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Static Cylindrical Symmetry and Conformal Flatness

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Abstract

We present the whole set of equations with regularity and matching conditions required for the description of physically meaningful static cylindrically symmmetric distributions of matter, smoothly matched to Levi–Civita vacuum spacetime. It is shown that the conformally flat solution with equal principal stresses represents an incompressible fluid. It is also proved that any conformally flat cylindrically symmetric static source cannot be matched through Darmois conditions to the Levi-Civita spacetime. Further evidence is given that when the Newtonian mass per unit length reaches 1/2 the spacetime has plane symmetry.

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1 Introduction

Cylindrical systems in Einstein's theory puzzles relativists since Levi–Civita found its vacuum solution in 1919 [1]. The precise meaning of its two independent parameters is still hard to grasp and, in particular, the one that describes the Newtonian energy per unit length σ looks the most elusive. The fact that there are two parameters while in its counterpart, Newtonian theory, has only one parameter looks a sufficient justification for deserving more research. But the importance of its research goes further if one notices the close link between Levi–Civita, γ and Schwarzschild spacetimes [2] and its peculiar properties. Besides, there has been renewed interest in cylindrically symmetric sources in relation with different, classical and quantum, aspects of gravitation (see [3] and references therein). Such sources may serve as test–bed for numerical relativity, quantum gravity and for probing cosmic censorship and hoop conjecture, among other important issues, and represent a natural tool to seek the physics that lies behind the two independent parameters in Levi–Civita metric.

The purpose of this work is twofold. On the one hand we would like to present systematically the field equations as well as all regularity and junction conditions required to ensure the correct behaviour of a source of a cylindrically symmetric spacetime (Levi–Civita). On the other hand we want to bring out the relationship between the Weyl tensor and different aspects of the source. This last question is in turn motivated by the very conspicuous link existing in the spherically symmetric case between the Weyl tensor, the inhomogeneity of the energy density and the anisotropy of pressure [4].

The paper is organized as follows: in section 2 we present the general form of the energy momentum tensor, the line element, the Einstein equations, the active gravitational mass and the Weyl tensor. The exterior space–time as well as junction and regularity conditions are discussed in section 3. In section 4 the consequences derived from the condition of conformal flatness are obtained. The non existence of conformally flat models satisfying Darmois conditions is given in section 5. Finally, some conclusions are presented in the last section.

2 Interior spacetime

We consider a static cylindrically symmetric anisotropic non–dissipative fluid bounded by a cylindrical surface Σ and with energy momentum tensor given by

$$T_{\alpha\beta} = (\mu + P_r)V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_\phi - P_r)K_\alpha K_\beta + (P_z - P_r)S_\alpha S_\beta, \quad (1)$$

where, μ is the energy density, P_r, P_z and P_ϕ are the principal stresses and V_α, K_α and S_α are vectors satisfying

$$V^\alpha V_\alpha = -1, \;\; K^\alpha K_\alpha = S^\alpha S_\alpha = 1, \;\; V^\alpha K_\alpha = V^\alpha S_\alpha = K^\alpha S_\alpha = 0. \tag{2}$$

We assume for the interior metric to Σ the general static cylindrically symmetric which can be written

$$ds^2 = -A^2 dt^2 + B^2(dr^2 + dz^2) + C^2 d\phi^2, \tag{3}$$

where A, B and C are all functions of r. To represent cylindrical symmetry, we impose the following ranges on the coordinates

$$-\infty \leq t \leq \infty, \;\; 0 \leq r, \;\; -\infty < z < \infty, \;\; 0 \leq \phi \leq 2\pi. \tag{4}$$

We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = z$ and $x^3 = \phi$ and we choose the fluid being at rest in this coordinate system, hence from (2) and (3) we have

$$V_\alpha = -A\delta^0_\alpha, \;\; S_\alpha = B\delta^2_\alpha, \;\; K_\alpha = C\delta^3_\alpha. \tag{5}$$

