CBPF-NF-019/08

Stochastic Quantization for Complex Actions

G. Menezes[1] *and N. F. Svaiter* [2]

Centro Brasileiro de Pesquisas Físicas-CBPF,
Rua Dr. Xavier Sigaud 150,
Rio de Janeiro, RJ, 22290-180, Brazil

Abstract

We use the stochastic quantization method to study systems with complex valued path integral weights. We assume a Langevin equation with a memory kernel and Einstein's relations with colored noise. The equilibrium solution of this non-Markovian Langevin equation is analyzed. We show that for a large class of elliptic non-Hermitian operators acting on scalar functions on Euclidean space, which define different models in quantum field theory, converges to an equilibrium state in the asymptotic limit of the Markov parameter $\tau \to \infty$. Moreover, as we expected, we obtain the Schwinger functions of the theory.

PACS numbers: 03.70+k, 04.62.+v

[1]e-mail: gsm@cbpf.br

[2]e-mail: nfuxsvai@cbpf.br

1 Introduction

The program of stochastic quantization, first proposed by Parisi and Wu [1], and the stochastic regularization was carried out for generic fields defined in flat, Euclidean manifolds. A brief introduction to stochastic quantization can be found in Refs. [2] [3] [4], and a complete review is given in Ref. [5]. Recently Menezes and Svaiter [6] implemented the stochastic quantization in the theory of self-interacting scalar fields in a static Riemannian manifold and also a manifold with a event horizon, namely, the Einstein and the Rindler manifold. First, these authors solved a Langevin equation for the mode coefficients of the field, then they exhibit the two-point function at the one-loop level. It was shown that it diverges and to regularize the theory they used a covariant stochastic regularization. The presence of the Markov parameter as an extra dimension allows the authors to implement a regularization scheme, which preserves all the symmetries of the theory under study. It is clear that the stochastic quantization program can be implemented without problems, if it is possible to perform the Wick rotation, obtaining a real Euclidean action.

The picture that emerges from the above discussion is that the implementation of the stochastic quantization in curved background is related to the following fact. For static manifold, it is possible to perform a Wick rotation, i.e., analytically extend the pseudo-Riemannian manifold to the Riemannian domain without problem. Nevertheless, for non-static curved manifolds we have to extend the formalism beyond the Euclidean signature, i.e., to formulate the stochastic quantization in pseudo-Riemannian manifold, not in the Riemannian space (as in the original Euclidean space) as was originally formulated. See for example the discussion presented by Huffel and Rumpf [7] and Gozzi [8]. In the first of these papers the authors proposed a modification of the original Parisi-Wu scheme, introducing a complex drift term in the Langevin equation, to implement the stochastic quantization in Minkowski spacetime. Gozzi studied the spectrum of the non-self-adjoint Fokker-Planck Hamiltonian to justify this program. See also the papers [9] [10]. Of course, these situations are special cases of ordinary Euclidean formulation for systems with complex actions.

The main difference between the implementation of the stochastic quantization in Minkowski spacetime and in Euclidean space is the fact that in the latter case the approach to the equilibrium state is a stationary solution of the Focker-Planck equation. In the Minkowski formulation, the Hamiltonian is non-Hermitian and the eigenvalues of such Hamiltonian are in general complex. The real part of such eigenvalues are important to the asymptotic behavior at large Markov time, and the approach to the equilibrium is achieved only if we can show its positive semi-definiteness. The crucial question is: what happens if the Langevin equation describes diffusion around complex action? Some authors claim that it is possible to obtain meaningful results out of Langevin equation diffusion processes around complex action. Parisi [11] and Klauder and Peterson [12] investigated the complex Langevin equation, where some numerical simulations in one-dimensional systems was presented. See also the papers [13] [14]. We would also like to mention the approach developed by Okamoto et al. [15] where the role of the kernel in the complex Langevin equation was studied.

