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PERTURBATIONS OF THE FRIEDMANN UNIVERSE

by

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ABSTRACT

Correcting and extending previous work by Hawking (1966) and Olson (1976) we derive and analyse the complete set of perturbation equations of a Friedmann Universe in the quasi-Maxwellian form. The formalism is then applied to scalar, vector and tensor perturbations of a phenomenological fluid, which is modelled such as to comprise shear and heat flux. Depending on the equation of state of the background we find that there exist unstable (growing) modes of purely rotational character. We further find that (to linear order at least) any vortex perturbation is equivalent to a certain heat flux vector. The equation for the gravitational waves are derived in a completely equivalent method as in case of the propagation, in a curved space-time, of electromagnetic waves in a plasma endowed with some definite constitutive relations.

1. INTRODUCTION

Several other methods exist to investigate perturbations of inhomogeneous cosmological models (Lifshitz, 1946; Lifshitz and Khalatnikov, 1963; Bardeen , 1980) but we prefer here a formalism which in its roots goes back to some work of Jordan and coworkers (Jordan et al, 1960; Kundt and Trümper 1962; Trümper 1967) which is described in accessible form in Ellis (1971). As noted by Hawking (1966) the quasi-Maxwellian form of Einstein's equations has the advantage that one is dealing only with physical (i.e., directly measurable) quantities and things simplify further if the Weyl tensor of the background metric vanishes, as is the case for a Friedmann Universe. The reason for this is related to the tensorial character of Weyl conformal tensor $W_{\alpha\beta\mu\nu}$. Indeed, once in the background geometry $W_{\alpha\beta\mu\nu}$ vanishes, any small quantity $\delta W_{\alpha\beta\mu\nu}$ represents a true perturbation and not merely a trivial coordinate transformation.

The difficulty of any such perturbation method,other than the quasi-Maxwellian approach, is to find the physical content of the result obtained. This problem has been discussed by many authors and re-examined recently by Bardeen (1980) insome detail and is inherent in many nonlinear theories. Apart from the intrinsic (unresolved) problem to tell with mathematical rigour what it means that two geometries differ by a certain infinitesimal amount, there is the additional difficulty how to relate (in a physically unique way) the two solutions. The first problem is related to the general covariance of the equations; the second is related to the gauge invariance of the physical system. It is mainly with the second category of problems that one is usually concerned. Let us consider as an illustrative example a classical (cold, collisionless) fluid. Let the four-velocity of each particle of the fluid be $\mu^{\alpha}(x)$ and consider a "perturbed flow" $\mu^{\alpha}(x) + \delta\mu^{\alpha}$ which comes about through the action of some force δK^{α}. Let $\xi^{\alpha}(x)$ be defined as the map which takes a particle at point X^{α} into $\tilde{X}^{\alpha} = X^{\alpha} + \delta X^{\alpha} = X^{\alpha} + \xi^{\alpha}(x)$ such that there the perturbed four velocity $\tilde{\mu}^{\alpha}$ due to the force δk^{α} is just $\tilde{\mu}^{\alpha} = \mu^{\alpha}(x+\xi)+\delta\mu^{\alpha}(x+\xi)$.This map relates every particle of the unperturbed flow to a particle of the perturbed flow, but it is not unique. Evidently, we are free to add to ξ^{α} an (infinitesimal) piece of the (unperturbed) trajectory $\varepsilon\mu^{\alpha}$ and we arrive at a new possible map

$$\tilde{\xi}^{\alpha} = \xi^{\alpha} + \epsilon\mu^{\alpha} \tag{1}$$

While the two maps are mathematically completely equivalent they are by no means physically equivalent, as the transformation (1) does not leave invariant the proper 3-volume element, and therefore the proper baryon density. As is evident therefore it does not make sense at this stage of approximation to talk of the amplitude of the perturbation. To be able to do that one must consider explicetely the external perturbing force (switched-on e.g. adiabatically) which gave rise to the perturbation. Different operational prescriptions have been proposed by Olson (1976) and Bardeen (1980) to obtain gauge independent amplitudes. The problem is of particular relevance in cosmology, where one considers perturbations of the entire universe, i.e. perturbations which may be larger (at early times, at least) than the particle horizon.

We shall follow here the conformal approach which as will become clear below has a number of distinct advantages for space times which are conformally flat. In this method, the especification of gauge independent quantities is a simple task (at least in the case of Friedmann background) as one can easily be convinced by just looking at the kinematical parameters which vanish in the background, besides the invariants quantities associated to the perturbed conformal tensor. For pure gauge transformations all these quantities (which are null in the background) remains null. This is an absolute criterion to eliminate unphysical modes of perturbations.

Our motivation for reconsidering the whole conformal approach is two fold: we shall generalize it to non-perfect fluids and we shall give an analysis of the complete set of perturbation equations. This is needed, once prior results obtained by Hawking and Olson are incomplete and even contains some erroneous conclusions.

2. THE UNPERTURBED FIELD EQUATIONS

The background geometry obeys Eintein's field equations

$$R^{\mu}{}_{\nu} - \frac{1}{2} R \delta^{\mu}{}_{\nu} = -T^{\mu}{}_{\nu} \quad (2)$$

with the energy-momentum tensor of a general fluid

$$T^{\mu}{}_{\nu} = \rho V^{\mu} V_{\nu} - p h^{\mu}{}_{\nu} + q^{\mu} V_{\nu} + q_{\nu} V^{\mu} + \Pi^{\mu}{}_{\nu} \quad (3)$$

in which

$$h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - V^{\mu} V_{\nu} \qquad (4)$$

$$V_{\mu} V_{\nu} g^{\mu\nu} = +1$$

q_{μ} is the heat flux and $\Pi^{\mu}{}_{\nu}$ the traceless anisotropic pressure. The gradient of the four-velocity of the matter V^{α} can be decomposed in its irreductible parts as follows:

$$V_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} - \dot{V}_{\mu} V_{\nu} \qquad (5)$$

where $\dot{V}^{\alpha} = V^{\alpha}{}_{;\mu} V^{\mu}$ is the acceleration, $\theta = V^{\alpha}{}_{;\alpha}$ is the expansion, $\sigma_{\mu\nu} = \frac{1}{2} V_{\alpha;\beta} h^{\alpha}{}_{(\mu} h^{\beta}{}_{\nu)} - \frac{1}{3}\theta h_{\mu\nu}$ is the shear and $\omega_{\mu\nu} = \frac{1}{2} V_{\alpha;\beta} h^{\alpha}{}_{[\mu} h^{\beta}{}_{\nu]}$ is the vorticity of the flow V^{α}. We define the vorticity vector ω^{α} as:

$$\omega^{\tau} = \frac{1}{2} \eta^{\alpha\beta\rho\tau} \omega_{\alpha\beta} V_{\rho} \qquad (6)$$

and $\eta^{\alpha\beta\mu\nu} = -\frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\mu\nu}$ is the completely anti-symmetric tensor. Here and henceforth a bracket [,] means antisymmetrization that is $[\alpha,\beta] = \alpha\beta - \beta\alpha$ and a parenthesis ( ) denotes symmetrization $(\alpha,\beta) = \alpha\beta + \beta\alpha$. We define further the "electric" and "magnetic" part of the gravitational field