For the Einstein field equations, $G_{\alpha\beta} = \kappa T_{\alpha\beta}$ with (1), (3) and (5) we have the non null components

$$G_{00} = -\left(\frac{A}{B}\right)^2 \left[\left(\frac{B'}{B}\right)' + \frac{C''}{C}\right] = \kappa\mu A^2, \tag{6}$$

$$G_{11} = \frac{A'}{A}\frac{C'}{C} + \left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{B'}{B} = \kappa P_r B^2, \tag{7}$$

$$G_{22} = \frac{A''}{A} + \frac{C''}{C} + \frac{A'}{A}\frac{C'}{C} - \left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{B'}{B} = \kappa P_z B^2, \tag{8}$$

$$G_{33} = \left(\frac{C}{B}\right)^2 \left[\frac{A''}{A} + \left(\frac{B'}{B}\right)'\right] = \kappa P_\phi C^2, \tag{9}$$

where the primes stand for differentiation with respect to r. Since we have four equations for seven unknown functions, three additional constraints (e.g. equations of state) should be given in order to uniquely determine a solution.

There are two compact expressions that can be obtained from (7-9),

$$\kappa(P_r + P_z)B^2 = \frac{(AC)''}{AC}, \tag{10}$$

$$\kappa(P_z - P_\phi)B^2 = \frac{h''}{h} + \left(\frac{A'}{A} + \frac{B'}{B}\right)\frac{h'}{h}, \tag{11}$$

where

$$h = \frac{C}{B}. \tag{12}$$

The conservation equation, $T^{\beta}_{r;\beta}=0$, with (1) and (3) becomes

$$(\mu+P_r)\frac{A'}{A}+P_r'+(P_r-P_z)\frac{B'}{B}+(P_r-P_\phi)\frac{C'}{C}=0, \tag{13}$$

which can substitute any of the independent field equations (6-9).

The Whittaker formula [5] for the active gravitational mass per unit length m of a static distribution of perfect fluid with energy density μ and principal stresses P_r, P_z and P_ϕ inside a cylinder of surface Σ is

$$m=2\pi\int_0^{r_\Sigma}(\mu+P_r+P_z+P_\phi)\sqrt{-g}dr, \tag{14}$$

where g is the determinant of the metric. Now substituting (3) and (6-9) into (14) we obtain

$$m=\frac{4\pi}{\kappa}\int_0^{r_\Sigma}\left(\frac{A''}{A'}+\frac{C'}{C}\right)A'C\ dr, \tag{15}$$

which can be recast into the simpler form

$$m=\frac{4\pi}{\kappa}\int_0^{r_\Sigma}(A'C)'dr. \tag{16}$$

The spacetime (3) has the following non-null components of the Weyl tensor $C_{\alpha\beta\gamma\delta}$

$$C_{1212}=-\left(\frac{B^2}{AC}\right)^2C_{0303}=\frac{B^2}{6}\left[\frac{A''}{A}-2\left(\frac{B'}{B}\right)'+\frac{C''}{C}-2\frac{A'}{A}\frac{C'}{C}\right], \tag{17}$$

$$C_{1313}=-\left(\frac{C}{A}\right)^2C_{0202}$$

$$=\frac{(C)^2}{6}\left[\frac{A''}{A}+\left(\frac{B'}{B}\right)'-2\frac{C''}{C}-3\left(\frac{A'}{A}-\frac{C'}{C}\right)\frac{B'}{B}+\frac{A'}{A}\frac{C'}{C}\right], \tag{18}$$

$$C_{2323}=-\left(\frac{C}{A}\right)^2C_{0101}$$

$$=\frac{(C)^2}{6}\left[-2\frac{A''}{A}+\left(\frac{B'}{B}\right)'+\frac{C''}{C}+3\left(\frac{A'}{A}-\frac{C'}{C}\right)\frac{B'}{B}+\frac{A'}{A}\frac{C'}{C}\right]. \tag{19}$$

We obtain from (17-19)

$$\left(\frac{C}{B}\right)^2C_{1212}+C_{1313}+C_{2323}=0, \tag{20}$$

hence we have only two independent components of the Weyl tensor for (3).