We would like to remark that there are many examples where Euclidean action is complex. The simplest case is the stochastic quantization in Minkowski spacetime, as we discussed. Other situations are systems, as for example QCD, with non-vanishing chemical potential at finite temperature; for $SU(N)$ theories with $N > 2$, the fermion determinant becomes complex and also the effective action. Complex terms can also appear in the Langevin equation for fermions, but a suitable kernel can circumvent this problem [16] [17] [18]. Another case that deserves our attention is the stochastic quantization of topological field theories. One of the peculiar features within these kind of theories is the appearance of a factor i in front of the topological actions in Euclidean space. In these topological theories, the path integral measure weighing remain to be e^{iS}, even after the Wick rotation. An attempt to use a Markovian Langevin equation with a white noise to quantize the theory fails since the Langevin equation will not tend to any equilibrium at large Markov parameter. In the literature there are different proposes to solve the above mentioned problem. In a pure topological Chern-Simons theory, Ferrari et al. [19] introduced a non-trivial kernel in the Langevin equation. Other approach was developed by Menezes and Svaiter [20]. These authors proved that, using a non-Markovian Langevin equation with a colored random noise, it is possible to obtain convergence towards equilibrium even with an imaginary Chern-Simons coefficient. An interesting application of this method can be found on Ref. [21], where a Langevin equation with a memory kernel was introduced in order to obtain the Schwinger functions for the self-interacting scalar model. In conclusion, although several alternative methods have been proposed to deal with interesting physical systems where the Euclidean action is complex [22] [23] [24] [25], these methods do not suggest any general way of solving the particular difficulties that arise in each situation. Here, we wish to report progress in the stochastic quantization of theories with imaginary action, introducing a memory kernel.

It is the purpose of the present paper to use the method of the stochastic quantization to study systems with complex valued path integral weights. We assume a Langevin equation with a memory kernel and Einstein's relations with colored noise [26]. We show that for a large class of elliptic non-Hermitean operators which define different models in quantum field theory converges in the asymptotic limit of the Markov parameter $\tau \to \infty$, and we obtain the Schwinger functions of the theory. In section II, we briefly discuss the Parisi-Wu stochastic quantization for the case of free scalar field. In section III we implement the stochastic quantization for scalar field with complex action using a non-Markovian Langevin equation. Conclusions are given in section IV. In this paper we use $\hbar = c = k_B = 1$.

2 Stochastic quantization for the free scalar field theory: the Euclidean case

In this section, we give a brief survey of the stochastic quantization. This technique in flat spacetime with trivial topology can be summarized by the following steps. First, starting from a field defined in Minkowski spacetime, after analytic continuation to imaginary time, the Euclidean counterpart, i.e., the field defined in an Euclidean space, is obtained. Second, it is introduced a monotonically crescent Markov parameter, called in the literature "fictitious time" and also a random noise field $\eta(\tau, x)$, which simulates the coupling between the classical system and a heat reservoir. It is assumed that the fields defined at the beginning in a d-dimensional Euclidean space also depends on the Markov parameter, therefore the field and a random noise field are defined in a $(d+1)$-dimensional manifold. One starts with the system out of equilibrium at an arbitrary initial state. It is then forced into equilibrium assuming that its evolution is governed by a Markovian Langevin equation with a white random noise field [27] [28] [29]. In fact, this evolution is described by a process which is stationary, Gaussian and Markovian. Finally, the n-point correlation functions of the theory in the $(d + 1)$-dimensional space are defined by performing averages over the random noise field with a Gaussian distribution, that is, performing the stochastic averages $\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n)\,\rangle_\eta$. The n-point Schwinger functions of the Euclidean d-dimensional theory are obtained evaluating these n-point stochastic averages $\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n)\,\rangle_\eta$ when the Markov parameter goes to infinity ($\tau \to \infty$), and the equilibrium is reached. This can be proved in different ways for the particular case of Euclidean scalar field theory. One can use, for instance, the Fokker-Planck equation [30] [31] associated with the equations describing the stochastic dynamic of the system. A diagrammatical technique [32] has also been used to prove such equivalence.

