clear that this partition function does not have the properties of the η-and Jacobi ϑ-functions which appear in the chiral theories. In other words one needs the chiral factorization in order to validitate Verlinde's differential geometric torus interpretations, i.e. causality and braid group aspects alone are not sufficient to produce modular $SL(2,R)$ relations in thermal partition functions of R_0.

3 Structural Discoveries in LQP and Murphy's Law

There are three ways in which Local Quantum Physics reacts to inventions of particle physicists. On the one hand there are those ideas which although not required by the general principles are nevertheless consistent because they allow for a model realization (perturbatively, or low-dimensional nonperturbatively). A good example is supersymmetry; in the absence of experimental indications the theoretical "naturality" of such an object may depend on what aspect of the formalism of QFT one considers as physically important. The majority of inventions in QFT, especially those which tried to modify QFT by cutoffs, nonlocalities etc. suffered a less lucky fate and were in most cases doomed by Murphy's law [17] ("what can go wrong, will go wrong") which asserts its strength in a rather swift way especially in the precision setting of local quantum physics.

On the opposite end are the attempts to reformulate the content of quantum field theory in a intrinsic field-coordinate independent way, using methods of operator algebras and aiming not at inventions with unclear relations to the physical principles, but rather unraveling structural properties as consequences of the physical principles (which may have escaped the standard quantization formalism). The present observation about a timelike algebraic structure in conformal theories may serve as an illustration of a structural property which is not imposed from the outside and hard to be seen by Lagrangian methods. Rather it is derived from requiring a local explanation for the dynamical timelike superselection structure which in turn is responsible for the appearance of anomalous dimensions. It has been a remarkably gratifying experience that all admissible structures which had been derived by classifying the modes

of realizations of the underlying principles sooner or later were realized through the construction of new models. For example in the case of the lightlike braid group structure of chiral models all serious attempts to find a model realization of admissible charge superselection with given fusions and braid group commutation relations (exchange algebras) were successful and lend weight to the validity of an "anti-Murphy" maxim: *every structure which is derived from the general principles of local quantum physics has (sooner or later) a model realization.*

The specific connection between the anomalous dimensional spectrum and the plektonic data not only depend on spacetime dimensions but for a given spacetime dimension they also depend on the nature of the spacelike superselections e.g. on the presence of group representation multiplicities; e.g. charge-carrying fields in current algebras have a different dimensional spectrum than say W-algebras even if the fusion laws and the braid group statistics may be connected. The dynamic aspect of timelike superselections nourishes the hope that there are no coupling parameters which cannot be encoded into the spectrum of anomalous dimensions[5]. This would uniquely relate the spacetime nets with the superselection structure (similar to the case of chiral theories) and significantly facilitate their actual construction.

It should be clear at this point that the main purpose of our interest in conformal theories is the actual construction of a nontrivial 4-dimensional field theory in order to show that the principles of 20 century QFT are sound and allow a mathematically consistent presentation as all previous areas in physics. Principles as causality/locality should be limited and superseded by new principles and not by an ad hoc cut-off in certain Feynman- or functional- integrals and the conceptual havoc created by cutoffs is most evident in conformal theories.

The standard perturbation theory is not directly applicable to zero mass situations[6] (e.g. the infrared divergencies of massless QED); the strategy to obtain perturbative conformal models has been to look out for massive models with vanishing Beta-functions ($\simeq$vanishing of coupling constant renormalization) which is certainly necessary for conformal invariance but also seems to create that "soft" mass dependence which guaranties the existence of the massless limit. Although every conformal theory is automatically renormalizable in the sense of an order-independent dominating short distance power law, the Lagrangian candidates for conformal field theories are extremely scarce; the only known candidates in d=1+1 are very special 4-Fermion couplings as the Thirring model whereas for d=1+3 one seems to be restricted to a particular supersymmetric Yang-Mills models (SYM) namely those with vanishing beta-functions. Most chiral models have no coupling strength which allows to deform them continuously into free theories;

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[5]This is the situation in chiral theories where e.g.the coupling strength of the massless Thirring model may be completely absorbed into the variable anomalous dimensions of its chiral components, i.e. the dynamics is completly accounted for by the anomalous dimension spectrum.