3 Exterior spacetime and junction conditions

For the exterior spacetime of the cylindrical surface Σ, since the system is static, we take the Levi-Civita metric [1],

$$ds^2=-a^2\rho^{4\sigma}dt^2+b^2\rho^{4\sigma(2\sigma-1)}(d\rho^2+dz^2)+c^2\rho^{2(1-2\sigma)}d\phi^2, \tag{21}$$

where a, b, c and σ are real constants. The coordinates t, z and ϕ in (21) can be taken the same as in (3) and with the same ranges (4). The radial coordinates in (3) and (21), r and ρ, are not necessarily continuous on Σ as we see below by applying the junction conditions. The constants a and b can be removed by scale transformations, while c cannot be transformed away if we want to preserve the range of ϕ in (21) [6]. The constant σ represents the Newtonian mass per unit length. (For a discussion of the number of constants in cylindrical spacetimes see [7, 8, 9].)

In accordance with the Darmois junction conditions [10], we suppose that the first fundamental form which Σ inherits from the interior metric (3) must be the same as the one it inherits from the exterior metric (21); and similarly, the inherited second fundamental form must be the same. The conditions are necessary and sufficient for a smooth matching without a surface layer.

The equation of Σ, for the interior and exterior spacetimes, can be written respectively as

$$f(r) = r - r_\Sigma = 0, \quad g(\rho) = \rho - \rho_\Sigma = 0, \tag{22}$$

where r_Σ and ρ_Σ are constants. From (22) we can calculate the continuity of the first and second fundamental forms, and we obtain,

$$A_\Sigma = a\rho_\Sigma^{2\sigma}, \quad B_\Sigma = b\rho_\Sigma^{2\sigma(2\sigma-1)}, \quad C_\Sigma = c\rho_\Sigma^{1-2\sigma} \tag{23}$$

$$\left(\frac{A'}{A}\right)_\Sigma = \frac{2\sigma}{\rho_\Sigma}, \quad \left(\frac{B'}{B}\right)_\Sigma = \frac{2\sigma(2\sigma-1)}{\rho_\Sigma}, \quad \left(\frac{C'}{C}\right)_\Sigma = \frac{1-2\sigma}{\rho_\Sigma}. \tag{24}$$

Considering (7) on the surface Σ and substituting into the junction conditions (22) we obtain

$$P_{r\Sigma} = 0, \tag{25}$$

as expected.

The Whittaker mass per unit length (16) after integration and using the junction conditions (23) and (24) becomes

$$m = \frac{4\pi}{\kappa}\left[2ac\sigma - (A'C)_0\right], \tag{26}$$

where the index 0 means the quantity evaluated at the axis of the mass distribution.

Next, regularity conditions on the the axis of symmetry imply [11]

$$B'(0) = A'(0) = C(0) = C''(0) = 0, \quad C'(0) = B(0) = 1, \tag{27}$$

hence, considering the gravitational coupling constant $G = 1$ then $\kappa = 8\pi$ and (26) reduces to

$$m = ac\sigma. \tag{28}$$

4 Conformally flat interior

The conformally flat condition imposes the vanishing of all Weyl tensor components, hence from (17-20) we have

$$S' + S^2 - \frac{2h'}{h}S + \frac{h''}{h} = 0, \tag{29}$$

$$S' + S^2 + \frac{h'}{h}S - \frac{2h''}{h} = 0, \tag{30}$$

where

$$S = \frac{A'}{A} - \frac{B'}{B}. \tag{31}$$

Then it follows

$$h'S - h'' = 0, \tag{32}$$

$$S' + S^2 - \frac{h''}{h} = 0, \tag{33}$$

which produces

$$h''' - \frac{h''h'}{h} = 0. \tag{34}$$

Let us now examine the two possible cases, $h' \neq 0$ and $h' = 0$

4.1 Case $h' \neq 0$

We obtain from (34) after integration

$$h = a_1 \exp(a_2 r) + a_3 \exp(-a_2 r), \tag{35}$$

where a_1, a_2 and a_3 are integration constants with the condition that

$$h^2 \geq 4a_1 a_3. \tag{36}$$

However, regularity conditions on the axis (27) require

$$a_1 = -a_3, \tag{37}$$

and (59) reduces to

$$h = a_1 \sinh(a_2 r), \tag{38}$$

where a_1 was redefined.

