Representations of the $1D$ N-Extended Supersymmetry Algebra*

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Abstract

I review the present status of the classification of the irreducible representations of the algebra of the one-dimensional $N-$ Extended Supersymmetry (the superalgebra of the Supersymmetric Quantum Mechanics) realized by linear derivative operators acting on a finite number of bosonic and fermionic fields.

1 The Superalgebra of the Supersymmetric Quantum Mechanics

The superalgebra of the Supersymmetric Quantum Mechanics ($1D$ N-Extended Supersymmetry Algebra) is given by N odd generators Q_i ($i = 1, \ldots, N$) and a single even generator H (the hamiltonian). It is defined by the (anti)-commutation relations

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij} H, \\ [Q_i, H] &= 0. \end{aligned} \quad (1)$$

The knowledge of its representation theory is essential for the construction of off-shell invariant actions which can arise as a dimensional reduction of higher dimensional supersymmetric theories and/or can be given by $1D$ supersymmetric sigma-models associated to some d-dimensional target manifold (see [1] and [2]).

Two main classes of (1) representations are considered in the literature:
i) the non-linear realizations and
ii) the linear representations.

Non-linear realizations of (1) are only limited and partially understood (see [3] for recent results and a discussion). Linear representations, on the other hand, have been recently clarified and the program of their classification can be considered largely completed. In this work I will review the main results of the classification of the linear representations and point out which are the open problems. The work here reviewed is based on the references [4–9]. Some material here presented is new and is an anticipation of a work in progress ([10]).

The linear representations under consideration are given by a finite number of fields, bosonic and fermionic, depending on a single coordinate t (the time). The generator H is represented by the time-derivative, while the Q_i's generators are linear operators (matrices) whose entries are either c-numbers or time-derivatives up to a certain power. The main result of [4] states that all such irreducible representations, for a given N, are expressed by

a) a fundamental irreducible representation, nowadays called in the literature “the root multiplet” or “root representation”, with equal number n of bosonic and fermionic fields and that

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b) all remaining irreducible representations of (1) are obtained by applying an operation, the "dressing transformation", to the root representation.

In the root representation the Q_i generators are given by

$$Q_i = \left(\begin{array}{cc} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{array} \right), \tag{2}$$

where the σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of a D-dimensional (with $D = N$) Clifford algebra relation

$$\Gamma_i = \left(\begin{array}{cc} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{array} \right) \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}. \tag{3}$$

The dressing transformations, acting on the root generators Q_i, are given by the operations

$$Q_i \mapsto \widehat{Q}_i = D Q_i D^{-1} \tag{4}$$

realized by some diagonal matrix D whose non-vanishing entries are given by the identity and by non-negative powers of the hamiltonian H.

The regularity condition requires that only the $\widehat{Q}_i$ generators which do not admit entries with $\frac{1}{H}$ poles are legitimate operators of an irreducible representation.

A corollary of the [4] results is that the total number n of bosonic fields entering an irreducible representation (which equals the total number of fermionic fields) is expressed, for any given N, by the following relation

$$\begin{array}{rcl} N & = & 8l + m, \\ n & = & 2^{4l} G(m), \end{array} \tag{5}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$. $G(m)$ appearing in (5) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(6)

The modulo 8 property of the irreducible representations of the N-extended supersymmetry is in consequence of the famous modulo 8 property of Clifford algebras.

A dimensionality d can be assigned to any field entering a linear representation (irreducible or not). The hamiltonian H maps a given field of dimension d into a new field of dimension $d+1$. Bosonic (fermionic) fields have integer (respectively, half-integer) dimensions. Each finite linear representation is characterized by its "fields content", the set of integers $(n_1, n_2, \ldots, n_l)$ specifying the number n_i of fields of dimension d_i ($d_i = d_1 + \frac{i-1}{2}$, with d_1 an arbitrary constant) entering the representation.

Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the length of the representation (a root representation has length $l = 2$).

Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa. In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed. A multiplet is bosonic (fermionic) if its n_1 component fields of lower dimension are bosonic (fermionic). The representation theory does not discriminate the overall bosonic or fermionic nature of the multiplet.

According to [4], if $(n_1, n_2, \ldots, n_l)$ specifies the fields content of an irreducible representation, $(n_l, n_{l-1}, \ldots, n_1)$ specifies the fields content of a dual irreducible representation. Representations such that $n_1 = n_l, n_2 = n_{l-1}, \ldots$ are called "self-dual representations". In [5] it was shown how to extract from the associated Clifford algebras the admissible fields content of the (1) linear finite irreducible representations. We discuss the results of [5] in the next Section.

