

On the Irreps of the N -Extended Supersymmetric Quantum Mechanics and Their Fusion Graphs.

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Abstract

In this talk we review the classification of the irreducible representations of the algebra of the N -extended one-dimensional supersymmetric quantum mechanics presented in hep-th/0511274. We answer some issues raised in hep-th/0611060, proving the agreement of the results here contained with those in hep-th/0511274. We further show that the fusion algebra of the $1D$ N -extended supersymmetric vacua introduced in hep-th/0511274 admits a graphical presentation. The $N = 2$ graphs are here explicitly presented for the first time.

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1 Introduction

The one-dimensional N -extended supersymmetric quantum mechanics is an important and active research subject in many areas of both mathematics and physics. We refer to [1] and [2] for more comprehensive recent reviews of some of its aspects which, due to space-time limitations, cannot be covered in the present talk. Here we will focus on two main topics. At first, we make a review of the current status of the classification of the irreducible linear representations of the algebra associated with the one-dimensional supersymmetric quantum mechanics. In the following it will be presented a graphical interpretation of the $1D$ fusion algebra introduced in [3].

Concerning the classification of the irreps, the basic references are [4] and [3]. In [4] it was proven, essentially, that the irreps fall into classes of equivalence characterized by an associated Clifford algebra irrep (the connection between Clifford algebras and extended supersymmetries of $1D$ quantum mechanics was previously shown in [5, 6, 7]). In [3] this result was used as the starting point to produce a classification of the irreps. In this work a presentation of the results of [3] will be made. Some issues raised in [8] (see also [9]) will be answered, proving the compatibility of their results with the [3] classification.

It seems appropriate to present this seminar in a Symposium in honour of Jerzy Lukierski. Even if our collaboration did not involve the topics here discussed, the results here presented, however, were made possible by applying a formalism first elaborated in our common works (especially [10]).

2 The classification of the irreps

The finite linear irreps of the $D = 1$ N -extended supersymmetry algebra

$$\{Q_i, Q_j\} = \delta_{ij}H \quad (1)$$

(where the Q_i 's, $i, j = 1, \dots, N$, are the supersymmetry generators and H is the hamiltonian) are expressed by the set of (n_1, n_2, \dots, n_l) symbols representing the field-contents of the irreps. The non-negative integers n_i 's specify the number of fields of dimension $d_i = d_1 + \frac{i-1}{2}$ entering an irrep (the constant d_1 can be arbitrarily chosen). The fields whose dimension differs by $\frac{1}{2}$ have opposite statistics (bosonic/fermionic). The number l specifies the number of different dimensions of the fields entering an irrep and is referred to as the “length” of the irrep; l must satisfy the condition $l \geq 2$, with the $l = 2$ irreps being known in the literature as the “minimal-length” or “root” multiplets. In [3] it was explicitly presented the complete list of the allowed (n_1, n_2, \dots, n_l) field contents for $N \leq 10$. An algorithmic construction for computing the field contents for arbitrarily large values of N was produced and selected $N > 10$ examples were given. The list in [3] is understood as follows: for any N , (n_1, n_2, \dots, n_l) is present if and only if there exists at least one N -irrep with the given field content. As an example, the length-4 $(1, 7, 7, 1)$ field content is present for $N = 5, 6, 7$, but not for $N = 8$, meaning that there are no irrep with the given field content for $N = 8$, but there is (at least one) such irrep for $N = 5, 6, 7$.

The construction of [3] heavily relied on the [4] results which we briefly summarize here. All N -irreps of length $l \geq 3$ are obtained from the set of \overline{Q}_i operators acting on root

multiplets after applying an *acceptable* dressing transformation $\bar{Q}_i \rightarrow D\bar{Q}_iD^{-1} = Q'_i$. The dressing operator D is a diagonal operator whose entries are either 1 or positive powers of H . “Acceptable” refers to the fact that the whole set of Q'_i transformed operators must be regular (that is, as matrix operators, they must not contain any entry with $\frac{1}{H}$ or higher poles). Two distinct acceptable operators D_1, D_2 leading to the same field content applied on the same set of \bar{Q}_i root multiplets operators are given by a permutation of their diagonal entries. D_1, D_2 are obviously related by a similarity transformation, $D_2 = SD_1S^{-1}$ (it is worth recalling that the exchange of the diagonal elements in, e.g., a 2×2 diagonal matrix D is recovered in terms of the 2×2 similarity matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Similarity transformations between two acceptable dressings of given field-content and for a fixed set of \bar{Q}_i operators acting on root multiplets form a group of transformations which corresponds to a subgroup \tilde{G} of the permutation group of the diagonal elements of the dressing transformations.