After this brief digression, let us consider a free neutral scalar field. The Euclidean action that usually describes such theory is

$$S_0[\varphi] = \int d^dx \, \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2(x)\right). \tag{1}$$

The simplest starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume a flat Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^{\,d}$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noise $\eta(x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \to \varphi(\tau, x)$ and $\eta(x) \to \eta(\tau, x)$. Therefore, the fields and the random noise are defined in a domain: $T^{\,d} \times R^{\,(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{\delta\, S_0}{\delta\,\varphi(x)}|_{\varphi(x)=\varphi(\tau,\,x)} + \eta(\tau,x), \tag{2}$$

where τ is a Markov parameter, $\eta(\tau,x)$ is a random noise field and S_0 is the usual free Euclidean action defined in Eq. (1). For a free scalar field, the Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2\,)\varphi(\tau,x) + \eta(\tau,x), \tag{3}$$

where Δ is the d-dimensional Laplace operator. The Eq. (3) describes a Ornstein-Uhlenbeck process and we are assuming the Einstein relations, that is:

$$\langle\, \eta(\tau,x)\,\rangle_\eta = 0, \tag{4}$$

and for the two-point correlation function associated with the random noise field

$$\langle\, \eta(\tau,x)\,\eta(\tau',x')\,\rangle_\eta \;= 2\delta(\tau-\tau')\,\delta^d(x-x'), \tag{5}$$

where $\langle\, ...\rangle_\eta$ means stochastic averages. The above equation defines a delta-correlated random process. In a generic way, the stochastic average for any functional of φ given by $F[\varphi]$ is defined by

$$\langle\, F[\varphi\,]\,\rangle_\eta = \frac{\int\,[d\eta]F[\varphi\,]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x)\Big]}{\int\,[d\eta]\exp\Big[-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(\tau,x)\Big]}. \tag{6}$$

Let us define the retarded Green function for the diffusion problem that we call $G(\tau-\tau',x-x')$. The retarded Green function satisfies $G(\tau-\tau',x-x') = 0$ if $\tau-\tau' < 0$ and also

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2\,)\right]G(\tau-\tau',x-x') = \delta^d(x-x')\delta(\tau-\tau'). \tag{7}$$

Using the retarded Green function and the initial condition $\varphi(\tau,x)|_{\tau=0} = 0$, the solution for Eq. (3) reads

$$\varphi(\tau,x) = \int_0^\tau d\tau'\int_\Omega d^dx'\,G(\tau-\tau',x-x')\eta(\tau',x'). \tag{8}$$

Let us define the Fourier transforms for the field and the noise given by $\varphi(\tau,k)$ and $\eta(\tau,k)$. We have respectively

$$\varphi(\tau,k) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int\,d^dx\,e^{-ikx}\,\varphi(\tau,x), \tag{9}$$

and

$$\eta(\tau,k)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dx\,e^{-ikx}\,\eta(\tau,x). \tag{10}$$

Substituting Eq. (9) in Eq. (1), the free action for the scalar field in the $(d+1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]\,|_{\varphi(k)=\varphi(\tau,\,k)}=\frac{1}{2}\int d^dk\,\varphi(\tau,k)(k^2+m_0^2)\varphi(\tau,k). \tag{11}$$

Substituting Eq. (9) and Eq. (10) in Eq. (3) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi(\tau,k)=-(k^2+m_0^2)\varphi(\tau,k)+\eta(\tau,k). \tag{12}$$

In the Langevin equation the particle is subject to a fluctuating force (representing a stochastic environment), where its average properties are presumed to be known and also the friction force. Note that the "friction coefficient" in the Eq. (12) is given by $(k^2+m_0^2)$.

The solution for Eq. (12) reads

$$\varphi(\tau,k)=\exp\Big(-(k^2+m_0^2)\tau\Big)\,\varphi(0,k)+\int_0^\tau d\tau'\exp\Big(-(k^2+m_0^2)(\tau-\tau')\Big)\,\eta(\tau',k). \tag{13}$$

Using the Einstein relation, we get that the Fourier coefficients for the random noise satisfies

$$\langle\,\eta(\tau,k)\,\rangle_\eta=0 \tag{14}$$

and

$$\langle\,\eta(\tau,k)\eta(\tau',k')\,\rangle_\eta=2\,\delta(\tau-\tau')\,\delta^d(k+k'). \tag{15}$$

It is possible to show that $\langle\,\varphi(\tau,k)\varphi(\tau',k')\,\rangle_\eta|_{\tau=\tau'}\equiv D(k,k';\tau,\tau')$ is given by:

$$D(k;\tau,\tau)=(2\pi)^d\delta^d(k+k')\frac{1}{(k^2+m_0^2)}\Big(1-\exp\Big(-2\tau(k^2+m_0^2)\Big)\Big). \tag{16}$$

where we assume $\tau=\tau'$. Therefore, for $\tau\to\infty$ we recover the Euclidean two-point function.