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu} V^{\mu} V^{\nu} \qquad (7a)$$

$$H_{\alpha\beta} = -\overset{*}{W}_{\alpha\mu\beta\nu} V^{\mu} V^{\nu} \qquad (7b)$$

The symbol * means the dual, that is

$$\overset{*}{W}_{\alpha\mu\beta\nu} = \frac{1}{2} \eta_{\alpha\mu}{}^{\varepsilon\lambda} W_{\varepsilon\lambda\beta\nu}. \tag{8}$$

The Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ can then be decomposed:

$$W^{\alpha\mu\beta\nu} = (\eta^{\alpha\mu\lambda\sigma} \eta^{\beta\nu\tau\varepsilon} - g^{\alpha\mu\lambda\sigma} g^{\beta\nu\tau\varepsilon}) V_\lambda V_\tau E_{\sigma\varepsilon} + $$
$$+ (\eta^{\alpha\mu\lambda\sigma} g^{\beta\nu\tau\varepsilon} + g^{\alpha\mu\lambda\sigma} \eta^{\beta\nu\tau\varepsilon}) V_\lambda V_\tau H_{\sigma\varepsilon} \tag{9}$$

in which $g_{\alpha\mu\beta\nu} = g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\beta\mu}$.

The tensor $E_{\mu\nu}$ satisfy the properties

$$\begin{aligned} &E_{\mu\nu} = E_{\nu\mu} \\ &E_{\mu\nu} V^\nu = 0 \\ &E_{\mu\nu} g^{\mu\nu} = 0 \\ &H_{\mu\nu} = H_{\nu\mu} \\ &H_{\mu\nu} g^{\mu\nu} = 0 \\ &H_{\mu\nu} V^\nu = 0 \end{aligned} \tag{10}$$

The Riemann tensor $R_{\alpha\beta\mu\nu}$ can be writen:

$$R_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu} + J_{\alpha\beta\mu\nu} - \frac{1}{6} R\, g_{\alpha\beta\mu\nu} \tag{11}$$

in which

$$J_{\alpha\beta\mu\nu} = \frac{1}{2} \{R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}\} \tag{12}$$

The Bianchi identities $R^{\alpha\beta}{}_{\{\mu\nu;\lambda\}} \equiv R^{\alpha\beta}{}_{\mu\nu;\lambda} + R^{\alpha\beta}{}_{\nu\lambda;\mu} + R^{\alpha\beta}{}_{\lambda\mu;\nu} = 0$ can be writen in an equivalent form, using Weyl tensor:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha} R^{,\beta]} \tag{13}$$

Using Einstein's equations (2) these can be writen as

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha}T^{,\beta]} \tag{14}$$

Using (9) and projecting appropriately we can re-write Eintein's equation in a form which is reminiscent of Maxwell's equations:

$$h^{\varepsilon\alpha} h^{\lambda\gamma} E_{\alpha\lambda;\gamma} + \eta^{\varepsilon}{}_{\beta\mu\nu} V^{\beta} H^{\nu\lambda} \sigma^{\mu}{}_{\lambda} + 3H^{\varepsilon\nu} \omega_{\nu} =$$

$$= \frac{1}{3} h^{\varepsilon\alpha} \rho_{,\alpha} + \frac{\theta}{3} q^{\varepsilon} - \frac{1}{2}(\sigma^{\varepsilon}{}_{\nu} - 3\omega^{\varepsilon}{}_{\nu}) q^{\nu} + \frac{1}{2} \Pi^{\varepsilon\mu} \dot{V}_{\mu} +$$

$$+ \frac{1}{2} h^{\varepsilon\alpha} \Pi_{\alpha}{}^{\nu}{}_{;\nu} \tag{15a}$$

$$h^{\varepsilon\alpha} h^{\lambda\gamma} H_{\alpha\lambda;\gamma} - \eta^{\varepsilon}{}_{\beta\mu\nu} V^{\nu} E^{\nu\lambda} \sigma^{\mu}{}_{\lambda} - 3E^{\varepsilon\nu} \omega_{\nu} =$$

$$= 2(\rho+p)\omega^{\varepsilon} - \eta^{\varepsilon\alpha\beta\lambda} V_{\lambda} q_{\alpha;\beta} + \eta^{\varepsilon\alpha\beta\lambda}(\sigma_{\mu\beta} + \omega_{\mu\beta}) \Pi^{\mu}{}_{\alpha} \tag{15b}$$

$$h_{\mu}{}^{\varepsilon} h_{\nu}{}^{\lambda} \dot{H}^{\mu\nu} + \theta H^{\varepsilon\lambda} - \frac{1}{2} H_{\nu}{}^{(\varepsilon} h^{\lambda)}{}_{\mu} V^{\mu;\nu} -$$

$$-\eta^{\lambda\nu\mu\sigma} \eta^{\varepsilon\tau\alpha\beta} V_{\mu} V_{\tau} H_{\alpha\sigma} \theta_{\nu\beta} - \dot{V}_{\alpha} E_{\beta}{}^{(\lambda} \eta^{\varepsilon)\sigma\alpha\beta} V_{\alpha} +$$

$$+ \frac{1}{2} E_{\beta}{}^{\mu}{}_{;\alpha} h_{\mu}{}^{(\varepsilon} \eta^{\lambda)\sigma\alpha\beta} V_{\sigma} = -\frac{3}{4} q^{(\varepsilon}\omega^{\lambda)} + \frac{1}{2} h^{\varepsilon\lambda} q^{\mu} \omega_{\mu} +$$

$$+ \frac{1}{4} \sigma_{\beta}{}^{(\varepsilon} \eta^{\lambda)\alpha\beta\mu} V_{\mu} q_{\alpha} + \frac{1}{4} h^{\mu(\varepsilon} \eta^{\lambda)\alpha\beta\mu} V_{\mu} \Pi_{\mu\alpha:\beta} \tag{15c}$$

