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The Bekenstein Bound in Strongly Coupled $O(N)$ Scalar Field Theory

T. Santos Magalhães[1] *and N. F. Svaiter*[2]

Centro Brasileiro de Pesquisas Físicas-CBPF
Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil

G. Menezes[3]

Instituto de Física Teórica, Universidade Estadual Paulista,
Rua Dr. Bento Teobaldo Ferraz 271, Bloco II, Barra Funda,
São Paulo, SP, 01140-070, Brazil

Abstract

We discuss the $O(N)$ self-interacting scalar field theory, in the strong-coupling regime and also in the limit of large N. Considering that the system is in thermal equilibrium with a reservoir at temperature β^{-1}, we assume the presence of macroscopic boundaries confining the field in a hypercube of side L. Using the strong-coupling perturbative expansion, we generalize previous results, i.e., we obtain the renormalized mean energy E and entropy S for the system in first order of the strong-coupling perturbative expansion, presenting an analytical proof that the specific entropy also satisfies in some situations a quantum bound. When considering the low temperature behavior of the specific entropy, the sign of the renormalized zero-point energy can invalidate this quantum bound. If the renormalized zero point-energy is a positive quantity, at intermediate temperatures and in the low temperature limit, there is a quantum bound.

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[1]e-mail: tsantos@cbpf.br
[2]e-mail: nfuxsvai@cbpf.br
[3]e-mail: gms@ift.unesp.br

1 Introduction

Quantum fields in the presence of macroscopic boundaries have received a great deal of attention in the literature. Since boundaries can introduce a characteristic size in the theory, nontrivial phenomena may arise. A well discussed effect is the electromagnetic Casimir effect [1], where neutral and perfectly conducting parallel plates in vacuum attract each other. The introduction of a pair of conducting plates into the vacuum of the electromagnetic field alters the zero-point fluctuations of the field and thereby produces an interaction between the plates [2] [3] [4] [5]. A basic question that has been discussed when quantum fields interact with boundaries, is about the issue that the ratio between global variables which define a macroscopic state, as the mean energy, entropy, etc of the systems may be subjected to certain fundamental bounds.

In black hole thermodynamics, the generalized second law of thermodynamics states that the sum of the black-hole entropy and the entropy of the matter outside the black-hole does not decreases. Motivated by considerations of gravitational entropy, Bekenstein proposed a bound that relates the entropy S and the energy E of the quantum system, respectively, with the size of the boundaries that confine the fields, even in the absence of gravitational fields [6] [7] [8] [9] [10]. It states that $S \leq 2\pi\, E\, R/\hbar\, c$, where R stands for the radius of the smallest sphere that circumscribes the system. Although analytical proofs of this quantum bound on specific entropy for free fields has been proposed in the literature, many authors in the past criticized the bound [11] [12] [13] [14] [15]. Many of these criticisms were answered by Bekenstein and collaborators [16] [17]. An strong argument used in one of these examples is based in the fact that the renormalized zero-point energy of some free quantum field could be negative. Some authors claim that, if we take into account the boundaries responsible for the Casimir energy, it is possible to compensate their negative energy, yielding a positive total energy which respects the Bekenstein bound, although this is far from a simple problem [18].

We may observe that a quite important situation has not been discussed systematically in the literature. A step that remains to be derived is the validity of the bound for the case of interacting fields [19] [20], which are described by non-Gaussian functional integrals, at least up to some order of the perturbation theory. In this paper we generalize previous results [21], showing that for the $O(N)$ self-interaction scalar field theory in the limit of large N, in which situations the specific entropy satisfies a quantum bound. In the strong-coupling regime, using the strong-coupling expansion [22] [23] [24] [25] [26] [27] [28] one can evaluate the mean energy and the canonical entropy of the system, obtaining the validity of the bound for the case of strongly coupled fields. Note that for a very large number of fields, general arguments said that at least in the weak coupling regime, there a critical N_c, such that for $N > N_c$ the bound is violated. This is known as the species problem. Our results show that the species problem does not appear in the strong-coupling regime.

Using the generating functional of complete Schwinger functions $Z[V, h]$, and assuming that the source is constant we can perform the strong-coupling expansion. In the strong-coupling regime, we perform the perturbative expansion around a independent-value generating function, up to the order $(g_0)^{-\frac{2}{p}}$. Up to this order, it is possible to split $\ln Z(V, h)$ in two contributions: one that contains only the independent-value generating function and other that contains the spectral zeta-function. Therefore, in order to obtain the thermodynamic quantities, one must proceed in two stages. First, one gives a operational meaning to the independent-value generating function; then, one consistently implements the boundary conditions in the strong-coupling regime. Since we are working in first order of perturbation theory, to implement boundary conditions, we use the spectral zeta-function method [29] [30] [31] [32]. Quite recently a very simple application of this formalism was presented [33], where it was considered an anharmonic oscillator in thermal equilibrium with a reservoir at temperature β^{-1}.

In this work we study the $O(N)$ $(g_0\,\varphi^p)_d$ self-interacting scalar field theory in the limit of large N, in the strong-coupling regime. We assume the presence of macroscopic boundaries that confine the field in a hypercube of side L and also that the system is in thermal equilibrium with a reservoir. Generalizing previous results, we present an analytic proof that, up to the order $(g_0)^{-\frac{2}{p}}$, the specific entropy satisfies in some situations a quantum bound. Defining $\varepsilon_d^{(r)}$ as the renormalized zero-point energy for the free theory per unit length, $\xi = \frac{\beta}{L}$ and $h_1(d)$ and $h_2(d)$ as positive analytic functions of d, for the case of high temperature, we get that the specific entropy satisfies the inequality $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{h_2(d)}\,\xi$. When considering the low temperature behavior of the specific entropy, we have $\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{\varepsilon_d^{(r)}}\,\xi^{1-d}$. We are establishing a bound for the specific entropy in the $O(N)$ model in the large N limit, describing a strong-coupled system in the following cases: in the high temperature limit and if the renormalized zero point-energy is a positive quantity, at intermediate temperatures and also in the low temperature limit. We would like to point out that the problem of the sign of the renormalized zero-point energy of free fields described by Gaussian functional integrals, which is crucial for the subject that we are interested to investigate in this paper, is still open question in the literature. See for example the Refs. [34] [35] [36] [37].