The self-interacting theory is beyond the scope of this paper, however it can be carried out in a straightforward way. In the next section we present a modification of the Langevin equation that allows us to treat systems with complex Euclidean actions. Although non-trivial, it is intuitively obvious that the method can be extended to interacting field theory.

3 Stochastic quantization for complex actions

As an application of the ideas discussed previously, in this section we show how it is possible to quantize a theory with a complex action using a non-Markovian Langevin equation. We will be following similar steps as in Ref. [21]. Consider the following Euclidean action:

$$S = \frac{1}{2}\int d^d x\, \varphi\, K\varphi, \tag{17}$$

with the following Markovian Langevin equation:

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -K\varphi + \eta(\tau, x). \tag{18}$$

where K is an elliptic operator (with some minor changes, our proof in this section can be made to hyperbolic operators). The function φ is a scalar field, for simplicity, but we can generalize our results to fields of higher spin. If we let K to be non-Hermitian, the action in Eq. (17) becomes complex. There are many approaches in the literature to deal with complex actions; one of them is to employ a complex Langevin equation, separating the field in a real part and in a imaginary part, $Re(\varphi) = \varphi_1$ and $Im(\varphi) = \varphi_2$ [5]. With this approach, we get two Langevin equations, one for each of the two fields φ_1 and φ_2. Another one is to use a modified Langevin equation:

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -\int d^d y\, \kappa(x, y)\frac{\delta\, S_0}{\delta\, \varphi(y)}|_{\varphi(y)=\varphi(\tau,\, y)} + \eta(\tau, x), \tag{19}$$

where a subsequent change in the second moment of the noise field is

$$\langle\, \eta(\tau, x)\, \eta(\tau', x')\,\rangle_\eta\; = 2\delta(\tau - \tau')\, \kappa(x, x'). \tag{20}$$

With these modifications, we may choose an appropriate kernel:

$$\kappa(x, x') = K_x^\dagger \delta(x - x'), \tag{21}$$

so the Langevin equation becomes

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -K.K^\dagger\varphi + \eta(\tau, x). \tag{22}$$

We see that we get a "bosonized" version of the Langevin equation given by Eq. (18) and the problem with the convergence towards an equilibrium disappears, since $K.K^\dagger$ is a Hermitian operator. This is the prescription usually employed in the literature to deal with the stochastic quantization of fermions [16] [17] [18]. In fact, the root of this problem lies in the fact that there exists no classical analogue of fermion fields.

Another approach can be used as well. Let us consider the following non-Markovian Langevin equation:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\int_0^\tau ds\, M_\Lambda(\tau-s)\frac{\delta\, S}{\delta\,\varphi(x)}|_{\varphi(x)=\varphi(s,\,x)} + \eta(\tau,x), \tag{23}$$

where M_Λ is a memory kernel and the stochastic random field $\eta(\tau,x)$ satisfies the modified Einstein's relations

$$\langle\, \eta(\tau,x)\,\eta(\tau',x')\,\rangle_\eta = 2M_\Lambda(|\tau-\tau'|)\,\delta^d(x-x'). \tag{24}$$

In this case where $M_\Lambda(|\tau-\tau'|)$ has a width in the fictitious time, the description is Gaussian in spite of being non-Markovian. For the case of Euclidean free scalar field theory we have that the generalized Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\int_0^\tau ds\, M_\Lambda(\tau-s)K\varphi(s,x) + \eta(\tau,x). \tag{25}$$

We shall prove in this section that this method leads to convergence towards equilibrium, even though we have a complex Langevin equation.