Substituting (38) into (32) and integrating we have

$$A = a_3 \cosh(a_2 r) B, \tag{39}$$

where a_3 is another integration constant.

Thus, conformal flatness reduce the total number of unknown functions by two, through (38) and (39). However, since the total number of variables is seven, we still need one condition in order to determine a solution uniquely. So, let us consider the three different cases of isotropy.

i) $P_z = P_\phi$

Then we obtain from (8), (9) and (12)

$$\frac{h''}{h} + \frac{h'}{h}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0, \tag{40}$$

which together with (32) yields that $A' = 0$, this in turn implies, because of (39) and assuming without lost of generality $A = 1$,

$$B = \frac{1}{\cosh(a_2 r)}, \tag{41}$$

where we chose $a_3 = 1$ to satisfy (27). Feeding back (38), (39) and (41) into (6–9) we obtain

$$P_r = P_z = P_\phi = -\frac{\mu}{3} = -\frac{a_2^2}{\kappa}. \tag{42}$$

Thus the solution represents an incompressible cylinder with isotropic (negative) stresses.

ii) $P_r = P_z$

From (7) and (8), we have

$$\frac{A''}{A} + \frac{B''}{B} - 2\left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{B'}{B} = 0, \tag{43}$$

and with (38) and (39) we obtain the equation for B,

$$\frac{B''}{B} - 2\left(\frac{B'}{B}\right)^2 + a_2^2 = 0, \tag{44}$$

and by choosing the integration constants to satisfy (27) its solution is

$$B = \frac{1}{\cosh(a_2 r)}. \tag{45}$$

From (39) and (45) and assuming $A = 1$ this case yields the same solution as the preceding one.

iii) $P_r = P_\phi$.

From (7) and (9), it follows

$$\frac{A''}{A} + \frac{B''}{B} - 2\left(\frac{B'}{B}\right)^2 - 2\frac{A'}{A}\frac{B'}{B} - \frac{h'}{h}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0, \tag{46}$$

and substituting into it (38) and (39) leads to the equation for B,

$$\frac{B''}{B} - 2\left(\frac{B'}{B}\right)^2 - a_2 \coth(a_2 r)\frac{B'}{B} = 0, \tag{47}$$

which has the solution satisfying the regularity conditions (27)

$$B = \frac{1}{a_4[\cosh(a_2 r) - 1] + 1}, \tag{48}$$

where a_4 is an integration constant. Then substituting (48) into (39) we get

$$A = \frac{a_3 \cosh(a_2 r)}{a_4[\cosh(a_2 r) - 1] + 1} \tag{49}$$

Using field equations (6-9) together with (38), (48) and (49), we can obtain the expressions for the physical variables, which are

$$\kappa\mu = 2a_2^2 a_4 \left[(1 - a_4)\cosh(a_2 r) + a_4 - 3\right] + 3a_2^2, \tag{50}$$

$$\kappa P_r = \kappa P_\phi = 2a_2^2 a_4 \left[(a_4 - 1)\tanh(a_2 r)\sinh(a_2 r) + 1\right] - a_2^2, \tag{51}$$

$$\kappa P_z = 2a_2^2 a_4 \left[\frac{1 - a_4}{\cosh^2(a_2 r)} + a_4 + 1\right] - a_2^2. \tag{52}$$

Observe that in this case the matter distribution is not completely isotropic in the stresses and the energy density is not homogeneous.