The organization of the paper is as follows: In section II we discuss the strong-coupling expansion for the $O(N)$ $(g_0\,\varphi^p)_d$ theory. In section III we discuss the free energy and the spectral zeta-function of the system. In section IV we discuss the contribution in $\ln Z$ coming from the independent-value generating function in the large N limit. Finally, section V contains our conclusions. To simplify the calculations we assume the units to be such that $\hbar = c = k_B = 1$.

2 The strong-coupling perturbative expansion for $O(N)$ scalar theory

Let us consider N scalar fields with a $(g_0\,\varphi^p)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time. In the Euclidean field theory, we have the generating functional of complete Schwinger functions. In a d-dimensional Euclidean space, the self-interaction contribution to the action is given by

$$S_I(\varphi) = \int d^d x\,\frac{g_0}{p!}\,\varphi^p(x). \tag{1}$$

Notice that, in that case, the field φ must be regarded as a N-isovector $\vec{\varphi} = (\varphi^{(1)}; \ldots; \varphi^{(N)})$ with $\varphi^p = (\varphi^2)^{p/2}$ and $\varphi^2 = \sum_{i=1}^{N} \varphi^{(i)}\,\varphi^{(i)}$.

The basic idea of the strong-coupling expansion at zero temperature is to treat the Gaussian part of the action as a perturbation with respect to the remaining terms of the action in the functional integral. Let us assume a compact Euclidean space with or without a boundary, where the volume of the Euclidean space is V. Let us suppose that there exists an elliptic and self-adjoint differential operator O acting on scalar functions on the Euclidean space. The usual example is $O = (-\Delta + m_0^2\,)$, where Δ is the d-dimensional Laplacian. The kernel $K(m_0;\, x, y) \equiv\, K(m_0;\, x - y)$ is defined by

$$K(m_0; x - y) = \left(-\Delta + m_0^2\right) \delta^d(x - y). \tag{2}$$

Using the fact that the functional integral which defines $Z(V, \vec{h}\,)$ is invariant with respect to the choice of the quadratic part, let us consider a modification of the strong-coupling expansion. We split the quadratic part in the functional integral which is proportional to the mass squared in two parts; one in the derivative terms of the action, and the other in the independent value generating functional. The Schwinger functional can be defined by a new formal expression for the functional integral given by [38]

$$Z(V, \vec{h}\,) = \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\int d^dx \int d^dy \frac{\delta}{\delta h_i(x)} K(m_0, \sigma; x - y) \frac{\delta}{\delta h_i(y)}\right) Q_0(\sigma, \vec{h}\,), \tag{3}$$

where $Q_0(\sigma, \vec{h}\,)$, the new independent value functional integral, is given by

$$Q_0(\sigma, \vec{h}\,) = \mathcal{N}\int [d\varphi]\, \exp\biggl(\int d^dx \left(-\frac{1}{2}\,\sigma\, m_0^2\, \varphi^2(x) - \frac{g_0}{p\,!}\,\varphi^p(x) + \vec{h}(x)\cdot\vec{\varphi}(x)\right)\biggr). \tag{4}$$

Notice that the external source $\vec{h} = (h^{(1)}\,;\, ...\,;\, h^{(N)})$ is also an isovector. The modified kernel $K(m_0, \sigma; x - y)$ that appears in Eq. (3), is defined by

$$K(m_0, \sigma; x - y) = \left(-\Delta + (1 - \sigma)m_0^2\right) \delta^d(x - y), \tag{5}$$

where σ is a complex parameter defined in the region $0 \leq$ Re $(\sigma) < 1$.

The factor $\mathcal{N}$ is a normalization that can be found using that $Q_0(\sigma, \vec{h}\,)|_{\vec{h}=0} = 1$. Observe that the non-derivative terms which are non-Gaussian in the original action do appear in the functional integral that defines $Q_0(\sigma, \vec{h}\,)$. At this point it is convenient to consider $\vec{h}(x)$ to be complex. Consequently $h_i(x) = \mathrm{Re}(h_i) + i\,\mathrm{Im}(h_i)$. In the paper we are concerned with the case $\mathrm{Re}(h_i) = 0$.

Since we are assuming a spatially bounded system in equilibrium with a thermal reservoir at temperature β^{-1}, the strong-coupling expansion can be used to compute the partition function defined by $Z(\beta, \Omega, \vec{h}\,)|_{\vec{h}=0}$, and we are defining the volume of the $(d-1)$ manifold as $V_{d-1} \equiv \Omega$. From the partition function we define the free energy of the system, given by $F(\beta, \Omega) = -\frac{1}{\beta}\ln\, Z(\beta, \Omega, \vec{h}\,)|_{\vec{h}=0}$. This quantity can be used to derive the mean energy $E(\beta, \Omega)$, defined as

$$E(\beta, \Omega) = -\frac{\partial}{\partial\beta}\ln Z(\beta, \Omega, \vec{h}\,)|_{\vec{h}=0}, \tag{6}$$

and the canonical entropy $S(\beta, \Omega)$ of the system in equilibrium with a reservoir with a finite size given by

$$S(\beta, \Omega) = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z(\beta, \Omega, \vec{h}\,)|_{\vec{h}=0}. \tag{7}$$

In the next section we will show that in the situation that we are interested, it is possible, up to the order $(g_0)^{-\frac{2}{p}}$ to split $\ln Z(\beta, \Omega, \vec{h}\,)$ in two parts: the first one that contains only information given by the independent-value generating function and the second one that has the information on the boundary condition. As we will see, the information from the boundary conditions comes from the imposition on the functional space where we perform the functional integrals. For Gaussian theories the functional integrals can be performed exactly and a powerful technique which gives the logarithm of the product of the eigenvalues of some elliptic operator is given by the derivative of the spectral zeta-function defined in the extended complex plane in $s = 0$. In the next section we will discuss these points carefully.