We can introduce a mode decomposition such as

$$\varphi(\tau,x) = \int d\tilde{\mu}(n)\varphi_n(\tau)u_n(x) \tag{26}$$

and

$$\eta(\tau,x) = \int d\tilde{\mu}(n)\eta_n(\tau)u_n(x), \tag{27}$$

where the measure $\tilde{\mu}(k)$ depends on the metric we are interested in. Each Fourier coefficient φ_n obeys a (non-Markovian) Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi_n(\tau) = -\lambda_n\int_0^\tau ds\, M_\Lambda(\tau-s)\varphi_n(s) + \eta_n(\tau), \tag{28}$$

where λ_n is an eigenvalue of the operator K and $\eta_n(\tau)$ obeys

$$\langle\, \eta_n(\tau)\,\eta_{n'}(\tau')\,\rangle_\eta = 2M_\Lambda(|\tau-\tau'|)\,\delta^d(n,n'). \tag{29}$$

Following Fox [33] [34], we define the Laplace transform of the memory kernel:

$$M(z) = \int_0^\infty d\tau\, M_\Lambda(\tau)\, e^{-z\tau}. \tag{30}$$

With the initial condition $\varphi_n(\tau)|_{\tau=0} = 0$, the solution of the Eq. (28) reads:

$$\varphi_n(\tau) = \int_0^\infty d\tau' \, G_n(\tau - \tau') \, \eta_n(\tau'), \tag{31}$$

where using the step function $\theta(\tau)$, the Green function $G_n(\tau)$ is defined by:

$$G_n(\tau) \equiv \Omega_n(\tau) \, \theta(\tau). \tag{32}$$

The $\Omega_n(\tau)$ function that appears in Eq. (32) is defined through its Laplace transform:

$$\Omega_n(\tau) = \Big(z + \lambda_n M(z)\Big)^{-1}. \tag{33}$$

¿From Eq. (31) and the modified Einstein relations, we get that the free scalar correlation function $D_n(\tau, \tau')$ is given by:

$$\begin{aligned} & D_n(\tau, \tau') = \\ & = 2\delta^d(n, n') \int_0^\infty ds \int_0^\infty ds' \, G_n(\tau - s) \, G_n(\tau' - s') \, M_\Lambda(| \, s - s' \, |) \\ & = 2\delta^d(n, n') \int_0^\tau ds \int_0^{\tau'} ds' \, \Omega_n(\tau - s) \, \Omega_n(\tau' - s') \, M_\Lambda(| \, s - s' \, |). \end{aligned} \tag{34}$$

To proceed we have to write $D_n(\tau, \tau')$ in a simplified way. Note that the double Laplace transform of the right hand side is given by:

$$\begin{aligned} & \int_0^\infty d\tau \, e^{-z\tau} \int_0^\infty d\tau' \, e^{-z\tau'} \int_0^\tau ds \int_0^{\tau'} ds' \, \Omega_n(\tau - s) \, \Omega_n(\tau' - s') \, M_\Lambda(| \, s - s' \, |) = \\ & = \Omega_n(z) \, \Omega_n(z') \int_0^\infty ds \int_0^\infty ds' \, e^{-z's'} e^{-zs} \, M_\Lambda(| \, s - s' \, |). \end{aligned} \tag{35}$$

Now, with simple manipulations, we get:

$$\int_0^\infty ds \int_0^\infty ds' \, e^{-z's'} e^{-zs} M_\Lambda(| \, s - s' \, |) = \frac{M(z) + M(z')}{z + z'}. \tag{36}$$

Therefore, we get the identity:

$$\begin{aligned} & \int_0^\infty d\tau \, e^{-z\tau} \int_0^\infty d\tau' \, e^{-z\tau'} \int_0^\tau ds \int_0^{\tau'} ds' \Omega_n(\tau - s) \, \Omega_n(\tau' - s') \, M_\Lambda(| \, s - s' \, |) = \\ & = \Omega_n(z) \, \Omega_n(z') \left(\frac{M(z) + M(z')}{z + z'} \right). \end{aligned} \tag{37}$$