$$h_\mu{}^\varepsilon h_\nu{}^\lambda \dot{E}^{\mu\nu} + \Theta E^{\varepsilon\lambda} - \frac{1}{2} E_\nu{}^{(\varepsilon} h^{\lambda)}{}_\mu V^{\mu;\nu} -$$

$$- \eta^{\lambda\nu\mu\sigma} \eta^{\varepsilon\tau\alpha\beta} V_\mu V_\tau E_{\alpha\sigma} \Theta_{\nu\beta} + \dot{V}_\alpha H_\beta{}^{(\lambda} \eta^{\varepsilon)\sigma\alpha\beta} V_\alpha -$$

$$- \frac{1}{2} H_\beta{}^\mu{}_{;\alpha} h_\mu{}^{(\varepsilon} \eta^{\lambda)\sigma\alpha\beta} V_\sigma = \frac{1}{6} h^{\varepsilon\lambda} (q^\mu{}_{;\mu} - q^\mu \dot{V}_\mu -$$

$$- \Pi^{\mu\nu} \sigma_{\mu\nu}) - \frac{1}{2}(\rho + p) \sigma^{\varepsilon\lambda} + \frac{1}{2} q^{(\varepsilon} \dot{V}^{\lambda)} -$$

$$- \frac{1}{4} h^{\mu(\varepsilon} h^{\lambda)\alpha} q_{\mu;\alpha} + \frac{1}{2} h_\alpha{}^\varepsilon h_\mu{}^\lambda \dot{\Pi}^{\alpha\mu} +$$

$$+ \frac{1}{4} \Pi_\beta{}^{(\varepsilon} \sigma^{\lambda)\beta} + \frac{1}{4} \Pi_\beta{}^{(\varepsilon} \omega^{\lambda)\beta} + \frac{1}{6} \Theta \Pi^{\varepsilon\lambda} \qquad (15d)$$

a dot means derivative in the direction of the four-velocity V^μ.

We further have from the contracted Bianchi identities, via Einstein's equations, the conservation laws:

$$\dot{\rho} + (\rho+p)\Theta + \dot{q}^\mu V_\mu + q^\alpha{}_{;\alpha} - \Pi^{\mu\nu} \Theta_{\mu\nu} = 0 \qquad (16a)$$

$$(\rho+p)\dot{V}_\alpha - p_{,\mu} h^\mu{}_\alpha + \dot{q}_\mu h^\mu{}_\alpha + \Theta q_\alpha + q^\nu \Theta_{\nu\alpha} +$$

$$+ q^\nu \omega_{\alpha\mu} + \Pi_\alpha{}^\nu{}_{;\nu} + \Pi^{\mu\nu} \Theta_{\mu\nu} V_\alpha = 0 \qquad (16b)$$

in which $\Theta^{\mu\nu} \equiv \sigma^{\mu\nu} + \frac{\Theta}{3} h^{\mu\nu}$.

Fron the definition of the Riemann curvature tensor

$$V_{\mu;\alpha;\beta} - V_{\mu;\beta;\alpha} = R_\mu{}^\varepsilon{}_{\alpha\beta} V_\varepsilon \qquad (17)$$

we obtain the equation of motion of the Kinematical quantities:

$$\dot{\Theta} + \frac{\Theta^2}{3} + 2\sigma^2 - 2\omega^2 - a^{\alpha}{}_{;\alpha} = R_{\mu\nu}\, V^{\mu}\, V^{\nu} \tag{18}$$

$$h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, \dot{\sigma}_{\mu\nu} + \frac{1}{3}\, h_{\alpha\beta}\left[-\omega^2 - 2\sigma^2 + a^{\lambda}{}_{;\lambda}\right] + a_{\alpha}\, a_{\beta} -$$

$$- \frac{1}{2}\, h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, (a_{\mu;\nu} + a_{\nu;\mu}) + \frac{2}{3}\, \Theta\, \sigma_{\alpha\beta} + \sigma_{\alpha\mu}\, \sigma^{\mu}{}_{\beta} - \omega_{\alpha}\omega_{\beta} =$$

$$= R_{\alpha\varepsilon\beta\nu}\, V^{\varepsilon}\, V^{\nu} - \frac{1}{3} R_{\mu\nu}\, V^{\mu}\, V^{\nu}\, h_{\alpha\beta} \tag{19}$$

$$h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, \dot{\omega}_{\mu\nu} - \frac{1}{2}\, h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu} (a_{\mu;\nu} - a_{\nu;\mu}) + \frac{2}{3}\Theta\, \omega_{\alpha\beta} +$$

$$+ \sigma_{\alpha\mu}\, \omega^{\mu}{}_{\beta} - \sigma_{\beta\mu}\, \omega^{\mu}{}_{\alpha} = 0 \tag{20}$$

One further obtains from (17) three constraint equations (not envolving any dot derivative):

$$\frac{2}{3}\, \Theta_{,\mu}\, h^{\mu}{}_{\lambda} - (\sigma^{\alpha}{}_{\gamma} + \omega^{\alpha}{}_{\gamma})_{;\alpha}\, h^{\gamma}{}_{\lambda} - a^{\alpha}(\sigma_{\gamma\alpha} + \omega_{\gamma\alpha}) = R_{\mu\nu}\, V^{\mu}\, h^{\alpha}{}_{\lambda} \tag{21}$$

$$\omega^{\alpha}{}_{;\alpha} + 2\omega^{\alpha}\, \dot{V}_{\alpha} = 0 \tag{22}$$

$$\frac{1}{2}\left[\sigma_{\beta(\tau} - \omega_{\beta(\tau}\right]_{;\gamma}\, \eta_{\sigma)}{}^{\gamma\beta\varepsilon}\, V_{\varepsilon} + \dot{V}_{(\tau}\, \omega_{\sigma)} = H_{\tau\sigma} \tag{23}$$

Equations (15)-(20) together with (21)-(23) are, for given initial date on a space-like hypersurface equivalent to the original Einstein's equations (2). They show in a physically understandable manner how specific properties of the matter creates the gravitational fields $E_{\mu\nu}$ and $H_{\mu\nu}$, and how expansion, shear and vorticity are propagated along the flow lines. The constitutive relations for $\Pi_{\mu\nu}$ and q_μ must be taken for each case from the physics of the system.

2. THE PERTURBED EQUATIONS

The above set of equations will be used to describe the evolution of small disturbances in Friedmann's background. The metric will be written in the standard from:

$$ds^2 = dt^2 - A^2(t)\ \{d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\} \qquad (24)$$

in which $\sigma(\chi)$ may take the values $\sin\chi$, χ or $\sinh\chi$.