3 The independent-value generating function and the spectral zeta-function

To obtain a generalization for the Bekenstein bound, it is necessary to obtain global quantities as the entropy and the mean energy. Therefore, for simplicity we assume that the external source $\vec{h}(x)$ is constant. In this situation we call $Z(V, \vec{h})$ as a generating function. Since we are introducing boundaries in the domain where the field is defined, the spectrum of the operator $D = (-\Delta + (1-\sigma)m_0^2)$ has a denumerable contribution. Since the spectrum is unbounded above, divergences will appear. An analytic regularization procedure will be used to control the divergences of the theory. As we discussed, in order to impose Dirichlet boundary conditions over the fields, the functional integral must be taken over functions restricted to the geometric configurations.

We will study the $O(N)$ model in the limit of large N. In that limit, it is possible to write the independent- value generating function as a function of $|h|$. For simplicity, we will denote $|h| \equiv h$. Using this fact, it can be proved that at zero temperature, in the leading- order approximation (up to the to theorder $(g_0)^{-2/p}$) we can write the logarithm of the generating function as

$$\ln Z(\beta, \Omega, h) = \frac{1}{Q_0(\sigma, h)} \frac{\partial^2}{\partial h^2} Q_0(\sigma, h) \left(-\frac{\alpha}{2} + \frac{1}{2} \frac{d}{ds} \zeta_D(s)|_{s=0} \right), \tag{8}$$

where α is a infinite constant and $\zeta_D(s)$ is the spectral zeta-function associated with the elliptic operator D.

Let us consider now the situation in which the system is finite along each one of the spatial dimensions, i.e., $x_i \in [0, L]$, $i = 1, 2, ..., d-1$. Considering that the system is in thermal equilibrium at temperature β^{-1}, for the Euclidean time we assume periodic boundary conditions (Kubo-Martin-Schwinger KMS [39] [40] conditions) and for the Euclidean spatial dimensions we assume Dirichlet boundary conditions. We call this latter situation "hard" boundaries. See for example the Ref. [41]. For different kinds of confining boundaries see [42] [43]. The choice of the hard boundary provides an easy solution to the eigenvalue problem.

It follows that the operator D has the spectrum given by $\lambda_{n_1, \dots, n_d}$ where

$$\lambda_{n_1, \dots, n_d} = \left[\left(\frac{n_1 \pi}{L} \right)^2 + \dots + \left(\frac{n_{d-1} \pi}{L} \right)^2 + \left(\frac{2 n_d \pi}{\beta} \right)^2 + (1-\sigma) m_0^2 \right], \tag{9}$$

$n_1, n_2, \dots, n_{d-1}$ are natural numbers different from zero, since we are choosing Dirichlet boundary conditions and n_d are integer numbers. The spectral zeta-function associated with the operator D in this situation reads

$$\zeta_D(s) = \sum_{n_1, \dots, n_d}^{\infty}{}' \lambda_{n_1, \dots, n_d}^{-s}, \tag{10}$$

where s is a complex parameter, and the prime sign means that the term $n_1 = 0, n_2 = 0, .., n_{d-1} = 0$ must be excluded. The series above converges for $\mathrm{Re}\, s > \frac{d}{2}$ and its analytic continuation defines a meromorphic function of s, analytic at $s = 0$. Since we should have to introduce an arbitrary parameter μ with dimension of a mass to implement the analytic procedure with dimensionless quantities, we have scaling properties.

Using n as a general index instead of $n_1, ..., n_d$, the scaling properties follows from the fact that

$$\zeta_{\mu D}(s) = \sum_{n}^{\infty} (\mu^{-2}\,\lambda_n)^{-s} = \mu^{2s} \sum_{n}^{\infty} \lambda_n^{-s} = \mu^{2s} \zeta_D(s)\,. \tag{11}$$

Therefore we have

$$\frac{1}{2}\frac{d}{ds}\,\zeta_{\mu D}(s)|_{\,s=0} = \frac{1}{2}\frac{d}{ds}\,\zeta_D(s)|_{\,s=0} + \frac{1}{2}\ln\mu^2\,\zeta_D(s)|_{\,s=0}. \tag{12}$$

Before continuing, we would like to discuss two points. The first one is the fact that for different boundary condition, as, for example, Neumann boundary conditions in all the hyperplanes or periodic boundary conditions in all the spatial directions, the presence of the zero-mode can make the calculations more involved. It is important to remark that this zero mode problem does not appear in the calculations that we are presenting, since we are choosing Dirichlet boundary conditions in all hyperplanes, excluding the possibility of the spatial zero mode. The second point is that it is possible to show that there is no scaling in the situation that we are interested in. For the case of hypercube with Dirichlet boundary conditions it is possible to prove that the spectral zeta-function in $s = 0$ is zero, consequently $B_{\frac{d}{2}} = 0$ and there is no scaling in the theory.

Let us study in Eq. (8) the contribution arising from the spectral zeta-function which takes into account the geometric constraints upon the scalar field. Using the spectrum of the D operator given by Eq. (9) and the definition of the spectral zeta-function given by Eq. (10), we get that the derivative of the spectral zeta-function in $s = 0$ yields

$$\frac{d}{ds}\,\zeta_D(s)|_{\,s=0} = -\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\Biggl(\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) + \ln\Bigl(1 + \frac{a^2\beta^2}{4n_d^2L^2 + q^2\beta^2}\Bigr)\Biggr), \tag{13}$$

where $\vec{n}_{d-1} = (n_1, n_2, ..., n_{d-1})$, $q^2 = n_1^2 + n_2^2 + ... + n_{d-1}^2$ and $a^2 = \Bigl(\frac{(1-\sigma)m_0^2L^2}{\pi^2}\Bigr)$. Note that in Eq. (13) we are using that $\zeta_D(s)|_{\,s=0} = 0$. Using the following identity [44]

$$\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) = \int_1^{(\frac{\pi\,\beta\,q}{L})^2} \frac{d\theta^2}{\theta^2 + (2\pi n_d)^2} + \ln\Bigl(1 + (2\pi n_d)^2\Bigr)\,, \tag{14}$$

we can see that the first term in the right hand side of Eq. (13) gives a divergent contribution. To proceed we use another useful identity given by

$$\sum_{n_d=-\infty}^{\infty} \frac{1}{\theta^2 + (2\pi n_d)^2} = \frac{1}{2\theta}\Bigl(1 + \frac{2}{e^\theta - 1}\Bigr). \tag{15}$$

