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RIASSUNTO

Continuando un lavoro precedente, determiniamo la forma delle sezione d'urto di produzione $\sigma_n(s)$ che soddisfano esattamente le leggi empiriche note alle alte energie per il primo, secondo e terzo momento. Il risultato è ottenuto per mezzo di una equazione differenziale lineare del second'ordine per $\sigma_n(s)$ che permette di calcolare esplicitamente tutti i momenti successivi. In particolare, il quarto momento è in ottimo accordo con i dati. La soluzione asintotica dell'equazione per $\sigma_n(s)$ è data in forma analitica mentre la soluzione completa è studiata in forma numerica e confrontata con il diagrama dello scaling KNO che risulta essere una proprietà della soluzione asintotica. Non visono parametri liberi nel confronto con i dati ad eccezione di un parametro di normalizzazione. Come ci si aspetta, lo scaling alla KNO si istituisce rapidamente al crescere di n e l'acordo con i dati diventa sempre migliore. Tale accordo è ottimo per l'intero intervallo di valori di $n/\langle n\rangle$ per cui esistono dati $(0 \leq n/\langle n\rangle \lesssim 4)$ al crescere di n ma già per $n \sim 3$ è molto buono fino ad $n/\langle n\rangle \simeq 2$. Risulta che la soluzione asintotica data analiticamente à un'eccellente approssimazione al dati e può essere usata in calcoli pratici invece della soluzione completa per $\sigma_n(s)$.

SUMMARY

As a continuation of previous work, we determine hadronic production cross sections $\sigma_n(s)$ satisfying exactly the high energy empirical laws known for the first, second and third multiplicity moments. The result is obtained in the form of a second order linear differential equation for $\sigma_n(s)$ which allows one to calculate explicitely all successive moments. In particular, the fourth moment is in excellent agreement with the data. The asymptotic solution of the equation for $\sigma_n(s)$ is given analytically. KNO scaling turns out to be an asymptotic property of our solution. The full solution for $\sigma_n(s)$ is studied numerically and the KNO plot is compared with the data. No free parameters are left to be adjusted except for an overall normalization constant. As expected, KNO scaling sets in rather quickly with increasing n and the agreement with the data is progressively good. This agreement becomes excellent for the whole interval of $\frac{n}{\langle n\rangle}$ for which data exist $(0<\frac{n}{\langle n\rangle}\lesssim 4)$ as n becomes larger but already for $n\sim 3$ the agreement is very good up to $\frac{n}{\langle n\rangle} \simeq 2$. It turns out that the asymptotic solution, given in analytic terms is an excellent approximation to the data and can thus be used for practical purposes instead of the full solution for calculating $\sigma_n(s)$.

1. Introduction

Recently [1], a method of investigating hadronic production cross sections $\sigma_n(s)$ has been proposed which automatically satisfies the empirical constraints known from high energy data on the first and second distribution moments (multiplicity and dispersion). The Ansatz in Ref.[1] consisted in assuming for $\sigma_n(s)$ a factorized form of the type

$$\sigma_n(s) = c_n \, g(s) \, f^n(s). \qquad (1.1)$$

Demanding that this form satisfies the phenomenological inputs known to be valid at high energies on both the multiplicity and dispersion (Wroblewski's law [2] i.e. [3])

$$<n> = \alpha\sigma_{in}(s) \qquad (\sigma_{in} \equiv \sum_{n=1} \sigma_n(s); \; \alpha = \text{const.}) \qquad (1.2)$$

and

$$D \equiv \sqrt{<n^2>-<n>^2} = A(<n>-1) \qquad (A \simeq 0.56 \div 0.57) \qquad (1.3)$$

one is led to the unique form

$$\sigma_n(s) = \frac{(1/A^2)^{1/A^2}}{\Gamma(1/A^2)} \, \frac{\sigma_{in}(s)}{<n>} \left(\frac{n}{<n>}\right)^{1/A^2-1} e^{-\frac{n}{A^2<n>}} \qquad (1.4)$$

which satisfies KNO scaling[4]. As it can be checked by direct integration, eq. (1.4) is the solution of the linear first order differential equation

$$a_0(x)\ \frac{d\sigma_n(x)}{dx} + b_0(x)\ \sigma_n(x) = n\ \sigma_n(x) \tag{1.5}$$

where

$$a_0(x) = \frac{A^2}{\alpha}(\alpha x-1)^2$$
$$b_0(x) = \alpha x - A^2\frac{(\alpha x-1)^2}{\alpha x} \quad . \tag{1.6}$$

In (1.5), instead of the energy s , we have introduced

$$x \equiv \sigma_{in}(s) = \sum_{n=1} \sigma_n(x) \tag{1.7}$$

as the independent variable ([5]).