2 The finite linear irreducible representations

The (1) superalgebra is not a simple Lie superalgebra. It admits reducible representations which nevertheless are indecomposable. One class of representations of (1) is given by the so-called "enveloping representations" (see [5]). For each N, the "enveloping representation" admits 2^{N-1} bosonic and 2^{N-1} fermionic states spanned by the monomials

$$\prod_{i=1}^{N} {Q_i}^{\alpha_i},$$

where the α_i's take the values 0 and 1. The fields content of the enveloping representation is given by the set of numbers entering the Newton's binomial. Up to $N \leq 3$, the enveloping representation is irreducible. For $N = 3$ its fields content is given by $(1, 3, 3, 1)$. The enveloping representation is the unique $N = 3$ irreducible representation with length $l = 4$. Starting from $N \geq 4$, the enveloping representation is no longer irreducible. The fields content of the $N = 4$ enveloping representation is $(1, 4, 6, 4, 1)$. It contains twice as many fields than the number entering the $N = 4$ irreducible representations. The $N = 4$ enveloping representation is reducible, but indecomposable.

We review now the [5] results on the classification of the fields content of the linear finite irreducible representations of (1). In [5] it was shown how to extract from the Clifford algebras associated to the root multiplets (for any given N) the information about the allowed fields contents. The results are the following. The complete list of the allowed fields contents is explicitly produced for all values $N \leq 10$. Some corollaries follow from the [5] construction. $N = 1, 2, 4, 8$ are the only values of N such that all its irreducible representations have length $l \leq 3$. Conversely, starting from $N \geq 10$, irreducible representations with length $l = 5$ fields contents are allowed.

In [5] the length $l = 4$ fields contents were listed for all values $N \leq 12$. A careful examination in [10] shows that the results are indeed correct for $N \leq 11$, while missing cases appear for $N = 12$. We took here the opportunity to correct the $N = 12$ results and present the list of its length $l = 4$ admissible fields contents. It is given by the following values. Let us denote with $(h, 64 - k, 64 - h, k)$ the fields contents of the length $l = 4$ $N = 12$ irreducible representations (their total number of bosonic or fermionic fields is 64). The allowed values for

h, k are expressed by

$$\begin{array}{ll} h = 1 \,\&\, k = 1, \ldots, 52 & h = 10 \,\&\, k = 1, \ldots, 28 \\ h = 2 \,\&\, k = 1, \ldots, 48 & h = 11 \,\&\, k = 1, \ldots, 24 \\ h = 3 \,\&\, k = 1, \ldots, 44 & h = 12 \,\&\, k = 1, \ldots, 24 \\ h = 4 \,\&\, k = 1, \ldots, 40 & h = 13 \,\&\, k = 1, \ldots, 20 \\ h = 5 \,\&\, k = 1, \ldots, 36 & h = 14 \,\&\, k = 1, \ldots, 20 \\ h = 6 \,\&\, k = 1, \ldots, 36 & h = 15 \,\&\, k = 1, \ldots, 20 \\ h = 7 \,\&\, k = 1, \ldots, 32 & h = 16 \,\&\, k = 1, \ldots, 18 \\ h = 8 \,\&\, k = 1, \ldots, 32 & h = 17 \,\&\, k = 1, \ldots, 16 \\ h = 9 \,\&\, k = 1, \ldots, 28 & h = 18 \,\&\, k = 1, \ldots, 16 \end{array} \tag{7}$$

together with the exchanged $h \leftrightarrow k$, dually related, admissible values.

The computational scheme of [5] allows to compute length $l = 4$ and higher length fields contents for any given value N. On the other hand, it is still an open problem whether a closed form algorithm exists allowing to iteratively compute, at increasing values of N, the allowed fields contents.

3 The graphical presentation of the linear representations

In [7] and [8] it was pointed out that the fields contents alone do not necessarily uniquely specify the finite linear irreducible representations of (1). This result is based on a notion of class of equivalence of the irreducible representations of the $1D$ N-Extended Superalgebra which is motivated by the set of moves acting on its graphical presentations. In [9] an association was made between N-colored oriented graphs and the linear supersymmetry transformations. With slight modifications from [9] we can describe the identification as follows. The fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $x-y$ plane. The y coordinate can be chosen to correspond to the dimensionality d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the x axis. The higher dimensional fields have positive, half-integer values of y. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the N Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented N-colored graph with $2n$ vertices (see (5)). The graph is such that precisely N edges, one for each color, are linked to any given vertex of the graph.

Despite the fact that the presentation of the graph is not unique, certain of its features only depend on the class of the supersymmetry transformations. We introduce now, following [6], the invariant characterization. An unoriented “color-blind” graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black). For simplicity we discuss here the invariant characterization of the graphs associated to a length $l = 3$ irreducible representation (the generalization of the invariant characterization to graphs of arbitrary length is straightforward, see [6]) with fields content $(k, n, n-k)$. The connectivity of the associated length $l = 3$ color-blind graph can be expressed through the connectivity symbol $m_{1_{s_1}} + m_{2_{s_2}} + \ldots + m_{r_{s_r}}$ expressing the partition of the n $\frac{1}{2}$-dimensional fields (vertices) into the m_j vertices with s_j edges connecting them to the $n-k$ 1-dimensional auxiliary fields. We have that $m_1 + m_2 + \ldots + m_r = n$, while $s_j \neq s_{j'}$ for $j \neq j'$. The connectivity symbol is an invariant characterization of the class of the irreducible supersymmetry transformations.