Concerning the length-2 root multiplets the situation is the following. They are recovered by an associated Clifford irrep of Weyl type (i.e., whose generators are in block antidiagonal form) through the following position

$$\bar{Q}_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i \cdot H & 0 \end{pmatrix}, \quad (2)$$

where σ_i and $\tilde{\sigma}_i$ are matrices entering the associated Clifford generators

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix}, \quad \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}. \quad (3)$$

The N Clifford generators entering (3) are recovered from the block-antidiagonal space-like generators of the $Cl(p, q)$ Clifford algebras (with (p, q) signature) according to the following scheme:

$$\begin{aligned} Cl(2 + 8m, 1) &\rightarrow N = 1 && \text{mod } 8 \\ Cl(3 + 8m, 2) &\rightarrow N = 2 && \text{mod } 8 \\ Cl(4 + 8m, 3) &\rightarrow N = 3 && \text{mod } 8 \\ Cl(5 + 8m, 0) &\rightarrow N = 3, 4 && \text{mod } 8 \\ Cl(6 + 8m, 1) &\rightarrow N = 5 && \text{mod } 8 \\ Cl(9 + 8m, 0) &\rightarrow N = 5, 6, 7, 8 && \text{mod } 8 \end{aligned} \quad (4)$$

The maximal value for N corresponds to $N = p - 1$ (the “oxidized” cases [3]). The reduced supersymmetries (for $N < p - 1$) are obtained by selecting a proper subset of the block antidiagonal $Cl(p, q)$ space-like generators. Notice that, unlike the other values of N , the $N = 3, 5 \text{ mod } 8$ cases can be recovered in two different ways.

For a fixed value of N , the N Clifford generators entering (3) can be uniquely chosen up to similarity transformations (this result is in consequence of the unicity of the real irreducible Clifford algebra representations for $p - q \neq 1, 5 \text{ mod } 8$).* This important property applies in particular to the reduced values of N , implying that two different choices

*Please notice that this new set of similarity transformations acts on supersymmetry operators for root multiplets; it should not be confused with the set of similarity transformations acting on dressing operators.

of the $N < p - 1$ proper subset of block antidiagonal space-like generators are equivalent. It also implies that the two ways in (4) of recovering the $N = 3, 5 \bmod 8$ supersymmetries are equivalent, producing root multiplets which are related by similarity transformations. For any given N the (3) Clifford generators associated to supersymmetric root multiplets can be canonically chosen. They can be presented as matrices whose non-vanishing entries are ± 1 . A group G of similarity transformations relates all choices of Clifford generators whose non-vanishing entries are ± 1 . The dressing transformations, applied to each set of Clifford generators of this type, produce irreps with the same field-contents. Taking into account these properties, the [3] classification of the field-contents produces a classification of the linear finite irreps of the $D = 1$ N -extended supersymmetry. The (n_1, \dots, n_l) symbol uniquely characterizes the irreps upon which the $D\overline{Q}_i D^{-1}$ supersymmetry operators act. The \overline{Q}_i operators acting on root multiplets are related by the group G of similarity transformations, while the acceptable dressing operators D are related by the group \tilde{G} of similarity transformations. Under this equivalence class of transformations, (n_1, \dots, n_l) uniquely specifies an irrep.

The complete list of (n_1, \dots, n_l) irreps for $N \leq 10$ is furnished in [3] and will not be reproduced here.

3 Irreps fusion algebras and the associated graphs

The notion of fusion algebra of the supersymmetric vacua of the N -extended one dimensional supersymmetry was introduced in [3]. The tensoring of two zero-energy vacuum-state irreps (irreps associated with the zero energy eigenvalue of the hamiltonian operator H) can be symbolically written as

$$[i] \times [j] = N_{ij}^k [k] \quad (5)$$

where N_{ij}^k are non-negative integers specifying the decomposition of the tensored-products irreps into its irreducible constituents. The N_{ij}^k integers satisfy a fusion algebra with the following properties

- 1) Constraint on the total number of component fields,

$$\forall i, j \quad \sum_k N_{ij}^k = 2d \quad (6)$$

where d is the number of bosonic (fermionic) fields in the given irreps.

- 2) The symmetry property

$$N_{ij}^k = N_{ji}^k \quad (7)$$

- 3) The associativity condition,

$$[i] \times ([j] \times [k]) = ([i] \times [j]) \times [k] \quad (8)$$

which implies the commutativity of the $(N_i)_j^k \equiv N_{ij}^k$ fusion matrices.

Fusion algebras can also be nicely presented in terms of their associated graphs. The $N = 1$ and $N = 2$ fusion graphs are produced in the Appendix (with two subcases for each

N , according to whether or not the irreps are distinguished w.r.t. their bosonic/fermionic statistics). Let us discuss here how to present the [3] results in graphical form. The irreps correspond to points. N_{ij}^k oriented lines (with arrows) connect the $[j]$ and the $[k]$ irrep if the decomposition (5) holds. The arrows are dropped from the lines if the $[j]$ and $[k]$ irreps can be interchanged. The $[i]$ irrep should correspond to a generator of the fusion algebra. This means that the whole set of $N_l = N_{lj}^k$ fusion matrices is produced as sum of powers of the $N_i = N_{ij}^k$ fusion matrix.