Eq. (1.5) can be used to calculate directly the successive moments to compare with the data. Good data exists (up to about $p_{lab} \simeq 300$ GeV/c) for the third and fourth moments $<n^3>$ and $<n^4>$. More exactly, the third order normalized cumulant (or "skewness")

$$\gamma_3 = <(n-<n>)^3>/<n>^3 \tag{1.8}$$

and the fourth order one ("kurtosis" or "excess")

$$\gamma_4 = [<(n-<n>)^4-3<(n-<n>)^2>^2]/<n>^4 \tag{1.9}$$

are known ([6]) to be nearly constant (up to about $p_{lab} \simeq 300$ GeV/c) and small. Specifically, we have $\gamma_{3exp} \simeq 0.080 \pm 0.015$ and $\gamma_{4exp} \simeq 0.015 \pm 0.015$ while the successive normalized cumulants are essentially zero.

The higher moments calculated with the solution of Ref.([1]) using eq.(1.4) or (1.5) turned out to be rather small but fairly larger than required by the data (i.e. $\gamma_3 \simeq 0.12$ and $\gamma_4 \simeq 0.09$). This is an indication that the power-like Anzatz (1.1) is too simple-minded as to be able to account also for higher moments while it can be made satisfy exactly the first two. In particular, from the physical meaning of γ_3 and γ_4, we can argue that the distribution (1.4) is too wide on the positive side of the mean and too spread out as compared with the actual one.

In this paper, we try to retain the nice features of the approach of Ref.([1]) leading to a compact form for higher moments and to a simple expression for the asymptotic production cross sections while improving on its numerical yields.

This is done by adding the skewness γ_3 as an explicit constraint and generalizing accordingly the previous Ansatz (in the differential form (1.5)) as to accomodate this new constraint. We are thus led (Sec.2) to a linear second order differential equation for $\sigma_n(s)$ (rather than a first order one) whose asymptotic solution for $\langle n\rangle/n \gg 1$ is given in closed form (Sec.3) while the full solution is studied numerically (Sec.4). The fourth order moment is, again, evaluated explicitely (Sec. 5) and we find that its numerical value is reduced down to (at $p_{lab} \simeq 300$ GeV/c)

$$\gamma_4 \simeq 0.0033 \div 0.027 \qquad (1.10)$$

according to whether one chooses $A \simeq 0.56$ or $A \simeq 0.57$. It is rather staggering that a percent variation of A should make γ_4 vary within a factor ten. The relevant point, however, is that γ_4 is now very small and fully compatible (within the error bars) with the experimental value previously quoted. Both the asymptotic solution of the equation for $\sigma_n(s)$ and the full solution are compared with the data on the KNO form (Sec.4). No parameters are to be adjusted here except for an overall normalization constant and the agreement in both cases is outstanding.

The method could, in principle, be further generalized to include as a constraint also the fourth moment but this would lead to a very cumbersome third order linear differential equation for $\sigma_n(x)$ while it could not improve much on the results we obtain.

The approach is very simple and gives so good results as to be a handy tool for numerical calculations and for comparison with physical models. We plan to discuss in future work to what extent our result may be non unique in the sense of the Stiltjes theory of moments.