In [6] the [5] framework to classify the fields content of the irreducible representations was put in place to classify the connectivity symbols of the graphs. The analysis was conducted for the $N \leq 8$ irreducible representations. The results are the following. The length $l = 2$ and $l = 4$ irreducible representations of a given fields content admit a unique connectivity symbol. Therefore, up to $N \leq 8$, the fields content uniquely specifies such irreducible representations. For what concerns the length $l = 3$ irreducible representations, the following results hold. For $N = 2, 3, 4, 7, 8$ the connectivity symbol is uniquely expressed for any given fields content. Since $N = 1$ does not admit length $l = 3$ irreducible representations, the only values of N admitting irreducible representations with same fields content and inequivalent connectivity are given by $N = 5, 6$. The complete list of $N \leq 8$ irreducible representations with same fields content and inequivalent connectivity symbol is given below. For $N = 5$ we have

$N = 5:$	*connectivities*
$(6,8,2)_A$	$4_2 + 2_1 + 2_0$
$(6,8,2)_B$	$2_2 + 6_1$
$(5,8,3)_A$	$4_3 + 3_1 + 1_0$
$(5,8,3)_B$	$1_3 + 5_2 + 2_1$
$(4,8,4)_A$	$4_4 + 4_1$
$(4,8,4)_B$	$1_4 + 3_3 + 3_2 + 1_1$
$(4,8,4)_C$	$4_3 + 4_2$
$(3,8,5)_A$	$1_5 + 3_4 + 4_2$
$(3,8,5)_B$	$2_4 + 5_3 + 1_2$
$(2,8,6)_A$	$2_5 + 2_4 + 4_3$
$(2,8,6)_B$	$6_4 + 2_3$

(8)

For $N = 6$ we have

$$\begin{array}{|c|c|}\hline N=6: & connectivities \\ \hline (6,8,2)_A & 6_2+2_0 \\ (6,8,2)_B & 4_2+4_1 \\ \hline (5,8,3)_A & 4_3+2_2+2_1 \\ (5,8,3)_B & 2_3+6_2 \\ \hline (4,8,4)_A & 4_4+4_2 \\ (4,8,4)_B & 2_4+4_3+2_2 \\ (4,8,4)_C & 8_3 \\ \hline (3,8,5)_A & 2_5+2_4+4_3 \\ (3,8,5)_B & 6_4+2_3 \\ \hline (2,8,6)_A & 2_6+6_4 \\ (2,8,6)_B & 4_5+4_4 \\ \hline \end{array} \quad (9)$$

Just like the computation of the admissible fields contents, the computation of the admissible connectivity symbols can be carried out for larger values of N. It is however unclear whether a closed form algorithm exists allowing to iteratively compute, at increased values of N, the admissible connectivity symbols.

4 Conclusions and open problems

In recent years the status of the irreducible representations of the one-dimensional N-extended supersymmetry algebra realized by linear derivative operators acting on a finite number of bosonic and fermionic fields has been vastly clarified. Several classifications are now available. They regard not only the total number of fields entering the irreducible representations, but also the dimensionality of the fields (information which is encoded in the so-called fields content) and the types of graphs (information which is encoded in the so called connectivity symbol) describing the supersymmetry transformations. The computational schemes can, at least in principle, be applied to arbitrarily large values of N, the only limitation coming from the computational power. Some open questions still exist and some are finding an answer. We limit here to mention some recent results which will soon appear ([10]) concerning in particular the nature of the dressing transformations (non-diagonal dressings of the length-2 root operators can be associated to graphs such that more than one edge of a given color meet at a given vertex) or the interpretation of the connectivity symbols in terms of subalgebras decompositions.

For physical applications the most important and poorly understood (for large values of N, typically $N > 8$) problem to be addressed consists in the construction of off-shell invariant actions. We can mention the example of the one-dimensional supersymmetric sigma models. Several techniques could be put in place. Multilinear invariants can be recovered by tensoring irreducible representations (the notion of the fusion algebra discussed in [11] could prove useful). Alternatively, the relation of the supersymmetry transformations with Clifford algebras and division algebras already demonstrated its usefulness [5] in identifying at least a new off-shell invariant action for $N = 8$.

Very recently a paper ([3]) appeared. It suggests the possibility that non-linear realizations of the (1) supersymmetry algebra could be understood from constraining at least two linear irreducible representations. The results here discussed could prove useful in addressing this problem whose importance lies in the fact that non-linear realizations seem to be an essential ingredient to produce one-dimensional supersymmetric sigma models admitting target manifolds which are not conformally flat.

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