Using both identities given by Eq. (14) and Eq. (15), it is possible to express the double summation that appears in Eq. (13) by a single summation given by

$$\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\Bigl(\Bigl(\frac{\pi\,\beta\,q}{L}\Bigr)^2 + (2\pi n_d)^2\Bigr) = 2\sum_{\vec{n}_{d-1}=1}^{\infty}\int_1^{\bigl(\frac{\pi\,\beta\,q}{L}\bigr)} d\theta\Bigl(\frac{1}{2} + \frac{1}{e^\theta - 1}\Bigr) + \alpha_1, \tag{16}$$

where $\alpha_1 = \sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\Big(1+(2\pi n_d)^2\Big)$. Carrying out the θ integration, we finally arrive that Eq. (16) can be written as

$$\sum_{\vec{n}_{d-1}=1}^{\infty}\sum_{n_d=-\infty}^{\infty}\ln\left(\left(\frac{\pi\,\beta\,q}{L}\right)^2+(2\pi n_d)^2\right)=2\sum_{\vec{n}_{d-1}=1}^{\infty}\left(\frac{\pi\,\beta\,q}{2L}+\ln\Big(1-e^{-\frac{\pi\,\beta\,q}{L}}\Big)\right)+\alpha_2\,, \tag{17}$$

where $\alpha_2=\alpha_1-\sum_{\vec{n}_{d-1}=1}^{\infty}\Big(1+2\ln(1-e^{-1})\Big)$. Since this divergent contribution α_2 is β-independent we will see that it can be eliminated using the third law of thermodynamics. The first term on the right side of Eq. (17) is a divergent contribution, corresponding to the zero-point energy term. Using the following mathematical result [45] [46] given by

$$\prod_{n=-\infty}^{\infty}\left(1+\frac{a^2}{n^2+b^2}\right)=\frac{\sinh^2(\pi\sqrt{a^2+b^2})}{\sinh^2(\pi\,b)}\,, \tag{18}$$

we can write the last term of Eq. (13) in a more manageable way. Using the Eq. (17) and Eq. (18), the derivative of the spectral zeta-function in $s=0$ can be rewritten as

$$\frac{d}{ds}\,\zeta_D(s)|_{s=0}=-2\sum_{\vec{n}_{d-1}=1}^{\infty}\left[\ln\left(\frac{\sinh\Big(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\Big)}{\sinh\Big(\frac{\pi\beta q}{2L}\Big)}\right)+\ln\Big(1-e^{-\frac{\pi\,\beta\,q}{L}}\Big)+\frac{\pi\,\beta\,q}{2L}\right]-\alpha_2\,. \tag{19}$$

In the next section we will evaluate the contribution of the O(N) model independent-value generating function. In order to give operational meaning to the independent-value generating function we will use Klauder's results.

4 Contribution of the large N $O(N)$ model independent-value generating function in $\ln Z$

To give meaning to the independent value generating functional $Q_0(\sigma,\vec{h}\,)$, we are using the Klauder's result as the formal definition of the independent-value generating functional derived for the O(N) model in a d-dimensional Euclidean space [28]. It is possible to show that the independent-value generating function can be written as

$$Q_0(\sigma,\vec{h}\,)=\exp\left(-\frac{c_N}{2V}\int d^dx\int\Big(1-\cos(\vec{h}\cdot\vec{u})\Big)\exp\left(-\frac{1}{2}\sigma\,{m_N}^2u^2-\frac{g_N}{p\,!}u^p\right)\frac{d^Nu}{|u|^N}\right) \tag{20}$$

where c_N, m_N and g_N are new parameters dependent of N. To solve the integral defined in Eq. (20), we notice that both $\vec{u}$ and $\vec{h}$ are isovectors of N components, i.e., $\vec{u}=(u_0,u_1,\,...\,,u_N)$ and $\vec{h}=(h_0,h_1,\,...\,,h_N)$.

If we use a N-dimensional polar coordinate system defined by

$$\begin{cases} u_0=|u|\cos\theta_1 \\ u_1=|u|\cos\theta_2\sin\theta_1 \\ \quad ... \\ u_{N-1}=|u|\sin\theta_{N-1}\,...\,\sin\theta_1 \end{cases} \tag{21}$$

with the N-dimensional volume element given by:

$$d^N u = |u|^{N-1} d|u|\, d\Omega_N \tag{22}$$

where

$$d\Omega_N = \prod_{l=1}^{N-1} \sin^{N-1-l}\theta_l d\theta_l, \tag{23}$$

we can rewrite Eq. (20) as:

$$\begin{aligned} Q_0(\sigma,h) &= c_N \int [1 - \cos(h\,|\,u|\,\cos\theta_1)] \times \\ &\times \exp\left(-\frac{1}{2}\sigma\, {m_N}^2 u^2 - \frac{g_N}{p\,!}u^p\right) \frac{d|\,u|}{|\,u|}\, \sin^{N-2}(\theta_1)\, d\theta_1\, d\Omega_{N-1}, \end{aligned} \tag{24}$$

where the integral over $d\Omega_{N-1}$ gives the surface of the unit sphere in $N-1$ dimensions. Using an approximation to the integration over θ_1 valid for very large N, and the change of variables $|\,u| \to v = |\,u|^2/(N-2)$, it is possible to write the independent-value generating functional as:

$$Q_0(\sigma,h) = \exp\left(-\frac{1}{2V}\int d^d x \int \frac{dv}{v}\left[1-\exp\left(-\frac{1}{2}h^2 v\right)\right]\exp\left(-\frac{1}{2}\sigma m^2 v - \frac{g}{p\,!}v^{p/2}\right)\right). \tag{25}$$

Here, the normalization parameter c_N, the mass m and the coupling constant g are rescaled in a way that $Q_0(\sigma,h)$ in Eq. (25) has no N-dependence. For more details of this discussion, see the Ref. [28].