2. The equation for $\sigma_n(s)$

The simplest generalization of the Ansatz of Ref.([1]) in its differential form (1.5) is given by

$$a(x)\,\frac{d\sigma_n(x)}{dx} + b(x)\,\sigma_n(x) + \int_d^x c(y)\,\sigma_n(y)dy = n\sigma_n(x) \tag{2.1}$$

where we use

$$x \equiv \sigma_{in}(s) = \sum_{n=1} \sigma_n(x) \qquad (2.2)$$

as the independent variable of the problem. Later on, we shall find it more convenient to use the KNO variable

$$Z = \frac{<n(s)>}{n} \qquad (2.3)$$

in terms of which we shall have asymptotically ([4]) (for large <n>)

$$\sigma_n(s) \simeq \frac{\sigma_{in}(s)}{<n(s)>} \psi \left(\frac{n}{<n(s)>} \right) \qquad (2.4)$$

The functions a(x), b(x) and c(x) in (2.1) will be determined by the high energy phenomenological constraints that we will impose. The solution of Ref.([1]) will be recovered in the case c(x)=0. The lower limit of integration in (2.1) will be assumed to be constant and determined later on but for all practical purposes it can be taken =0 when working at large x.

The empirical high energy constraints on the first, second and third moments <n>, $<n^2>$ and $<n^3>$ will be written as

$$\sum_{n=1} n\sigma_n(s) \equiv <n>\sigma_{in} \equiv \alpha(x) = \alpha x^2 \qquad (2.5)$$

$$\sum_{n=1} n^2\sigma_n(s) \equiv <n^2>\sigma_{in} \equiv \beta(x) = \alpha^2 x^3 + A^2 x(\alpha x-1)^2 \qquad (2.6)$$

and

$$\sum_{n=1} n^3\sigma_n(s) \equiv \langle n^3\rangle\ \sigma_{in} \equiv \gamma(x) = (\gamma+1)\alpha^3x^4+3A^2\alpha x^2(\alpha x-1)^2 \tag{2.7}$$

where, according to eqs.(1.3) and (1.8) and to the experimental value quoted for the skewness γ_3, we shall use the numerical values (2,6)

$$\begin{aligned} A &= 0.56 \div 0.57 \\ \gamma &= 0.08 \end{aligned} \tag{2.8}$$

while the dimensional proportionality constant α will never come into play.

If we now differentiate (2.1) with respect to x, we find

$$a(x)\ \frac{d^2\sigma_n}{dx^2} + B(x)\ \frac{d\sigma_n}{dx} + C(x)\sigma_n = n\ \frac{d\sigma_n}{dx} \tag{2.9}$$

where we have defined

$$\begin{aligned} B(x) &= b(x) + \frac{da(x)}{dx} \\ C(x) &= c(x) + \frac{db}{dx} \end{aligned} \tag{2.10}$$

We now sum eq.(2.9) over n upon having multiplied it respectively by 1, n and n^2 and we find the following system of linear algebraic equations for a(x), B(x) and C(x)

$$\begin{aligned} B(x)+ xC(x) &= \alpha_x'(x) \\ \alpha_x''(x)\ a(x)+ \alpha_x'(x)\ B(x)+ \alpha(x)\ C(x) &= \beta_x'(x) \\ \beta_x''(x)\ a(x)+ \beta_x'(x)\ B(x)+ \beta(x)\ C(x) &= \gamma_x'(x) \end{aligned} \tag{2.11}$$

where the coefficients $\alpha''_x(x),\ldots\gamma'_x(x)$ (denoting second and first order derivatives of $\alpha(x)$, $\beta(x)$, $\gamma(x)$ with respect to x) follow from the constraint equations (2.5-7).Substituting their functional form and solving the system (2.11) with the usual Cramer's method, we obtain

$$a(x)=\frac{\alpha x^2}{A^2+1}(2\gamma+A^2-3A^4)+\frac{xA^2}{A^2+1}(7A^2-2)+\frac{A^2}{\alpha(A^2+1)}(1-5A^2)+\frac{A^4}{\alpha^2x(A^2+1)}$$

$$B(x)=\frac{\alpha x}{A^2+1}(1-4\gamma+2A^2+9A^4)-\frac{18A^4}{A^2+1}+\frac{A^2}{\alpha x}\frac{11A^2-1}{A^2+1}-\frac{2A^4}{\alpha^2x^2(A^2+1)}$$

$$C(x)=\frac{\alpha}{A^2+1}(1+4\gamma-9A^4)+\frac{18A^4}{A^2+1}\frac{1}{x}-\frac{A^2}{\alpha x^2}\frac{11A^2-1}{A^2+1}+\frac{2A^4}{\alpha^2x^3(A^2+1)}$$