Let us first consider equations (15a,b,c,d). Since $E = H = \omega = \sigma = \dot{V} = 0$ are all zero we do not write a δ in front. The only non vanishing quantities of the background are ρ, p and θ. For these we use $\delta\rho$, δp and $\delta\theta$ for the perturbed quantities: $\rho = \rho_0 + \delta\rho$. We obtain after straightforward manipulations

$$\dot{E}^{\mu\nu}\, h_\mu{}^\rho\, h_\nu{}^\lambda + \Theta E^{\rho\lambda} - \frac{1}{2} E_\nu{}^{(\rho}\, h^{\lambda)}{}_\mu\, V^{\mu;\nu} -$$

$$- \frac{\Theta}{3}\, \eta^{\lambda\nu\mu\varepsilon}\, \eta^{\rho\tau\alpha\beta}\, V_\mu V_\tau E_{\varepsilon\alpha} h_{\beta\nu} -$$

$$- \frac{1}{2} H_\beta{}^\mu{}_{;\lambda}\, h_\mu{}^{(\rho}\, \eta^{\lambda)\tau\alpha\beta}\, V_\tau = -\frac{1}{2}(\rho + p)\,\delta\sigma^{\rho\lambda} \qquad (25)$$

$$\dot{H}^{\mu\nu} h_\mu{}^\rho h_\nu{}^\lambda + \Theta H^{\rho\lambda} - \frac{1}{2} H_\nu{}^{(\rho}\, h^{\lambda)}{}_\mu\, V^{\mu;\nu} -$$

$$- \frac{\Theta}{3}\, \eta^{\lambda\nu\mu\varepsilon}\, \eta^{\rho\tau\alpha\beta}\, V_\mu V_\tau H_{\varepsilon\alpha} h_{\beta\nu} + \frac{1}{2} E_\beta{}^\mu{}_{;\lambda}\, h_\mu{}^{(\rho}\, \eta^{\lambda)\tau\alpha\beta}\, V_\alpha = 0$$

$$(26)$$

$$H_{\alpha\mu;\nu}\, h^{\alpha\varepsilon}\, h^{\mu\nu} = (\rho+p)\,\delta\omega^{\varepsilon} \tag{27}$$

$$E_{\alpha\mu;\nu}\, h^{\alpha\varepsilon}\, h^{\mu\nu} = \frac{1}{3}\,\delta\rho_{,\alpha}\, h^{\alpha\varepsilon} - \frac{1}{3}\,\dot{\rho}\,\delta V^{\beta} \tag{28}$$

Similarly we obtain from (18), (19) and (20):

$$(\delta\Theta)^{\bullet} + \frac{2}{3}\,\Theta\delta\Theta - \dot{V}^{\alpha}{}_{;\alpha} = -\,\frac{(1+3\lambda)}{2}\,\delta\rho \tag{29}$$

$$\dot{\sigma}_{\mu\nu} + \frac{1}{3}\, g_{\mu\nu}\, \dot{V}^{\alpha}{}_{;\alpha} - \frac{1}{2}\,\dot{V}_{(\mu;\nu)} + \frac{2}{3}\,\Theta\,\sigma_{\mu\nu} = -E_{\mu\nu} \tag{30}$$

$$\dot{\omega}^{\mu} + \frac{2}{3}\,\Theta\,\omega^{\mu} = \frac{1}{2}\,\eta^{\alpha\mu\rho\sigma}\,\dot{V}_{\rho;\sigma}\, V_{\alpha} \tag{31}$$

The constraint equations give:

$$\frac{2}{3}\,\delta\Theta_{,\mu} - \frac{2}{3}\,\dot{\Theta}\delta V_{\mu} - (\sigma^{\alpha}{}_{\rho} + \omega^{\alpha}{}_{\rho})_{;\alpha}\, h^{\rho}{}_{\mu} = 0 \tag{32}$$

$$\omega^{\alpha}{}_{;\alpha} = 0 \tag{33}$$

$$H_{\mu\nu} = -\,\frac{1}{2}\, h^{\alpha}{}_{(\mu}\, h^{\beta}{}_{\nu)}\, (\sigma_{\alpha\rho;\lambda} + \omega_{\alpha\rho;\lambda})\, \eta_{\beta}{}^{\varepsilon\rho\lambda}\, V_{\varepsilon} \tag{34}$$

Finally equations (16) yields:

$$(\delta\rho)^{\bullet} + \Theta(\delta\rho+\delta p) + (\rho+p)\,\delta\Theta + q^{\alpha}{}_{;\alpha} = 0 \tag{35}$$

$$\dot{p}\,\delta V_{\mu} - \delta p_{,\beta}\, h^{\beta}{}_{\mu} + (\rho+p)\dot{V}_{\mu} + h_{\mu\alpha}\,\dot{q}^{\alpha} + \frac{4}{3}\Theta\, q_{\mu} + h_{\mu\alpha}\,\Pi^{\alpha\beta}{}_{;\beta} = 0 \tag{36}$$

The method to solve this apparently rather involved set of equations goes back to the work of Lifshitz (1946), see Lifshitz and Khalatnikov (1963). Our method here is a variant of his. There are three different types of perturbations: scalar, vector and tensor.

SCALAR PERTURBATIONS

In the case of scalar perturbations we make the ansatz:

$$\begin{aligned}
\delta\Theta &= R(t)\; Q \\
\delta\rho &= N(t)\; Q \\
\delta V_\mu &= V(t)\; h_\mu{}^\nu\; Q_{,\nu} \\
E_{\mu\nu} &= E(t)\; \hat{P}_{\mu\nu} \\
H_{\mu\nu} &= H(t)\; \hat{P}_{\mu\nu} \\
\sigma_{\mu\nu} &= \Sigma(t)\; \hat{P}_{\mu\nu} \\
\Pi_{\mu\nu} &= \Pi(t)\; \hat{P}_{\mu\nu} \\
q_\mu &= q(t)\; h_\mu{}^\nu\; Q_{,\nu} \\
\omega_\mu &= 0
\end{aligned} \tag{37}$$

Inserting into the above equations we find that Q has to obey

$$h^{\mu\nu}\, \hat{\nabla}_\mu\, \hat{\nabla}_\nu\, Q = \frac{K^2}{A^2}\, Q \tag{38}$$

$$\text{where}\quad K^2 = \begin{cases} K^2 \geq 0 & \text{for the flat universe} \\ n^2-1\;,\; n=1,2,\ldots & \text{for the closed universe} \\ q^2+1 \quad \text{for } q^2 \geq 0 & \text{for the open universe} \end{cases}$$