[6]Note that interacting conformal theories are not theories of particles since a massless particle has necessarily an interpolating field with ("protected") canonical dimension and that in turn leads to an interaction-free sector, i.e. all fields whose applications to the vacuum do not lead out of this sector (which includes in particular all composites of the interpolating field) are also void of interactions [1].

rather such coupling constants are only expected to arise in a Zamolodchikov perturbation around the conformal islands; there is no reason to expect an improvement of this situation in d=1+3, which means that the above scarcity would be a perturbative fake.

At this point one may think that the much discussed AdS-CQFT connection could enrich the d=1+3 conformal situation by applying the perturbation on the AdS side where there is no (Beta-function) restriction besides the SO(4,2) invariance. Indeed the rigorously proven isomorphism[7] [18] does secure the existence of a conformal theory but unfortunately the holographic images of all AdS pontlike field theories are conformal theories which violate the causal propagation (causal shadow) property since the pointlike AdS theory contains too many degrees of freedom for a physically acceptable conformal theory [13]. On the other hand a conformal theory with the correct causal propagation leads to a AdS theory in which the best localized generators are "strands" i.e. stringlike objects which however do not increase the conformal degrees of freedom as the dynamical strings of string theory would do.

I believe that this new timelike structure will just leave enough space to sail between the scilla of trivial free field theories (as expressed by the triviality of an imposed interacting particle structure) and the charybdis of mathematically uncontrollable "normal" quantum field theories. In this way I expect 4-dimensional conformal field theories to be the first nontrivial nonperturbatively controllable and explicitly constructed QFT in physical spacetime. It would be nice if finally, after almost 50 years of strenuous efforts, local quantum field theory could enjoy the same status as any other mathematically and conceptually established physical theory.

References

[1] B. Schroer, Phys. Lett. **B 494,** (2000) 124, hep-th/0005110

[2] D. Buchholz, "Mathematical Physics Towards the 21st Century", Proceedings Beer-Sheva 1993, Ben Gurion University Press 1994

[3] B. Schroer, in preparation

[4] A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241**, (1984) 333

[5] R. Haag, *Local Quantum Physics*, Springer Verlag (1992)

[6] K. Fredenhagen, K-H Rehren and B. Schroer, Rev. Math. Phys., **SI1** (1992) 113, and references therein.

[7] K.-H. Rehren, Ann. Henri Poincaré **1**, (2000) 607

[8] M. Lüscher and G. Mack, Commun. Math. Phys. **41**, (1975) 203

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[7]In the formulation of the Maldacena-Witten conjecture it is tacitely assumed that to one conformal theory there corresponds exactly one AdS theory, but without leaving the realm of pointlike fields it is not possible to prove this.

[9] Schroer and J. A. Swieca, Phys. Rev. **D10**, (1974) 480

[10] B. Schroer, J. A. Swieca and A. H. Voelkel, Phys. Rev. **D11**, (1975) 11

[11] J. Kupsch, W. Ruehl and B. C. Yunn, Ann. Phys. (N.Y.) **89**, (1975) 141

[12] Nicolay M. Nikolov and Ivan T. Todorov, *Rationality of conformally invariant local correlation functions on compactified Minkowski space*, hep-th/0009004

[13] B. Schroer, *Anomalous Scale Dimensions from Timelike Braiding*, hep-th/0005134

[14] K.-H. Rehren, private communication

[15] K.-H. Rehren, "Braid group statistics and their superselection rules" in *The Algebraic Theory of Superselection Rules*, edited by D. Kastler (World Scientific, Singapore), 1990

[16] C. Fronsdal, Phys. Rev. **D10**, (1974) 589

[17] A. Held and P Yordzis, Gen. Rel. Grav. **13**, (1981) 873

[18] K.-H. Rehren, *Local Quantum Observables in the Anti de Sitter-Conformal QFT Correespondence,* hep-th/0003120