(2.12)

or,using eq.(2.10), we find for b(x) and c(x)

$$b(x)=\frac{\alpha x}{A^2+1}(1-8\gamma+15A^4)+\frac{A^2}{A^2+1}(2-25A^2)+\frac{A^2}{\alpha x}\frac{11A^2-1}{A^2+1}-\frac{A^4}{\alpha^2x^2(A^2+1)}$$

(2.13)

$$c(x)=\frac{12\alpha}{A^2+1}(\gamma-2A^4)+\frac{18A^4}{A^2+1}\frac{1}{x}$$

Notice, incidentally, that when $x>>1$, c(x) reduces to zero when $\gamma=2A^4$ which was, indeed, the asymptotic value $x\to\infty$ found for $\gamma_3(x)$ in Ref.([1]).

With the explicit functional form of the coefficients a(x), B(x) and C(x), we can now study the properties of $\sigma_n(x)$ as solution of the differential equation (2.9). Before doing so,we will still determine the lower limit of integration d in the integral term in (2.1). It should, however, be realized that the small x behavior is in principle devoided of physical significance in our approach since the constraint equations we have used (2.5-7) are valid in the large x limit. Any coincidence of our approach with the physical picture in the low x domain is, therefore, to be considered, at the very best, as an extra bonus or rather, as a manifestation of the precocious setting of KNO scaling.

To determine d in (2.1), we notice that summing eq.(2.1) over n, we get

$$\langle n\rangle x - a(x) - xb(x) = \int_d^x c(y)y\,dy \qquad (2.14)$$

which, upon using eqs.(2.12,13) and integrating, gives for αd the algebraic equation

$$(\alpha d)^2(\gamma - 2A^4) + 3A^4\alpha d - A^4 = 0 \qquad (2.15)$$

whose positive root, using (2.8) is

$$\alpha d \simeq 0.40 \qquad (2.16)$$

almost independent of the numerical value (2.8) used for A .

The consistency of the value thus obtained for d can be checked from

$$x\langle n^2\rangle - \alpha_x'(x)\ a(x) - \alpha(x)\ b(x) = \int_d^x c(y)\ \alpha(y)\ dy \qquad (2.17)$$

which obtains multiplying (2.1) by n and summing over n. The new equation for αd has one root (1/3) practically coincident with (2.16).

3. Study of the asymptotic differential equation for $\sigma_n(x)$.

Let us now consider the second order linear differential equation (2.9) which we have found for $\sigma_n(x)$.

Given the complicated form of the coefficients a(x), B(x) and C(x), it is clear that the full solution of (2.9) cannot be given in closed form. For this reason, $\sigma_n(x)$ is studied numerically in Sec.4.

To get an idea of the properties of $\sigma_n(x)$ we first of all notice that eq.(2.9) has not singular points (i.e. zeros of a(x)) for finite positive x values. Thus, the only singular points of eq.(2.9) for non negative x values are x=0 and $x=\infty$. As the constraint equations are valid in the large x domain, if we take the asymptotic limit $x\to\infty$ on (B(x)-n)/a(x) and on C(x)/a(x) and we obtain from (2.9)

$$z^2\delta \frac{d^2\sigma_n^{(as)}(z)}{dz^2} + (\varepsilon z-1)\frac{d\sigma_n^{(as)}(z)}{dz} + \eta\, \sigma_n^{(as)}(z) = 0 \qquad (3.1)$$

In the above equation $z=\frac{<n>}{n}$ is the KNO variable previously introduced (2.3) and δ, ε and η are constant coefficients given by

$$\delta = \frac{2\gamma+A^2-3A^4}{A^2+1} \simeq 0.137 \div 0.127$$

$$\varepsilon = \frac{1+2A^2+9A^4-4\gamma}{A^2+1} \simeq 1.667 \div 1.722 \qquad (3.2)$$

$$\eta = \frac{1+4\gamma-9A^4}{A^2+1} \simeq 0.333 \div 0.278$$

The above numerical values correspond to the two choices for A= 0.56 ÷0.57 (see(2.8)). To the extent that KNO scaling sets in precociously, the solution of the asymptotic equation (3.1) should be a good approximation to the full solution of eq.(2.9). That this is indeed so will be seen in the next section when comparing with the data.