Here $$\hat{P}_{\mu\nu} = \frac{1}{K^2}\, h_\mu{}^\alpha\, h_\nu{}^\beta\, \hat{\nabla}_\alpha\, \hat{\nabla}_\beta\, Q - \frac{1}{3A^2}\, h_{\mu\nu}\, Q \tag{39}$$

The operator $\hat{\nabla}_\mu$ is the co-variant gradient projected in the rest-space orthogonal to V^μ. We have for an arbitrary vector f^μ

$$\hat{\nabla}_\mu f^\alpha = h_\mu{}^\lambda h^\alpha{}_\sigma \nabla_\lambda f^\sigma \equiv h_\mu{}^\lambda h^\alpha{}_\sigma f^\sigma{}_{;\lambda} \qquad (40)$$

Using this ansatz into equations of perturbation of $\delta\rho$ and $\delta\theta$ we obtain an equation for the density of contrast $\mu \equiv \frac{\delta\rho}{\rho}$:

$$\frac{1}{1+\lambda}(A^2 \dot{\mu})^{\cdot\cdot} - \frac{(1+3\lambda)}{2}(A^2\rho\mu)^{\cdot} + \lambda\theta\left[-\frac{(A^2\dot{\mu})^{\cdot}}{(1+\lambda)} + \frac{(1+3\lambda)}{2}A^2\rho\mu\right] +$$

$$+ \frac{\dot{\theta}}{\theta}\left[\frac{(1+3\lambda)}{2}A^2\rho\mu - \frac{(A^2\dot{\mu})^{\cdot}}{(1+\lambda)}\right] + \frac{\lambda}{(1+\lambda)}k^2\left[\dot{\mu} - \frac{\dot{\theta}}{\theta}\mu\right] = 0 \qquad (41)$$

It can be shown that one integral of this equation is

$$\mu_s = -(1+\lambda)R_o\,\theta \qquad (42)$$

in conformity with Lifshitz. This solution is fictitious and can be eliminated by the gauge transformation

$$t \to \tilde{t} = t + R_o$$

Reducing the order of the differential equation by the ansatz $\mu = \mu_s F$ we obtain, with $M = \dot{F}$

$$\theta\ddot{M} + \left[\left(\frac{2}{3} - \lambda\right)\theta^2 - (1+3\lambda)\rho\right]\dot{M} + (1+3\lambda)\left[2\lambda\theta^2 - \frac{\rho}{2}(1+3\lambda)\right]\frac{\rho}{\theta}M +$$

$$+ \lambda\frac{K^2}{A^2}\theta M = 0 \qquad (43)$$

with the general solution

$$\mu = \mu_{(1)}\,t^{2/3} + \mu_{(2)}\,t^{-1}, \text{ for euclidean section}$$

$$\mu = \frac{3\ \sin\eta}{A_o(1-\cos\eta)^2}\ \left[(\sin\eta - 3\eta) f_o + f_1\right] + \frac{3 f_o}{A_o(1-\cos\eta)}$$

for closed section

(η is defined by $dt = A(\eta) d\eta$)

$$\mu = f_o\ \sinh^{-2}\ (\frac{\eta}{2})\ (1 - \frac{\eta}{2}\ \coth\ \frac{\eta}{2}) + 3\ f_b$$

for open section

in case of dust; and in case of radiation;

$$\mu = 2\alpha\left[\frac{\sin\phi}{\phi} + (\frac{1}{\phi^2} - \frac{1}{2})\cos\phi\right] + 2\beta\left[\frac{\cos\phi}{\phi} - (\frac{1}{\phi^2} - \frac{1}{2})\ \sin\phi\right]$$

for euclidean section

in which $\phi \equiv \frac{2}{\sqrt{3A_o}}\ K\ t^{1/2}$;

for the closed section:

$$\mu = \theta \int J\ A(\eta) d\eta$$

with

$$J(\eta) = \eta^{3/2}\{c_1\ J_{1/2}(q\eta) + c_2\ Y_{1/2}(q\eta)\}$$

in which $q^2 = \frac{K^2}{3}$ and $J_{1/2}$ and $Y_{1/2}$ are Bessel's functions.

We can obtain the result for the open section by the formal map $A \to iA$, $\eta \to i\eta$ and $k \to iK$. Just for the completeness we list below the remaining set of perturbation equations

$$E = \frac{K^2 A^2}{2(3\epsilon + K^2)}\ \left[N - \dot{\rho} + \theta q + \frac{3\epsilon + K^2}{A^2}\ \Pi\right]$$

$$H = 0$$

$$\dot{R} - \frac{K^2 \dot{V}}{A^2} + \frac{2}{3}\theta R = -\frac{1}{2}(1+3\lambda)N$$

$$\dot{\Sigma} - K^2 \dot{V} = -E - \frac{\Pi}{2}$$

$$\frac{2}{3}(3\epsilon + K^2)\frac{\Sigma}{A^2} - 2K^2\left[\left(\frac{\varepsilon}{A^2} + \frac{1+\lambda}{2}\right)V - \frac{2}{3}R\right] = K^2 q$$

$$\dot{N} + \theta(1+\lambda)N + (1+\lambda)\rho R + \frac{K^2 q}{A^2} = 0$$

$$-\lambda\left[N - \dot{\rho}V\right] + (1+\lambda)\rho\dot{V} + \left[\dot{q} + \theta q + \frac{2}{3A^2}\left(\frac{3\epsilon + K^2}{K^2}\right)\right]\Pi = 0$$

In case of perfect fluid these equations reduce to the ones investigated by Lifshitz et al previously.