In order to study $Q_0(\sigma,h)$ let us define $E(m,\sigma,g,h)$ given by

$$E(m,\sigma,g,h) = \int \frac{dv}{v}\left[1-\exp\left(-\frac{1}{2}h^2 v\right)\right]\exp\left(-\frac{1}{2}\sigma m^2 v - \frac{g}{p\,!}v^{p/2}\right) \tag{26}$$

Using a series representation for $\exp x$ and using the fact that the series obtained ($\sum_{k=1}^{\infty} c_k\, f_k(u)$) not only converges on the interval $[0,\infty)$, but also converges uniformly there, the series can be integrated term by term. It is not difficult to show that

$$E(m,\sigma,g,h) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}\frac{h^{2k}}{k!}\int_{-\infty}^{\infty} dv\, v^{k-1}\exp\left(-\frac{1}{2}\sigma m^2 v - \frac{g}{p\,!}v^{p/2}\right) \tag{27}$$

Now let us use the fact that the σ parameter can be choose in such a way that the calculations becomes tractable. Let us choose $\sigma = 0$. Therefore we have

$$E(m,\sigma,g,h)|_{\sigma=0} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}\frac{h^{2k}}{k!}\int_{-\infty}^{\infty} dv\, v^{k-1}\exp\left(-\frac{g}{p\,!}v^{p/2}\right) \tag{28}$$

The sum in Eq. (28) has odd and even terms. For even $p > 4$, the integral of the odd terms will be zero and, remembering that the integrated function will be even for $k = (1,3,5,\,...\,)$, the integral of the even terms can be rewritten as:

$$E(m,\sigma,g,h)|_{\sigma=0} = 2\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{2k+1}\frac{h^{2k+1}}{(2k+1)!}\int_{0}^{\infty} dv\, v^{2k}\exp\left(-\frac{g}{p\,!}v^{p/2}\right) \tag{29}$$

Let us use the following integral representation for the Gamma function [45]

$$\int_0^\infty dx\, x^{\nu-1} \exp(-\mu\, x^p) = \frac{1}{p}\, \mu^{-\frac{\nu}{p}}\, \Gamma\left(\frac{\nu}{p}\right), \quad \mathrm{Re}(\mu) > 0 \quad \mathrm{Re}(\nu) > 0 \;\; p > 0. \tag{30}$$

At this point it is clear that the $(g_0\,\varphi^p)$ theory, for even $p > 4$, can easily handle applying our method. Using the result given by Eq. (30) in Eq. (29) we have

$$E(m, \sigma, g, h)|_{\sigma=0} = \sum_{k=0}^{\infty} j(p,k) \frac{h^{2k+1}}{g^{(2k+1)/p}}, \tag{31}$$

where the coefficients $j(p,k)$ are given by

$$j(p,k) = \left(-\frac{1}{2}\right)^{2k-1} \Big[p\,(2k+1)!\Big]^{-1} (p\,!)^{(2k+1)/p}\, \Gamma\left(\frac{4k+2}{p}\right) \tag{32}$$

Substituting the Eq. (31) and Eq. (32) in Eq. (25) we obtain that the independent-value generating function $Q_0(\sigma,h)|_{\sigma=0}$ can be written as

$$Q_0(\sigma,h)|_{\sigma=0} = \exp\left[-\frac{1}{2\Omega\beta} \int_0^\beta d\tau \int d^{d-1}x \sum_{k=0}^{\infty} j(p,k) \frac{h^{2k+1}}{g^{(2k+1)/p}}\right]. \tag{33}$$

It is easy to calculate the second derivative for the independent-value generating function with respect to h. Note that $Q_0(\sigma,h)\big|_{h=\sigma=0} = 1$. Thus we have

$$\begin{aligned} \frac{\partial^2}{\partial h^2} Q_0(\sigma,h)|_{\sigma=0} \;\; &= \;\; \left(\frac{1}{4} \sum_{k,\,q=0}^{\infty} j(p,k,q)\,(2k+1)(2q+1)\, \frac{h^{2k+2q}}{g^{(2k+2q+2)/p}}\right) \times \\ &\quad \times \exp\left(-\frac{1}{2}\sum_{k=0}^{\infty} j(p,k) \frac{h^{2k+1}}{g^{(2k+1)/p}}\right) + J(g,p,h), \end{aligned} \tag{34}$$

where $J(g,p,h)$ is given by

$$J(g,p,h) = \left(-\frac{1}{2}\sum_{k=1}^{\infty} j(p,k)(2k+1)(2k) \frac{h^{2k-1}}{g^{(2k+1)/p}}\right) \exp\left(-\frac{1}{2}\sum_{k=0}^{\infty} j(p,k) \frac{h^{2k+1}}{g^{(2k+1)/p}}\right), \tag{35}$$

and $j(p,k,q) = j(p,k)\, j(p,q)$.

We are interested in the case $h = 0$, therefore $J(g,p,h)$ does not contribute to the Eq. (34), since $lim_{h\to 0} J(h) = 0$. Using the fact that we are interested in the case $h = 0$, we have the simple result that in the Eq. (34) only the term $k = q = 0$ contributes. We get that the independent-value generating function $Q_0(\sigma,h)$ satisfies $Q_0(\sigma,h)\big|_{h=\sigma=0} = 1$, and

$$\frac{\partial^2}{\partial h^2} Q_0(\sigma,h)|_{h=\sigma=0} = \frac{1}{p^2} \left(\frac{p\,!}{g}\right)^{\frac{2}{p}} \left[\Gamma\left(\frac{2}{p}\right)\right]^2. \tag{36}$$

We will define $\Phi(p) = \frac{1}{p^2}\, p\,!^{\frac{2}{p}} \left[\Gamma\left(\frac{2}{p}\right)\right]^2$. In the next section we show that it is possible to obtain a quantum bound in the spatially bounded system described by the $O(N)$ self-interacting scalar field theory in the strong-coupling regime in the large N limit. In high temperatures the bound is always correct, nevertheless, for the cases of intermediate or low temperatures, the sign of the renormalized zero-point energy is crucial for the validity of the bound for the specific entropy.