It is easy to see that if we set

$$\sigma_n^{(as)}(z) = z^{1-\varepsilon/2\delta}\, e^{-1/2\delta z}\, W_n\left(\frac{1}{\delta z}\right) \qquad (3.3)$$

eq.(3.1) converts into a confluent hypergeometric equation of the Whittaker type (7)

$$\frac{d^2W_n(t)}{dt^2} + \left\{-\frac{1}{4} + \frac{k}{t} + \frac{1/4-m^2}{t^2}\right\} W_n(t) = 0 \qquad (3.4)$$

where $t= 1/\delta z$ and

$$k = \frac{\varepsilon}{2\delta} - 1 \simeq 5.13 \div 5.78$$

$$m = \frac{1}{2\delta}\left[(\varepsilon-\delta)^2 - 4\eta\delta\right]^{1/2} \simeq 5.40 \div 6.10 \qquad (3.5)$$

Of the two independent solutions of (3.4), we choose the one which reduces to (1.4) in the case of Ref.([1]) when $\gamma=2A^4$ (in which limit, from (3.2) we have $\delta_0= A^2$, $\varepsilon_0= 1+A^2$, $\eta_0=1-A^2$) and we find

$$\begin{aligned} \sigma_n^{(as)}(z) &= K \frac{\sigma_{in}}{<n>} z^{1-\varepsilon/2\delta} e^{-1/2\delta z} W_{k,m}\left(\frac{1}{\delta z}\right) \\ &= K' \frac{\sigma_{in}}{<n>} z^{1/2-\varepsilon/2\delta-m} e^{-1/\delta z} \psi\left(a,c;\frac{1}{\delta z}\right) \end{aligned} \qquad (3.6)$$

where $\psi(a,c;y)$ is the hypergeometric function regular ([6]) at $y\simeq\infty$, and

$$\begin{aligned} a &= \frac{1}{2} + m - k \simeq 0{\cdot}777 \div 0{\cdot}823 \\ c &= 1 + 2m \simeq 11.80 \div 13.21 \end{aligned} \qquad (3.7)$$

In (3.6) K and K' are dimensionless parameters satisfying the normalization condition

$$\sum_n \sigma_n^{(as)}(z) = \sigma_{in}^{(as)} \simeq x \qquad (3.8)$$

and related by

$$K' = K\, \delta^{-1/2-m} \qquad (3.9)$$

Just as it was the case in Ref.([1]), the asymptotic solution (3.6) satisfies automatically KNO scaling as a consequence of the asymptotic equation (3.1) depending only on $z= <n>/n$ and not on $<n>$ and n separately. Furthermore, it is immediate to recover form (3.6) the solution (1.4) of Ref.([1]) in the case $\gamma = 2A^4$ from the behavior

$$\sigma_n^{(as)}(z) \underset{z\to 0}{\simeq} K\, z^{2(1-\varepsilon/2\delta)}\, e^{-1/\delta z} \tag{3.10}$$

Before giving an estimate of the constant K (or K'),we would like to point out that the asymptotic form (3.6) is indeed positive definite as one should have it for a cross section.This comes from general theorems on the zeros of confluent hypergeometric functions ([7]) stating that when both parameters a and c are positive (as it is the case from (3.7)) no zeros of $\psi(a,c;y)$ are to be found on the positive real axis. Thus, if the function is positive, say around $y=\infty$, it remains positive down to $y=0$.

To estimate the constant K in (3.6), we proceed first to convert the sum over n into an integral over a continuous variable ν to obtain with the use of(3.8)

$$K^{-1} \simeq \int_0^\infty dy\; y^{\varepsilon/2\delta-1}\, e^{-y/2\delta}\, W_{k,m}(y/\delta) \tag{3.11}$$

which gives

$$K \simeq \left(\frac{1}{\delta}\right)^{\varepsilon/2\delta} \left[\Gamma\left(\frac{\varepsilon}{2\delta}+m+\frac{1}{2}\right)\Gamma\left(\frac{\varepsilon}{2\delta}-m+\frac{1}{2}\right)\right]^{-1} \tag{3.12}$$

As the expression (3.12)([8]) is the normalization for the asymptotic solution (3.6), one can check from (3.6,12) that one gets, asymptotically, the correct form for the constraint inputs (2.5-7). This can be seen multiplying (3.6) by n, n^2, n^3 (i.e.