VECTOR PERTURBATIONS

We set

$$\delta V_\mu = V(t)\ \hat{S}_\mu$$

$$q_\mu = q(t)\ \hat{S}_\mu \tag{44}$$

$$\Pi_{\mu\nu} = \Pi(t)\ \hat{\Sigma}_{\mu\nu}$$

$\hat{S}_\mu$ is defined in the 3-dimensional rest space orthogonal to V^μ by means of the projector operator $h_\mu{}^\nu$. That is

$$V^\mu\ \hat{S}_\mu = 0$$

$$h_\mu{}^\nu\ \hat{S}_\nu = \hat{S}_\mu$$

$\hat{S}_\mu$ is an eigen-vector of the Laplace operator

$$h^{\mu\nu}\ \hat{\nabla}_\mu\ \hat{\nabla}_\nu\ \hat{S}_\alpha = \frac{K^2}{A^2}\ \hat{S}_\alpha \tag{45}$$

(K is an integer). It is stationary, that is

$$\hat{S}_{\alpha;\mu} V^{\mu} = 0 \tag{46}$$

and divergence-free:

$$\hat{\nabla}_{\alpha} \hat{S}^{\alpha} = 0 \tag{47}$$

From this vector we can construct tensors $\hat{\Sigma}_{\alpha\beta}$ and $\hat{F}_{\alpha\beta}$ by taking the symmetric and the antisymmetric parts of the derivative:

$$\hat{\Sigma}_{\mu\nu} = \hat{\nabla}_{(\mu} \hat{S}_{\nu)} \tag{48}$$

$$\hat{F}_{\mu\nu} = \hat{\nabla}_{[\mu} \hat{S}_{\nu]} \tag{49}$$

Besides, we can define another associated quantity* $\hat{S}_{\mu}$ and $*\hat{\Sigma}_{\mu\nu}$ which will be very useful in dealing with the perturbation equations. We set

$$*\hat{S}^{\mu} = \eta^{\mu\varepsilon\beta\lambda} \hat{S}_{\beta;\lambda} V_{\varepsilon} \tag{50}$$

Using the properties (45), (46) and (47) we can show the additional relations:

$$h^{\mu\nu} \hat{\nabla}_{\nu} \hat{\Sigma}_{\mu\rho} = \frac{2\epsilon + K^2}{A^2} \hat{S}_{\rho} \tag{51}$$

$$\hat{\Sigma}_{\mu\nu;\lambda} V^{\lambda} + \frac{\theta}{3} \hat{\Sigma}_{\mu\nu} = 0 \tag{52}$$

$$h^{\mu}_{(\rho}\ h^{\nu}_{\ \alpha)}\ \eta_{\mu}^{\ \beta\gamma\lambda}\ V_{\lambda}\hat{\nabla}_{\gamma}\hat{\Sigma}_{\nu\beta} = h^{\mu}_{(\rho}\ h_{\alpha)}^{\ \ \nu}\ \hat{\nabla}_{\nu}{}^{*}\hat{S}_{\mu} \tag{53}$$

$$h^{\mu}_{(\rho}\ h^{\nu}_{\ \alpha)}\ \eta_{\mu}^{\ \beta\gamma\lambda}\ V_{\lambda}\hat{\nabla}_{\gamma}\hat{F}_{\nu\beta} = -h^{\nu}_{(\rho}\ h^{\mu}_{\ \alpha)}\ \hat{\nabla}_{\nu}{}^{*}\hat{S}_{\mu} \tag{54}$$

$$h^{\alpha}_{\ \rho}h^{\gamma\nu}\ \hat{\nabla}_{\nu}\hat{\nabla}_{\gamma}{}^{*}\hat{S}_{\alpha} = \frac{K^2}{A^2}\ {}^{*}\hat{S}_{\rho} \tag{55}$$

$$h_{(\alpha}^{\ \ \mu}\ h_{\beta)}^{\ \ \nu}(\hat{\nabla}_{\mu}{}^{*}\hat{S}_{\nu})^{\bullet} + \frac{2}{3}\theta\ h_{(\alpha}^{\ \ \mu}\ h_{\beta)}^{\ \ \nu}\ \hat{\nabla}_{\mu}{}^{*}\hat{S}_{\nu} = 0 \tag{56}$$

$$h_{(\alpha}^{\ \ \mu}\ h_{\beta)}^{\ \ \nu}\ (\hat{\nabla}_{\mu}\ \hat{S}_{\nu})^{\bullet} + \frac{\theta}{3}\ h_{(\alpha}^{\ \ \mu}\ h_{\beta)}^{\ \ \nu}\ \hat{\nabla}_{\mu}\ \hat{S}_{\nu} = 0 \tag{57}$$

$$h^{\mu(\rho}\eta^{\sigma)\lambda\nu\alpha}\ V_{\lambda}\ \hat{\nabla}_{\nu}{}^{*}\Sigma_{\alpha\mu} = (2\epsilon-K^2)h^{\mu(\sigma}\ h^{\rho)\nu}\ \hat{S}_{\nu;\mu} \tag{58}$$

in which ϵ is defined in terms of the 3-scalar of curvature ${}^{(3)}R$ by

$$\hat{R}_{\alpha\beta\mu\nu} = -\frac{\epsilon}{A^2}\ h_{\alpha\beta\mu\nu} \tag{59}$$

$$h_{\alpha\beta\mu\nu} \equiv h_{\alpha\mu}\ h_{\beta\nu} - h_{\alpha\nu}\ h_{\beta\mu}$$

From the definition of the vorticity $\omega_{\mu\nu} = \frac{1}{2}\ \delta V_{[\mu,\nu]}$ and (44)

$$\omega_{\mu\nu} = \frac{1}{2}\ V(t)\ \hat{F}_{\mu\nu} \tag{60}$$

For the shear and the electric tensor $E_{\mu\nu}$ we set

$$\sigma_{\mu\nu} = L(t)\ \hat{\Sigma}_{\mu\nu} \tag{61}$$

$$E_{\mu\nu} = E(t)\ \hat{\Sigma}_{\mu\nu} \tag{62}$$

The acceleration $\dot{V}^{\mu}$ is given by:

$$\dot{V}^{\mu} = (\delta V^{\mu})^{\bullet} + \frac{\theta}{3}\,\delta V^{\mu}$$

$$= (\dot{V} + \frac{\theta}{3} V)\ \hat{S}_{\mu} \tag{63}$$

Using this value into equation (36) yields :

$$\dot{V} + \frac{(1-3\lambda)}{3}\,\theta V + \frac{1}{1+\lambda}\,\frac{1}{\rho}\left[\dot{q} + \frac{4}{3}\,\theta\, q + \frac{2\epsilon+K^2}{A^2}\,\Pi\right] = 0 \tag{64}$$

Now, Raychandhuri equation (29) gives

$$\dot{\theta} = -3\left[\frac{\epsilon}{A^2} + \frac{1+\lambda}{2}\,\rho\right] \tag{65}$$

Using this value into equation (32):

$$L = \frac{A^2}{2\epsilon+k^2}\,(q+(1+\lambda)\rho V) + \frac{V}{2} \tag{66}$$

From (28):

$$E = \frac{1}{3(2\epsilon+k^2)}\,A^2\,\theta\left[(1+\lambda)\rho V + q\right] + \frac{\Pi}{2} \tag{67}$$

Let us now turn to the calculus of $H_{\mu\nu}$. Equation (34) gives directly

$$H_{\alpha\beta} = -\frac{1}{2}(L - \frac{V}{2})\ h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ \hat{\nabla}_{\nu}\ {}^{*}\hat{S}_{\mu} \tag{68}$$

This induce us to define the expansion

$$H_{\alpha\beta} = H(t)\,{}^{*}\hat{\Sigma}_{\alpha\beta}$$

and consequently

$$H(t) = - \frac{A^2}{2(2\epsilon+k^2)} [(1+\lambda)\rho V + q] \tag{69}$$

Discussion of the results.