Remembering Eq. (33), we can show that:

$$\Omega_n(z)\,\Omega_n(z')\biggl(\frac{M(z)+M(z')}{z+z'}\biggr)=\frac{1}{\lambda_n}\biggl(\frac{\Omega_n(z)+\Omega_n(z')}{z+z'}-\Omega_n(z)\,\Omega_n(z')\biggr). \tag{38}$$

So, in parallel with result given by Eq. (36), we finally obtain a very simple expression for $D_n(\tau,\tau')$ in terms of $\Omega_n(\tau)$. We have

$$D_n(\tau,\tau')=\frac{2}{\lambda_n}\,\delta^d(n,n')\,\Bigl(\Omega_n(\mid\tau-\tau'\mid)-\Omega_n(\tau)\,\Omega_n(\tau')\Bigr). \tag{39}$$

Now, we need an expression for the memory kernel in order to investigate the convergence of Eq. (39). A series of kernels were proposed in the literature:

$$M_\Lambda^{(m)}(\tau)=\frac{1}{2m!}\Lambda^2(\Lambda^2\mid\tau\mid)^m\exp(-\Lambda^2\mid\tau\mid). \tag{40}$$

For simplicity, we shall take the case for $m=0$. Then, from Eq. (30), Eq. (33) and Eq. (40), and applying the inverse Laplace transform [35], we obtain the following expression for the Ω-function:

$$\Omega_n(\tau)=\biggl(\frac{\Lambda^2}{\beta}\sinh\Bigl(\frac{\beta\tau}{2}\Bigr)+\cosh\Bigl(\frac{\beta\tau}{2}\Bigr)\biggr)\,\exp\Bigl(-\tau\frac{\Lambda^2}{2}\Bigr), \tag{41}$$

where we have defined a quantity β given by:

$$\beta\equiv x+iy, \tag{42}$$

where

$$x=\sqrt{\frac{\Lambda^4+\alpha_R+|z|}{2}}, \tag{43}$$

$$y=\alpha_I\sqrt{\frac{1}{2(\Lambda^4+\alpha_R+|z|)}}, \tag{44}$$

and, finally:

$$|z|=\sqrt{(\Lambda^4+\alpha_R)^2+\alpha_I^2}, \tag{45}$$

with $\alpha=-2\Lambda^2\lambda_n$ and we have written α as $\alpha=\alpha_R+i\alpha_I$, where α_R and α_I are real quantities. Similarly, we have, for the Green function:

$$G_n(\tau)=\biggl(\frac{\Lambda^2}{\beta}\sinh\Bigl(\frac{\beta\tau}{2}\Bigr)+\cosh\Bigl(\frac{\beta\tau}{2}\Bigr)\biggr)\,\exp\Bigl(-\tau\frac{\Lambda^2}{2}\Bigr)\theta(\tau). \tag{46}$$

Now, in order to have convergence, we must demand that $G_n \to 0$ when $\tau \to \infty$. Using hyperbolic identities we see that, in order to have $G_n \to 0$ when $\tau \to \infty$, we shall have:

$$|\lambda_n^R| > \frac{(\lambda_n^I)^2}{2\Lambda^2}, \tag{47}$$

where we have written the eigenvalues as $\lambda_n = \lambda_n^R + i\lambda_n^I$. Since Λ is, in principle, arbitrary, we see that the condition given by Eq. (47) does not seem to pose any serious restrictions on the eigenvalues of the operator K. However, we have another restriction. Note that, with this prescription, the function $M(x-y;\tau)$ defined in the Ref. [21] as

$$M(x-y,\tau) \equiv \int_0^\tau ds\, M_\Lambda(\tau-s)\, G(\tau-s, x-y), \tag{48}$$

where $G(\tau - s, x)$ is the retarded Green function for the diffusion problem, whose Fourier transform is given by:

$$\begin{aligned} M(k,\tau) \quad &= \quad \frac{\Lambda^2}{2}\frac{1}{9\,\Lambda^4+\beta^2}\Big\{8\,\Lambda^2 - \\ & 4\,\exp\Big(-\frac{3\Lambda^2}{2}\tau\Big)\Big[2\Lambda^2\cosh\Big(\frac{\beta\tau}{2}\Big) + \Big(\frac{3\Lambda^4}{2\beta} - \frac{\beta}{2}\Big)\sinh\Big(\frac{\beta\tau}{2}\Big)\Big]\Big\}, \end{aligned} \tag{49}$$