Let us discuss explicitly the $N = 2$ case. We obtain the following list of four irreps (if we discriminate their statistics):

$$[1] \equiv (2, 2)_{Bos}; [2] \equiv (1, 2, 1)_{Bos}; [3] \equiv (2, 2)_{Fer}; [4] \equiv (1, 2, 1)_{Fer} \quad (9)$$

The corresponding $N = 2$ fusion algebra is realized in terms of four 4×4 , mutually commuting, matrices given by

$$N_1 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv X; N_2 = N_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Y; N_3 = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \equiv Z. \quad (10)$$

The fusion algebra admits three distinct elements, X, Y, Z and one generator (we can choose either X or Z), due to the relations

$$Y = \frac{1}{8}(X^3 - 2X) \quad , \quad Z = -\frac{1}{4}(X^3 - 6X^2 + 4X). \quad (11)$$

The vector space spanned by X, Y, Z is closed under multiplication

$$\begin{aligned} X^2 &= Z^2 = ZX = X + 2Y + Z, \\ XY &= Y^2 = YZ = 4Y. \end{aligned} \quad (12)$$

This fusion algebra corresponds to the ‘‘smiling face’’ graph (Figure 4 in the Appendix).

4 Conclusions

The supersymmetric quantum mechanism is a fascinating subject with several open problems. The potentially most interesting one concerns perhaps the construction of off-shell invariant actions with the dimension of a kinetic term for large values of N (let’s say $N > 8$). They could provide some hints towards an off-shell formulation of higher-dimensional supergravity and M -theory. The fusion algebras, which encode the information of the decomposition of tensor representations, could provide useful in attacking this problem.

Concerning the representation theory itself, some questions are still opened. The authors of [8] pointed out the existence of inequivalent (starting from $N \geq 5$) supersymmetry irreps with the same field content. They explicitly discussed the $N = 5$ $(6, 8, 2)$ and the $N = 6$ $(6, 8, 2)$ irreps, producing in both cases two inequivalent irreps. These results are in agreement with those in [3]. At first it must be noticed that $(6, 8, 2)$ is an admissible field content for both $N = 5$ and $N = 6$ irreps, see [4]. The inequivalences obtained in [8] correspond to a different notion of the equivalence relation than the one here discussed (their equivalence class is w.r.t.

the general linear transformations of the supersymmetry generators and/or fields). It produces a refinement of the equivalence relation here employed. To spot the differences, one can use the valid analogy of the classification of simple Lie algebras. Simple Lie algebras over the complex numbers are classified by the Dynkin's diagrams, while simple Lie algebras over the reals are obtained by the real forms. The (n_1, \dots, n_l) field contents work as Dynkin's diagrams, uniquely specifying the irreps under the class of equivalence here discussed.

Concerning the classification of irreps, the present status is the following. The complete classification of the irreps under the equivalence relation here discussed was produced in [3] (explicit results were furnished for $N \leq 10$). At present, no classification of irreps is yet available under the [8] notion of the equivalence relation.

Appendix: the $N = 1, 2$ fusion graphs.

We present here four fusion graphs of the $N = 1$ and $N = 2$ supersymmetric quantum mechanics irreps. The "A" cases below correspond to ignore the statistics (bosonic/fermionic) of the given irreps. In the "B" cases, the number of fundamental irreps is doubled w.r.t. the previous ones, in order to take the statistics of the irreps into account. The construction of the graphs is discussed in the main text.



Figure 1: Fusion graph of the $N=1$ superalgebra, A case, 1 irrep $((1,1))$, no bosons/fermions distinction.

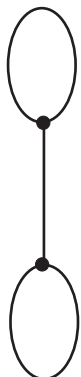


Figure 2: Fusion graph of the $N=1$ superalgebra, B case, 2 irreps $((1, 1)_{Bos}$ and $(1, 1)_{Fer}$) with bosons/fermions distinction.

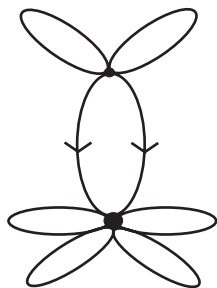


Figure 3: Fusion graph of the $N=2$ superalgebra, A case, 2 irreps $((2, 2)$ and $(1, 2, 1))$, no bosons/fermions distinction.

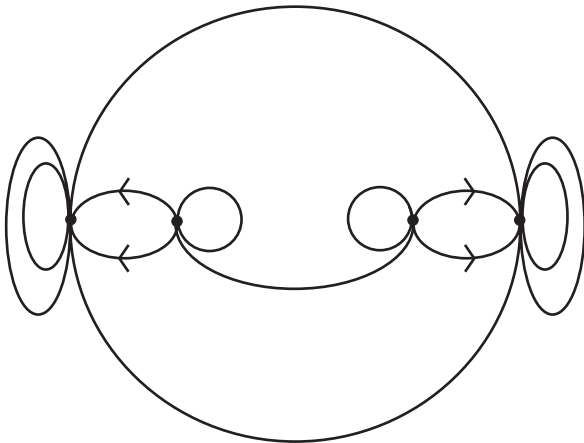


Figure 4: Fusion graph of the $N=2$ superalgebra, B case, 4 irreps, bosons/fermions distinction, “the smiling face”. From left to right the four points correspond to the $[2] - [1] - [3] - [4]$ irreps, respectively. The lines are generated by the $N_1 \equiv X$ fusion matrix, see (10).

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