ν, ν^2, ν^3) and integrating it. The results coincide with the leading order expressions from eq.(2.5,6,7) at large x.

4. Numerical study of $\sigma_n(x)$

In this Section, we study the numerical solution $\sigma_n(s)$of eq.(2.9) corresponding to the asymptotic solution (3.6,12)and we compare it with the data ([9]) for the KNO universal function. To this aim, we first of all recall that the KNO form obtains in the asymptotic regime. Thus, if we rewrite (2.9) using the variable ([10])

$$y = n/\langle n\rangle \tag{4.1}$$

we see that the equation becomes independent of n for large n when (2.4) holds. It is, however, rather instructive to inspect the rate at which KNO scaling sets in. We will also plot the solution of eq.(3.1) to see what becomes the comparison with the KNO form using directly the asymptotic limit of our solution.

In Fig.1 we show the plot of eq.(3.6) as compared with the data. The agreement is staggering given that the curve is not a fit but the result of our calculation of $\sigma_n^{(as)}$ satisfying the three first moments constaints. The overall normalization constant has been adjusted at the maximum.

Fig. 2 shows the result of the curve for $\sigma_n(z)$ as obtained for various values of n from a numerical integration of equation (2.9) with the boundary conditions

$$\sigma_n(y=\frac{n}{<n>}) \underset{y\to\infty}{\simeq} O(\frac{1}{y}) \text{ or } O(\frac{1}{y^2}) \qquad (4.2)$$

which are consequence of eqs.(2.12,13).

Either of the above boundary conditions (4.2) give practically the same numerical yields and for $y \gtrsim 5$ the solution is very stable by changing the value y_0 of y at which the solution was nummerically joined with the boundary condition(4.2). For simplicity we have chosen $y_0 = 5$.

Aside from the curve n=1 (which, quite understandably cannot have any asymptotic property), already for n=3 the agreement with the asymptotic curve (i.e. with the data) is good up to $\frac{n}{<n>} \lesssim 2$. As expected, the agreement improves dramatically with increasing n(at fixed $\frac{n}{<n>}$ this means with increasing $<n>$) and becomes excellent for large n. This is a direct check of the extent to which precocious KNO scaling sets in.

5. The normalized cumulant of fourth order.

Multiplying eq.(2.1) by n^3 and summing over n, we get

$$x<n^4> = a(x)\gamma_x'(x)+b(x)\gamma(x)+\int_d^x c(y)\gamma(y)dy \qquad (5.1)$$

in terms of which, the fourth normalized multiplicity moment γ_4 (eq.(1.9)) is given by

$$\gamma_4 = [<n^4>-4<n^3><n>-3<n^2>^2+12<n^2><n>^2-6<n>^4]/<n>^4 \qquad (5.2)$$

(where all moments so far introduced (2.5-7) appear also. Using eqs.(2.5-7) and (2.12,13) it is just matter of some algebra to compute explicitely γ_4 in closed form. Neglecting terms of order $\frac{1}{\langle n\rangle^3}$ and lower (which turn out to be negligibly small) one obtains

$$\gamma_4 = \frac{3}{5(A^2+1)}\,(4\gamma^2-\gamma+12A^2\gamma-3A^4\gamma+2A^4-16A^6)+$$

$$+\frac{3}{2}\,\frac{A^2}{A^2+1}\,\frac{1}{\langle n\rangle}(23A^4+5A^2\gamma-3A^2-8\gamma)+ \tag{5.3}$$

$$+\frac{3A^2}{A^2+1}\,\frac{1}{\langle n\rangle^2}\,(\gamma+4A^2-3A^2\gamma-12A^4) + 0(\frac{1}{\langle n\rangle^3})$$

Using the numerical values (2.8) for A^2 and γ, the above formula gives

$$\gamma_4 \simeq (-0.0333+\frac{0.286}{\langle n\rangle}+\frac{0.061}{\langle n\rangle^2}) \div (-0.0162+\frac{0.349}{\langle n\rangle}+\frac{0.023}{\langle n\rangle^2}) \tag{5.4}$$

according to whether one chooses A=0.56 or A=0.57.