will no longer be Hermitian. But, if we impose the following restriction:

$$|\lambda_n^R| < 5\Lambda^2, \tag{50}$$

its real part will remain positive, which implies that the real part of the eingenvalues of the Fokker-Planck Hamiltonian defined therein are positive, assuring, therefore, convergence to equilibrium, i.e., the system reaches its ground state. It seems that Eq. (47) and Eq. (50) can be imposed simultaneously without restrictions on Λ. In particular, we are allowed to take arbitrarily large values for Λ, which would imply that our approach works for almost any value for the eigenvalues λ_n. It can be easily proved that the presence of the zero modes destroys the convergence to equilibrium.

¿From the results above, it is easy to see that the free two-point function is given by:

$$\begin{aligned} & D_n(\tau,\tau') = \\ & = \frac{2}{\lambda_n}\,\delta^d(n,n')\Big[\Big(\frac{\Lambda^2}{\beta}\,\sinh\Big(\frac{\beta(\tau-\tau')}{2}\Big) + \cosh\Big(\frac{\beta(\tau-\tau')}{2}\Big)\Big)\,\exp\Big(-\frac{\Lambda^2}{2}\,|\,(\tau-\tau')\,|\Big) \\ & -\Big(\frac{\Lambda^2}{\beta}\,\sinh\Big(\frac{\beta\tau}{2}\Big) + \cosh\Big(\frac{\beta\tau}{2}\Big)\Big)\Big(\frac{\Lambda^2}{\beta}\,\sinh\Big(\frac{\beta\tau'}{2}\Big) + \cosh\Big(\frac{\beta\tau'}{2}\Big)\Big)\,\exp\Big(-\frac{\Lambda^2}{2}(\tau+\tau')\Big)\Big] \end{aligned} \tag{51}$$

For $\tau = \tau'$, we get:

$$D_n(\tau,\tau) = \frac{2}{\lambda_n}\,\delta^d(n,n')\left(1 - \left(\frac{\Lambda^2}{\beta}\,\sinh\!\left(\frac{\beta\tau}{2}\right) + \cosh\!\left(\frac{\beta\tau}{2}\right)\right)^2 \exp(-\Lambda^2\,\tau)\right). \tag{52}$$

So, in the limit $\tau \to \infty$, we obtain the following result:

$$D_n(\tau,\tau) = \frac{2}{\lambda_n}\,\delta^d(n,n'). \tag{53}$$

Therefore, apart from an unimportant constant, we have obtained that, in the asymptotic limit $\tau \to \infty$, we have reached convergence towards the expected equilibrium, and the two-point Schwinger function was obtained.

4 Conclusions and perspectives

There are many examples where the Euclidean field theory is defined for an imaginary action. Since in this case the path integral weight is not positive definite, the stochastic quantization in this situation is problematic. Parisi and Klauder proposed complex Langevin equations [11] [12], and some problems of this approach are the following. First of all, complex Langevin simulations do not converge to a stationary distribution in many situations. Besides, if it does, it may converge to many different stationary distributions. The complex Langevin equation also appears when the original method proposed by Parisi and Wu is extended to include theories with fermions [16] [17] [18]. The first question that appears in this context is if make sense the Brownian problem with anticommutating numbers. It can be shown that, for massless fermionic fields, there will not be a convergence factor after integrating the Markovian Langevin equation. Therefore the equilibrium is not reached. One way of avoiding this problem is to introduce a kernel in the Langevin equation describing the evolution of two Grassmannian fields.

In this paper, we have used the method of the stochastic quantization to study systems with complex valued path integral weights. We assumed a Langevin equation with a memory kernel and Einstein's relation with colored noise. The equilibrium solution of such Langevin equation was analyzed. We have shown that for a large class of elliptic non-Hermitean operators which define different models in quantum field theory converges in the asymptotic limit of the Markov parameter $\tau \to \infty$, and we have obtained the Schwinger functions of the theory. Although non-trivial, the method proposed can be extended to interacting field theory with complex actions. The generalization of the method for this situation is under investigation by the authors.