It seems worth to remark that although we used only from equations (Raychanduri equation, the conservation of the energy-momentum tensor, the constraint relation which connects $H_{\mu\nu}$ with spatial derivatives of shear and vorticity; and the equation of the divergence of $E_{\mu\nu}$) we have solved our problem of finding the evolution of vortex perturbation. All others equations are identicaly satisfied. This can be shown by a rather long but straightforward substitution of the values we obtained for E, L and H into the remaining equations(*).

These results are generalizations of results obtained by Lifshitz for nonvanishing stress and heat flux and agree with Lifshitz' result in case the later are zero. Let us remark that in order to compare our analysis with those of Lifshitz we have to express our results in his notation. In other words, we have to find explicitly $\delta g_{\mu\nu}$. This is not dificult and may be obtained just by looking at the expression which defines the shear. We have

$$\sigma_{\mu\nu} = \frac{1}{2} h_\mu{}^\alpha h_\nu{}^\beta \delta V_{(\alpha;\beta)} - h_\mu{}^\alpha h_\nu{}^\beta (\delta\Gamma^\rho{}_{\alpha\beta}) V_\rho - \frac{1}{3} h_{\mu\nu}\delta\theta - \frac{1}{3}\theta\delta g_{\mu\nu} \tag{70}$$

(*) J.M. Salim - PhD Thesis (1982) unpublished

After this we can easily obtain $\delta g_{\mu\nu}$.

Thus, it remains only - to exhaust our analysis - to integrate eq.(64). This can be made after the constitutive relations of the stress and the heat flux is given. Let us examine two particular cases.

Case i: Perfect Fluid

We set $q = \Pi = 0$

Integrating (64) gives

$$V = V_o A^{3\lambda-1} \qquad (V_o \text{ is a constant}) \tag{71}$$

and consequently

$$E = \frac{1+\lambda}{3(2\epsilon+K^2)} \rho_0 V_o \theta A^{-2} \tag{72}$$

$$H = -\frac{2}{3}\frac{E}{\theta} \tag{73}$$

$$L = \frac{1+\lambda}{2\epsilon+k^2} \rho_0 V_o + \frac{1}{2} V_o A^{3\lambda-1} \tag{74}$$

(we used $\rho = \rho_0 A^{-3\lambda-3}$)

Remark that for a typical expansion factor ($\theta \sim t^{-1}$), in later eras all gravitational energy (proportional to $E^2 + H^2$) becomes "magnetic" (E/H goes to zero as $t \to \infty$), a result which depends on the presence of vorticity in the perturbation.

In the case of radiation ($\lambda = 1/3$) the velocity perturbation and the corresponding shear and vorticity are constant, a result which was known since Lifshitz paper. Remark however that for stiff matter, in case $\frac{1}{3} < \lambda < 1$ the vorticity (and the shear) increases as time goes on, once in the standard Friedmann background A(t) is a monotonic function.

Case ii: Stokes (linear) Fluid

We set $q = 0$, $\Pi_{\mu\nu} = n\,\sigma_{\mu\nu}$. In this case equation (64) gives

$$V = V_0 \exp\left[-nt - \frac{3}{8}\,\frac{n}{\rho_0}(2\epsilon + k^2)\int A^2\,dt\right] \qquad (75)$$

(V_0 is a constant).

In general, for a Stokes (linear) fluid the constant α is restricted to be positive. This guarantees, by the ad hoc use of the second law of thermodynamics that entropy can only increase in the direction of the arrow of time (defined by the expansion of the universe). However in the case $\Pi_{\mu\nu}$ is just a small quantity and we restrict perturbations to be linear, equation (35) implies that such condition on n can be relaxed, once the contribution of the anisotropic energy to variation of entropy is a second order effect, which can be neglected. Thus the instability of this perturbation, which occurs in case of negative n, is not forbidden.

Let us make one more remark on the general features of vortex perturbation. Suppose we intend to consider a pure electric perturbation by setting $H = 0$. The equations of motion imply that then $\Pi = E = 0$. That is, the perturbed geometry is again conformally flat. This is possible only if there is a heat flux such that $q = -(1+\lambda)\rho V$ in which case the shear is given by $L = \frac{V}{2}$. This shows that we can prescribe arbitrarily function V(t). We have just to proceed as follows. Take L(t) as a given function of the distortion of the cosmic fluid (either by theory or by any kind of observation).

Then, obtain V(t). From this evaluate the amount of heat which is necessary to satisfy the complete system of perturbed equations. Note that this result is completely independent of the wavelength of the perturbation. The unique remaining task is to justify, through a physical model the presence of such heat. This, of course, has to be examined for each case individually.

TENSOR PERTURBATIONS

We set

$$\begin{aligned} \delta\theta &= 0 \\ \delta\rho &= 0 \\ \delta V_\mu &= 0 \\ E_{\mu\nu} &= E(t)\hat{U}_{\mu\nu} \\ \sigma_{\mu\nu} &= L(t)\hat{U}_{\mu\nu} \\ \pi_{\mu\nu} &= \Pi(t)\hat{U}_{\mu\nu} \end{aligned} \tag{76}$$

in which $\hat{U}_{\mu\nu}$ satisfy the following conditions:

$$h^{\alpha\beta}\,\hat{\nabla}_\alpha\,\hat{\nabla}_\beta\,\hat{U}_{\mu\nu} = \frac{k^2}{A^2}\,\hat{U}_{\mu\nu} \tag{77}$$

and the constant k has the corresponding spectrum:

$0 < |k| < \infty$ for the euclidean section

$k^2 = q^2+3$ $0 < q < \infty$ for the open section

$k^2 = n^2-3$, $n = 3,4\ldots$ for the closed section.

$$\hat{U}_{\mu\nu;\alpha}\,V^\alpha = 0 \tag{78}$$

$$h^{\mu\nu} \hat{\nabla}_{\alpha} \hat{U}_{\mu\nu} = 0 \tag{79}$$

$$h^{\mu\nu} \hat{U}_{\mu\nu} = 0 \tag{80}$$

We remark that due to the tensorial character of the kinematical quantities the unique survival component of the perturbation of the fluid is the shear $\sigma_{\mu\nu}$. It affects the evolution of $E_{\mu\nu}$ and $H_{\mu\nu}$ by equations (30), (34) and the quasi-Maxwellian set.