5 The specific entropy for strongly coupled $(g_0\,\varphi^p)_d$ theory

In this section we compute the specific entropy $\frac{S}{E}$ of the system. For simplicity, let us define $\ln Z(\beta,\Omega,h)|_{h=0} = \ln Z(\beta,\Omega)$. From Eq. (6) and Eq. (7), and using for simplicity that the mean energy $E(\beta,\Omega)=E$ and the entropy $S(\beta,\Omega)=S$, the specific entropy is given by

$$\frac{S}{E} = \beta - \ln Z(\beta,\Omega)\Big(\frac{d}{d\beta}\ln Z(\beta,\Omega)\Big)^{-1}. \tag{37}$$

Substituting Eq. (19) and Eq. (36) in Eq. (8) we have that $\ln Z(\beta,\Omega)$ is given by

$$\ln Z(\beta,\Omega) = -\Phi(p)\,g^{-2/p}\left(\frac{\alpha'}{2}+I_2(\beta)\right), \tag{38}$$

where $\alpha' = \alpha+\alpha_2$ and the quantity $I_2(\beta)$ is given by

$$I_2(\beta) = \sum_{\vec{n}_{d-1}=1}^{\infty}\left[\ln\left(\frac{\sinh\!\left(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\right)}{\sinh\!\left(\frac{\pi\beta q}{2L}\right)}\right)+\ln\!\Big(1-e^{-\frac{\pi\,\beta\,q}{L}}\Big)+\frac{\pi\,\beta\,q}{2L}\right]. \tag{39}$$

Defining C_1 and $C_2 = -\frac{2C_1}{\alpha'}$ that depend only of p and g_0 and do not depend on β as

$$C_1 = -\frac{\alpha'}{2}\,\Phi(p)\,g^{-2/p}, \tag{40}$$

the quantity $\ln Z(\beta,\Omega)$ can be written in a general form as

$$\ln Z(\beta,\Omega) = C_1 - C_2\,I_2(\beta). \tag{41}$$

It is worth to mention that the quantity C_1 corresponds to a divergent expression, C_2 is finite and the summation term in the right-hand side of Eq. (19) is proportional to the zero-point energy. In order to renormalize $\ln Z(\beta,\Omega)$ we first can use the third law of thermodynamics. The derivative of $\ln Z(\beta,\Omega)$ with respect of β yields

$$\frac{d}{d\beta}\ln Z(\beta,\Omega) = -C_2\,\frac{d}{d\beta}I_2(\beta), \tag{42}$$

where the derivative of $I_2(\beta)$ with respect to β is given by

$$\frac{d}{d\beta}I_2(\beta) = \frac{\pi}{2L}\sum_{\vec{n}_{d-1}=1}^{\infty}\left(\left(\sqrt{q^2+a^2}\,\coth\!\Big(\frac{\pi\beta}{2L}\sqrt{q^2+a^2}\Big)-q\,\coth\!\Big(\frac{\pi\beta\,q}{2L}\Big)\right)+\frac{2\,q}{e^{\frac{\pi\,\beta\,q}{L}}-1}+q\right). \tag{43}$$

Using the definition of the mean energy given by Eq. (6), Eq. (40), Eq. (42) and Eq. (43) we have that the unrenormalized mean energy in the massless case $(a=0)$ is given by

$$E(\beta,\Omega)|_{a=0} = \frac{\pi}{2L}\Phi(p)\,g^{-2/p}\sum_{\vec{n}_{d-1}=1}^{\infty}\left(\frac{2\,q}{e^{\frac{\pi\,\beta\,q}{L}}-1}+q\right). \tag{44}$$

The formula above has the simple interpretation of being phase space sums over the mean energy of each mode, where the zero-point energy is included. Note that the divergence that appear in the mean energy given by the Eq. (44) is coming from the zero-point energy, which is given by

$$E_0 = \frac{\pi}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} (n_1^2 + n_2^2 + ... + n_{d-1}^2)^{\frac{1}{2}}\,, \tag{45}$$

and its sign is given by the ratio between the first and the second terms of the right-hand side of Eq. (44), for a negative zero-point energy. An analytic regularization gives the renormalized zero-point energy. Using the definition of the Epstein-zeta function given by

$$A(a_1, a_2, ..., a_k; 2s) = \sum_{\vec{n}_k=-\infty}^{\infty} (a_1 n_1^2 + a_2 n_2^2 + ... + a_k n_k^2)^{-s}\,, \tag{46}$$

we can find the analytic extension of the Epstein-zeta function in the complex plane, in particular at $s = -\frac{1}{2}$, to define the Casimir energy [34] [47] [48] [49], as the finite part of a meromorphic function that possesses simple poles. Although, in general situations, there is an ambiguity in the renormalization procedure, in our case $\zeta_D(s)|_{s=0} = 0$. Therefore there is no scaling in the theory and consequently the renormalized zero-point energy does not depend on the renormalized scale μ. Note that although in the expression for the renormalized mean energy, up to the order $(g_0)^{-\frac{2}{p}}$, the coupling constant appears, we are interested only in the ratio $\frac{S}{E}$, and the dependence of the coupling constant disappears.