Working at $p_{lab}\simeq$ 300 GeV/c where $\langle n\rangle\simeq 8$, we find, approximately

$$\gamma_4 \simeq 0.0033 \div 0.027 \tag{5.5}$$

values which cover entirely the range of the experimental estimate [6] $\gamma_{4exp} \simeq 0.015 \pm 0.015$. Taking (5.4) at face value, one finds that γ_4 vanishes somewhere between $\langle n\rangle\simeq 9$ and $\langle n\rangle\simeq 20$ and that it tends to a small negative value as $\langle n\rangle\to\infty$. Both remarks are, however, of very little significance due to the large variations that occur in γ_4 as we vary A of about one percent.

6. Concluding remarks

We have shown with an explicit calculation how high energy phenomenological inputs can be used as constraints to derive hadronic production cross sections. Data on moments up to the third have been used and the fourth one is calculated and predicted right where one finds it experimentally.

The asymptotic form $\sigma^{(as)}$ has been derived and shown to obey KNO scaling and to reproduce extremely well the experimental KNO form. The extent at which KNO scaling sets in precociously has also been discussed.

It is felt that the method outlined in the paper may be an interesting way to impose asymptotic constraints on high energy production cross sections and a very handy way of explicetely constructing them.

The questions which remains open and which we hope to investigate in the future is whether the present approach can be extended to other more complicated situations and to what extent the solution we find compares with the theory of moments. Another interesting point which we plan to investigate is the conection between our present approach and the equations of the group of renormalization ([11]).

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(2) For the up to date situation, see A. Wroblewski: Proceedings of the 10^{th} Int. Symposium on Multiparticle Dynamics; Editors S.N. Ganguli, P.K. Malhotra and A. Subramanian, Goa, India (September 1979); p.191.

(3) In the paper we have used $<n> = 2\,_{in}$ whereas the data would rather require $<n> = \alpha\sigma_{in}+\beta$. Since, however, this is used as an independent variable, it can be checke that nothing would have changed in our result had we used $<n> = \alpha\sigma_{in}+\beta$ instead of $<n> = \alpha\sigma_{in}$.

(4) Z. Koba, H.B. Nielsen and P. Olesen: Nucl. Phys. B40 317(1972).

(5) The specification of the growth with energy of $\sigma_{in}(s)$ is totally immaterial for our purposes but it is generally accepted that $\sigma_{in}(s) \underset{s\to\infty}{\simeq} O(\ln^2 s)$.

(6) Z. Koba and D. Weingarten: Lettere al Nuovo Cimento 8 303(1973); A. Giovannini, P. Autich, E. Calligarich, G. Cecchet, R.Dolfini, F. Impellizzeri and S. Ratti: Nuovo Cimento 24A 421(1974).

(7) F. Tricomi: Funzioni ipergeometriche confluenti; Edizioni Cremonese-Roma 1954.

(8) Since we are ignoring the production of neutral particles in our treatment, we expect that the actual normalization factor

Kexp. should be a factor of 1.5 to 2 larger than the value (3.12) obtained theoretically.

(9) P. Slattery: Phys. Rev. Lett. <u>26</u> 1624(1972); A.J. Buras and Z. Koba: Lettere al Nuovo Cimento <u>6</u> 629(1973); E.H. de Groot: Phys. Lett. <u>B57</u> 159(1975); Z. Koba - Proceedings of the 1973 CERN-JINR School of Physics - CERN - Yellow report 73-12 (24-September (1973).

(10) In this Section, we find it preferable to use $y = n/\langle n \rangle$ as independent variable rather than $z = \langle n \rangle/n$ since it is in the former that the data are plotted to check the KNO form (see Ref. (9).

(11) W. Ernst and I. Schmitt, Nuovo Cimento <u>31A</u>, 109(1973)

FIGURES CAPTION

Fig.1 - $\sigma^{as}(y)$ as obtained from eq.(3.6) compared with the data, for A= 0.56 and y_0= 5.