From equation (34) we obtain

$$H_{\mu\nu} = \frac{1}{2} L \; h^{\alpha}{}_{(\mu} \; h^{\beta}{}_{\nu)} \; \eta_{\beta}{}^{\varepsilon\lambda\rho} \; V_{\varepsilon} \; \hat{\nabla}_{\lambda} \; \hat{U}_{\rho\alpha} \tag{81}$$

This induce us to define the star-operator $^{*}U_{\mu\nu}$:

$$^{*}\hat{U}_{\mu\nu} = P_{\mu\nu} [\hat{U}] \tag{82}$$

Then

$$H_{\mu\nu} = L^{*}\hat{U}_{\mu\nu} \tag{83}$$

The dynamics of the complete system reduces to the set:

$$\dot{E} + \theta E - \left[\frac{1}{2}(\rho - p) - \frac{1}{3} \theta^2 + \frac{k^2}{A^2} \right] L = \frac{1}{2} \dot{\Pi} + \frac{1}{6} \theta \Pi \tag{84}$$

$$\dot{L} + \frac{2}{3} \theta L + E = - \frac{1}{2} \Pi \tag{85}$$

This simple form of the equations of evolution of the perturbation is a remarkable consequence of decomposition (76) and of the properties of $^{*}\hat{U}_{\mu\nu}$. This set of equation (84,85) is very similar to the propagation of electromagnetic waves in a Friedmann background. The term of stress (rhs of equations (84) and (85)) are related to the current form-vector j^{μ} of the charged particles by a simple relation.

The exam of the set (84) and (85) can be simplified if we note that it corresponds to a dynamical set of the standard form

$$\dot{E} = F_1(L,E,t)$$
$$\dot{L} = F_2(L,E,t) \qquad (86)$$

However the fact that this system is not autonomous (the explicit dependence on t can be eliminated only in very simple cases) difficults such examination.

In case the electric and the magnetic modes are not excited simultaneously the analysis is greatly simplified. Indeed, set $L = H = 0$ to reduce (86) to:

$$2E = -\Pi = E_0 A^{-2} \qquad (88)$$

which implies that the gravitational energy goes like A^{-4}, as it should be.

In the other simple case $E = 0$, the system reduces to

$$\dot{L} + \frac{2}{3}\,\theta L = -\,\frac{\Pi}{2} \qquad (89)$$

$$\left[\frac{1}{2}(\rho - p) - \frac{\theta^2}{3} + \frac{K^2}{A^2}\right]L = -\,\frac{1}{2}\,\dot{\Pi} - \frac{1}{6}\,\theta\Pi \qquad (90)$$

From (89) and (90) we obtain an equation for L:

$$\ddot{L} + \dot{L}\theta + L\left[- \frac{5}{6}\,\rho - \frac{1}{2}\,p + \frac{1}{3}\,\theta^2 - \frac{k^2}{A^2}\right] = 0 \qquad (91)$$

For short wavelength and substituting the known values for ρ, p and θ we can write this equation (91) under the generic form for the euclidean section:

$$\frac{d^2L}{dt^2} + f_1(t)\,\frac{dL}{dt} + f_2(t)L = 0 \qquad (92)$$

$$\text{with } f_1(t) = \frac{a_o}{t} \quad \text{and } f_2(t) = \frac{b_o}{t^2}$$

for f_o and q_o being constants. Equation (92) for different functions of p, ρ, θ and A can be solved by means of confluent hypergeometric functions [Morse-Feshback]. Coming back to the general case, let us divide eq. (84) by L and define a new variable $\phi \equiv \frac{E}{L}$. Then, eq. (84) reduces to a non-linear equation for ϕ. From this we can evaluate the shear L by means of the integral

$$L = \exp - \int \left(\frac{2}{3}\,\theta + \frac{q}{2} + \phi\right)\, dt \qquad (93)$$

in which we have used the constitutive relation $\Pi_{\mu\nu} = q\,\sigma_{\mu\nu}$ (Klimek).

The equation for ϕ takes the form

$$\dot{\phi} - \phi^2 + \frac{\dot{A}}{A}\,\phi = M(t) - \frac{1}{2}q\,\frac{\dot{A}}{A} - \frac{q^2}{4} \qquad (94)$$

in which $M(t) \equiv \frac{1}{2}(\rho - p) - \frac{1}{3}\theta^2 + \frac{K^2}{A^2}$.

Remark that ϕ being a ratio of two small quantities, it is not necessarily small. Thus we cannot neglect the quadratic term ϕ^2 which makes this treatment of the perturbation to depend on a non-linear equation. Fortunately, in some cases of interest this difficulty can be overtaken. For instance in cases of large wavelenghts, that is, for $\frac{K}{A} << 1$ we can neglect the term $(\frac{K}{A})^2$ in expression for M(t). In this case $\phi \sim t^{-1}$ and also $L \sim t^{-1}$. We encounter here the same assymptotic situation we faced before in case of pure vortex perturbation, that is, for large times the gravitational perturbation becomes purely magnetic.

It seems woth to point out a remarkable fact which seems very little if any noticed before, that is: gravitational disturbances are present if and only if the fluid acquires a shear deformation.

REFERENCES

Hawking S.W., 1966, Astrophys. J. 145 544

Olson D.W., 1976, Phys. Rev. D14, 327

Lifshitz E.M., 1946 J. Phys. USSR 10, 116

Lifshitz E.M., Khalatnikov I.M., 1963 Adv. Phys. 12, 185

Bardeen J.M:, 1980, Phys. Rev. D22, 1882

Jordan P., Ehlers J., Kundt W., 1960, Abh. Wiss and Lit. Mainz Mat. Nat. Kl nº2

Kundt W. Trümper M., 1962, nº12

Trümper M., 1967, Zeits. Astrophys. 66, 215

Ellis G.F.R., 1971, Proceedings of the International School of Physics E. Fermi, course XLVII, pp104-182, Academic Press, N.Y.

Novello, M., Salim J.M., To appear in Fundamentals of Cosmic Physics

Harrison E.R., Rev. Mod. Phys. 39 (1967) 862

Peebles P.J.E., Cosmology: The Physics of large scale Structure (Princeton Univ. Press, Princeton 1980)

Morse P.M. and Feshback H., Methods of Theoretical Physics (1953) McGraw Hill Book Company Inc.

Klimek Z., Acta Astronomica, 25, 79 (1975)