After this discussions we are able to present the entropy of the system. Substituting Eq. (41) and Eq. (42) in the definition of the entropy given by Eq. (7), we have that the entropy of the system can be written as

$$S = C_1 - \beta C_2 \left(\frac{I_2(\beta)}{\beta} - \frac{d}{d\beta} I_2(\beta) \right)\,. \tag{47}$$

A system with a unique ground state corresponds to a state of vanishing entropy at zero temperature. Since at zero temperature the system goes to a non-degenerate ground state, the entropy must go to zero. The expression of the entropy given by Eq. (47) must satisfy the third law of thermodynamics, i.e., the entropy of a system has a limiting property that $\lim_{\beta\to\infty} S = 0$. To proceed, lets analyze the limit given by

$$\lim_{\beta\to\infty} \frac{I_2(\beta)}{\beta} = \lim_{\beta\to\infty} \frac{d}{d\beta} I_2(\beta) = \frac{\pi a^2}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} \frac{1}{\sqrt{q^2+a^2}+q} + \frac{\pi}{2L} \sum_{\vec{n}_{d-1}=1}^{\infty} q\,. \tag{48}$$

Substituting Eq. (48) in Eq. (47), and using the third law of thermodynamics, we get

$$\lim_{\beta\to\infty} S = C_1 = 0\,. \tag{49}$$

Therefore the first step to find a finite result for $\ln Z(\beta, \Omega)$ was achieved, since we were able to renormalize C_1 to zero using the third law of thermodynamics. After this step we have

$$\ln Z(\beta, \Omega) = -C_2\, I_2(\beta)\,. \tag{50}$$

Substituting Eq. (50) in Eq. (37) we can see that for the case $a = 0$, i.e., the massless case, the quotient $\frac{S}{E}$ yields

$$\frac{S}{E} = 2\pi R\, T_d(\xi)\,, \tag{51}$$

where we are defining the dimensionless variable ξ given by $\xi = \beta/L$. Since the field is confined in a hypercube, the radius of the smallest $(d-1)$-dimensional sphere that circumscribes this system should be given by $R = \frac{1}{2}\sqrt{(d-1)}\,L$. The function $T_d(\xi)$ defined in Eq. (51) is given by

$$T_d(\xi) = \frac{1}{\pi\sqrt{d-1}}\,\frac{\xi\,P_d(\xi) + R_d(\xi)}{\varepsilon_d^{(r)} + P_d(\xi)}\,, \tag{52}$$

where $\varepsilon_d^{(r)} = LE_0^{(r)}$ and the positive functions $P_d(\xi)$ and $R_d(\xi)$ are defined respectively by

$$P_d(\xi) = \sum_{\vec{n}_{d-1}=1}^{\infty} \pi\,q\left(e^{\pi\,\xi\,q} - 1\right)^{-1} \tag{53}$$

and

$$R_d(\xi) = -\sum_{\vec{n}_{d-1}=1}^{\infty} \ln\left(1 - e^{-\pi\,\xi\,q}\right)\,. \tag{54}$$

It is interesting to study the behavior of the specific entropy for low and high temperatures. For the case of high temperatures, using the results obtained in Ref. [21] we get

$$\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{h_2(d)}\,\xi\,. \tag{55}$$

where the functions $h_1(d)$ and $h_2(d)$ are given respectively by

$$h_1(d) = \frac{S_{d-1}}{\pi^d\sqrt{d-1}}\,\zeta(d)\Big(\Gamma(d) + \Gamma(d-1)\Big) \tag{56}$$

and

$$h_2(d) = \frac{S_{d-1}}{\pi^{d-1}}\Big(\Gamma(d)\,\zeta(d) - f(d)\Big)\,. \tag{57}$$

The quantity $f(d)$ that appears in the definition of $h_2(d)$ is given by

$$f(d) = \sum_{l=0}^{\infty} \frac{B_l}{(d+l-1)l!}. \tag{58}$$

At high temperatures the dimension in the imaginary direction shrinks to zero and the system behaves like a classical system in $(d-1)$ dimensions where quantum fluctuations are absent. This behavior of the specific entropy increasing with β in the high-temperature limit was obtained by Deutsch in Ref. [13]. Bekenstein using the condition $\beta \ll R$ (high temperature limit) also obtained the same behavior in Ref. [10]. Since the thermal energy can compensate the negative renormalized zero-point energy, the quantum bound holds.

When considering the low temperature behavior of the specific entropy, we can see that the problem of the sign of the renormalized zero-point energy can invalidate the quantum bound. In this limit also using the results obtained in Ref. [21] we have

$$\frac{S}{E} < 2\pi R\,\frac{h_1(d)}{\varepsilon_d^{(r)}}\,\xi^{1-d}\,. \tag{59}$$

Although some authors claim that the energy of the boundaries of such systems can compensate the negative renormalized-zero point energy yielding a net positive energy, for us this is still an open question that deserves further investigation.

6 Conclusions and perspectives

In this paper we studied self-interacting scalar fields in the strong-coupling regime in equilibrium with a thermal bath, also in the presence of macroscopic boundaries. Using the Klauder representation for the independent-value generating functional, and up to the order $(g_0)^{-\frac{2}{p}}$, we show that it is possible to obtain a quantum bound in the system described by the $O(N)$ self-interacting scalar field in the strong-coupling regime in the limit of large N. Note that for a very large number of fields, general arguments said that at least in the weak coupling regime, there a critical N_c, such that for $N > N_c$ the bound is violated. Our results show that this problem does not appear in the strong-coupling regime. Our results are quite similar to the previous one obtained for the one component scalar model.

We have shown that, for fields confined in a hypercube of size L, in the strong-coupling regime, at low and intermediate temperatures ($\beta \approx L$), the quantum bound depends on the sign of the renormalized zero-point energy given by $E_0^{(r)}$. Defining $\varepsilon_d^{(r)}$ as the renormalized zero-point energy for the free theory per unit length, we get the following functional dependencies. For low temperatures we get $\frac{S}{E} < 2\pi R \frac{h_1(d)}{\varepsilon_d^{(r)} \xi^{d-1}}$, where R is the radius of the smallest sphere circumscribing the system. For the case of high temperature, we get that the specific entropy always satisfies a quantum bound, given by $\frac{S}{E} < 2\pi R \frac{h_1(d)}{h_2(d)} \xi$. Since in our approach, we have to evaluate spectral zeta-function, our results can easily be generalized fort geometries where the analytic form of the spectrum is known. For domains with unknown spectrum, the problem is more involved. Also for scalar fields in non-isometric domains but isospectral situation, the ratio S/E must obey the Bekenstein bound.