4.2 Case $h' = 0$

Then we have from (33)

$$S = \frac{1}{b_1 + r} \tag{53}$$

where b_1 is an integration constant. Using (53) in (31), we obtain after integration

$$A = Bb_2(b_1 + r), \tag{54}$$

where b_2 is another integration constant. However regularity conditions (27) imply from (55) that $A = 0$, which is obviously unacceptable.

So far we have only assumed the spacetime to be conformally flat at the interior, and regularity conditions to be satisfied. However as it can be easily checked, neither of the models above satisfy the Darmois conditions (23-24). As a matter of fact, and as it will be shown in the next section, there is no conformally flat interior solutions satisfying Darmois (and regularity) conditions.

5 Non existence of conformally flat solution satisfying Darmois conditions

As we have seen if the cylinder has a matter content that is conformally flat and satisfies regularity conditions on the axis then,

$$h = a_1 \sinh(a_2 r), \tag{55}$$

if $h' \neq 0$.

Now considering the junction conditions (23) and (24) we obtain from (55)

$$a_1 = \frac{c\rho_\Sigma^{1-4\sigma^2}}{b\sinh(a_2 r_\Sigma)}, \tag{56}$$

$$a_2 r_\Sigma \coth(a_2 r_\Sigma) = 1 - 4\sigma^2. \tag{57}$$

Since always $a_2 r_\Sigma \coth(a_2 r_\Sigma) > 1$ then the condition (57) can never be satisfied.

But if $h' = 0$, then we have from (23) and (24)

$$\sigma = \frac{1}{2}, \quad h = \frac{c}{b}. \tag{58}$$

When $\sigma = 1/2$ there are strong evidences that the spacetime has plane symmetry [9, 11, 12, 13].

Hence we can state that *any static cylindrical source matched smoothly to the Levi-Civita spacetime does not admit conformally flat solution.*

6 Conclusions

We have deployed the equations describing the static cylinder, as well as the regularity and matching conditions. Then the consequences derived from the assumption of conformal flatness were obtained. It was shown that there exist no interior conformally flat solution which satisfies regularity conditions and matches smoothly to Levi–Civita spacetime on the boundary surface. Of course if we relax Darmois conditions and allow for the existence of shells at the boundary surface, the latter conclusion does not hold.

It was also shown that the conformally flat, isotropic (in the stresses) cylinder is necessarily incompressible (μ =constant). Inversely, since the solution for the incompressible isotropic cylinder is unique (there are four equations for four variables) then it is clear that such solution is also conformally flat.

So, if we look for an incompressible cylinder matching smoothly to Levi–Civita (hence not conformally flat), we have to relax the condition of isotropy in the stresses . Thus for example one could assume μ =constant, $P_z = P_\phi \neq P_r$, then we can integrate (11) to obtain

$$ABh' = c_1, \tag{59}$$

where c_1 is an integration constant. By considering junction conditions we get

$$c_1 = ac(1 - 4\sigma^2). \tag{60}$$

From (28) and (60) it follows that as $\sigma \rightarrow 1/2$, $m \rightarrow \infty$. This result gives further evidence that the spacetime at this limit for σ has plane symmetry [9, 11, 12, 13]. Of course to fully specify a solution another condition has to be given.

Finally it is worth noting the diferences and the similarities between this case and the the spherically symmetric situation. For spherical symmetry there is only one independent component of the Weyl tensor, while for cylindrical symmetry there are two independent components. For spherical symmetry the conditions of incompressibility and isotropic pressure lead also to a unique solution, the interior Schwarzschild solution, which is conformally flat [14], however unlike our present case, that solution can be matched smoothly on the boudary surface to the exterior solution. If the condition of isotropic pressure is relaxed in the spherically symmetric case, conformally flat solutions matching smoothly to Schwarzschild spacetime exist, but are not incompressible [15]. The same happens in the cylindrically symmetric case with $P_r = P_\phi \neq P_z$, however in this case the solution does not satisfy Darmois conditions.

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