5 Acknowlegements

This paper was supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnológico do Brazil (CNPq).

References

[1] G. Parisi and Y. S. Wu, Sci. Sin. **24**, 483 (1981).

[2] B. Sakita, *"Quantum Theory of Many Variable Systems and Fields"*, World Scientific Publishing (1985).

[3] P. Damgaard and H. Heiffel, *"Stochastic Quantization"*, World Scientific (1988).

[4] M. Namiki, Prog. Theor. Phys. Suppl. **111**, 1 (1993).

[5] P. Damgaard and H. Heiffel, Phys. Rep. **152**, 227 (1987).

[6] G. Menezes and N. F. Svaiter, Jour. of Phys. **A**, Math. Theor. **40**, 8548 (2007).

[7] H. Huffel and H. Rumpf, Phys. Lett. **148B**, 104 (1984).

[8] E. Gozzi, Phys. Lett. **150B**, 119 (1985).

[9] H. Huffel and Landshoff, Nucl. Phys. **B260**, 545 (1985).

[10] D. J. E. Callaway, F. Cooper, J. R. Klauder and H. A. Rose, Nucl. Phys. **B262**, 19 (1985).

[11] G. Parisi, Phys. Lett. **B131**, 393 (1983)

[12] J. R. Klauder and W. P. Peterson, J. Stat. Phys. **39**, 53 (1985).

[13] J. R. Klauder, Phys. Rev. **A29**, 2036 (1984).

[14] J. Ambjorn and S. K. Yang, Phys. Lett. **165**, 140 (1985).

[15] H. Okamoto, K. Okano. L. Schulke and S. Tanaka, Nucl. Phys. **B342**, 684 (1989).

[16] T. Fukai, H. Nakazato, J. Ohba, K. Okano and Y. Yamanaka, Prog. Theor. Phys. **69**, 1600 (1983).

[17] P. H. Damgaard and K. Tsokos, Nucl. Phys. **B 235**, 75 (1984).

[18] P. H. Damgaard, Prog. Theor. Phys. Suppl. **111**, 43 (1993).

[19] F. Ferrari and H. Hufel, Phys. Lett. **B261**, 47 (1991).

[20] G. Menezes and N. F. Svaiter, Jour. of Math. Phys. **47**, 073507 (2006).

[21] G. Menezes and N. F. Svaiter, Physica **A374**, 617 (2007).

[22] L. L. Salcedo, Phys. Lett. **B304**, 125 (1993).

[23] H. Gausterer, J. Phys. **A27**, 1325 (1994).

[24] H. Gausterer, Nucl. Phys. **A642**, 239 (1998).

[25] C. Pehlevan and G. Guralnik, *"Complex Langevin Equations and Schwinger-Dyson Equations"*, arXiv: 0710.3756 (2007).

[26] R. Zwanzig, K. S. J. Nordholm and W. C. Mitchell, Phys. Rev. **A5**, 2680 (1972).

[27] M. Kac and J. Logan, *"Fluctuations"*, in *"Fluctuation Phenomena"*, E. W. Montroll and J. L. Lebowitz (Editors), North Holland Publishing Company (1974).

[28] R. Zwanzig, *"Non-Equilibrium Statistical Mechanics"*, Oxford University Press, New York (2001).

[29] R. Kubo, M Toda and N. Hashitsume, *"Statistical Physics"*, Springer Verlag, Heidelberg (1991).

[30] G. Parisi and N. Sourlas, Nucl. Phys. **B206**, 32 (1982).

[31] E. Floratos and J. Iliopoulos, Nucl. Phys. **B214**, 392 (1983).

[32] W. Grimus, Z. Phys. **18C** (1983) 129.

[33] R. F. Fox, J. Stat. Phys. **16**, 259 (1977).

[34] R. F. Fox, J. Math. Phys. **18**, 2331 (1977).

[35] H. Bateman, *"Tables of Integral Transformations"*, Vol.1, California Institute of Technology (1954).