A natural extension of this paper is to extend the results, i.e., the validity of the bound for the case of interacting field theory described by asymptotically free models [50] [51] [52] [53] [54], at least up to some order of perturbation theory. This situation of field theory with asymptotically free behavior, defined in a small compact region of space may occur in QCD-the confinement-deconfinement phase transition at high temperatures or if usual matter is strongly compressed. In ultra-relativistic heavy ion collisions we expect that the plasma of quarks and gluons can be produced, just after the collision, hot and compressed nuclear matter is confined in a small region of space. The validity of the Bekenstein bound in systems defined in a compact spatial region, described by asymptotically free theories is under investigation by the authors.

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References

[1] H. B. G. Casimir, Proc. Kon. Ned. Akad. Wekf. **51**, 793 (1948).

[2] G. Plunien, B. Müller and W. Greiner, Phys. Rep. **134**, 87 (1986).

[3] A. A. Grib, S. G. Mamayev and V. M. Mostepanenko, *"Vacuum Quantum Effects in Strong Fields"*, Friedman Laboratory Publishing. St. Petesburg (1994).

[4] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).

[5] K. A. Milton, *"The Casimir Effect: Physical Manifestation of Zero-Point Energy"*, World Scientific (2001).

[6] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973).

[7] J. D. Bekenstein, Phys. Rev. **D23**, 287 (1981).

[8] J. D. Bekenstein, Phys. Rev. **D30**, 1669 (1984).

[9] M. Schiffer and J. D. Bekenstein, Phys. Rev. **D39**, 1109 (1989).

[10] J. D. Bekenstein, Phys. Rev. **D49**, 1912 (1994).

[11] D. N. Page, Phys. Rev **D26**, 947 (1982).

[12] D. Unwin, Phys. Rev. **D26**, 944 (1982).

[13] D. Deutch, Phys. Rev. Lett. **48**, 286 (1982).

[14] W. G. Unruh, Phys. Rev. **D42**, 3596 (1990).

[15] R. Bousso, J. High Energy Phys. **04**, 035 (2004).

[16] J. D. Bekenstein, Foundations of Physics 35, 1805 (2005).

[17] M. Schiffer and J. D. Bekenstein, Physical Review **D42**, 3598 (1990).

[18] R. Bousso, *Bound States and the Bekenstein Bound* ArXiv hep-th/0310148 (2003).

[19] J. D. Bekenstein and E. I. Guendelman, Phys. Rev. **D35**, 716 (1987).

[20] J. D. Bekenstein and M. Schiffer, Int. J. Mod. Phys. **C1**, 355 (1990).

[21] M. Aparicio Alcalde, G. Menezes and N. F. Svaiter, Phys. Rev. **77D**, 125024-1 (2008).

[22] J. R. Klauder, Acta Phys. Aust. **41**, 237 (1975).

[23] R. Menikoff and D. H. Sharp, J. Math. Phys. **19**, 135 (1978).

[24] J. R. Klauder, Ann. Phys. **117**, 19 (1979).

[25] S. Kovesi-Domokos, Il Nuovo Cim. **33A**, 769 (1976).

[26] C. M. Bender, F. Cooper, G. S. Guralnik and D. H. Sharp, Phys. Rev. **D19**, 1865 (1979).

[27] N. F. Svaiter, Physica **A345**, 517 (2005).

[28] J. R. Klauder, *"Beyond Conventional Quantization"*, Cambridge University Press, Cambridge (2000).

[29] S. W. Hawking, Comm. Math. Phys. **55**, 133 (1977).

[30] A. Voros, Comm. Math. Phys. **110**, 439 (1987).

[31] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, *"Zeta Regularization Techniques and Applications"*, World Scientific, Singapure (1994).

[32] K. Kirsten, *"Spectral Functions in Mathematics and Physics"*, Chapman and Hall/CRC, Florida (2002).

[33] N. F. Svaiter, Physica **A386**, 111 (2006).

[34] J. Ambjorn and S. Wolfram, Ann. Phys. **147**, 1 (1983).

[35] F. Caruso, N. P. Neto, B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D43**, 1300 (1991).

[36] R. D. M. De Paola, R. B. Rodrigues and N. F. Svaiter, Mod. Phys. Lett. **A34**, 2353 (1999).

[37] L. E. Oxman. N. F. Svaiter and R. L. P. G. Amaral, Phys. Rev. **D72**, 125007 (2005).

[38] N. Parga, D. Toussaint and J. R. Fulco, Phys. Rev. **D20**, 887 (1979).

[39] R. Kubo, J. Phys. Soc. Jap. **12**, 570 (1957).

[40] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

[41] N. F. Svaiter and B. F. Svaiter, Jour. Phys. **A25**, 979 (1992).

[42] F. Caruso, R. De Paola and N. F. Svaiter, Int. Jour. Mod. Phys. **A14**, 2077 (1999).

[43] L. H. Ford and N. F. Svaiter, Phys. Rev. **58**, 065007-1 (1998).

[44] J. I. Kapusta, *"Finite-temperature Field Theory"*, Cambridge University Press (1989).

[45] I. S. Gradshteyn and I. M. Ryzhik, *"Tables of Integrals, Series and Products"*, Academic Press Inc., New York (1980).

[46] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, *"Integrals and Series"*, Vol. 1 and 2, Gordon and Breach Science Publishers (1986).

[47] L.H. Ford, Phys. Rev. **D21**, 933 (1980).

[48] K. Kirsten, J. Math. Phys. **32**, 3008 (1991).

[49] L. H. Ford and N. F. Svaiter, Phys. Rev. **D51**, 6981 (1995).

[50] D. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).

[51] H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

[52] D. Gross and F. Wilczek, Phys. Rev. **D8**, 3633 (1973).

[53] D. Gross and F. Wilczek, Phys. Rev. **D9**, 980 (1974).

[54] H. D. Politzer, Phys. Rep. **146**, 129 (1974).