Fig.2 - $\sigma_n(y)$ as obtained from numerical integration of eq.(2.9) with n= 1,3,8,15,30,50. The dashed line is the asymptotic solution shown in fig.1.

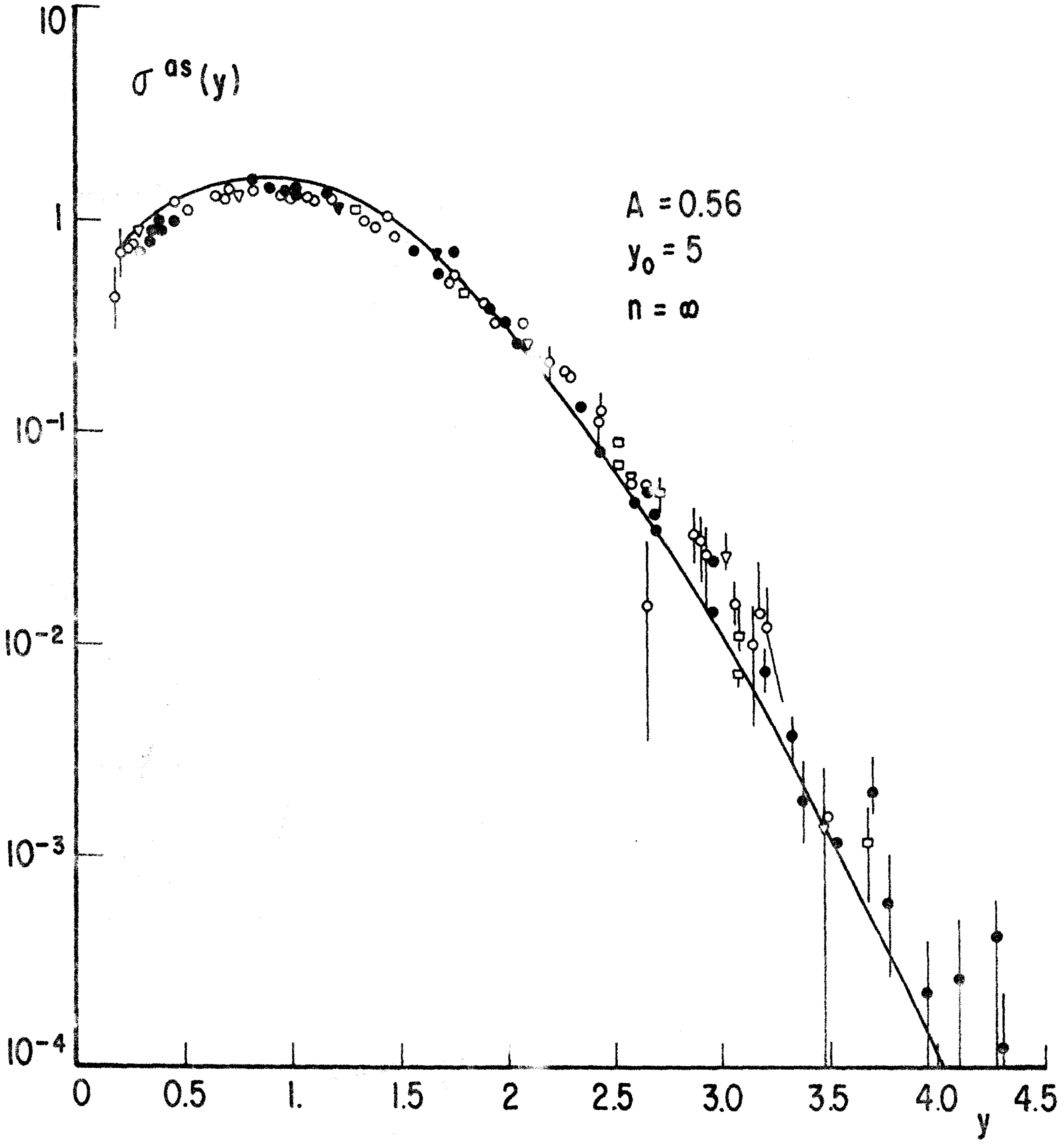
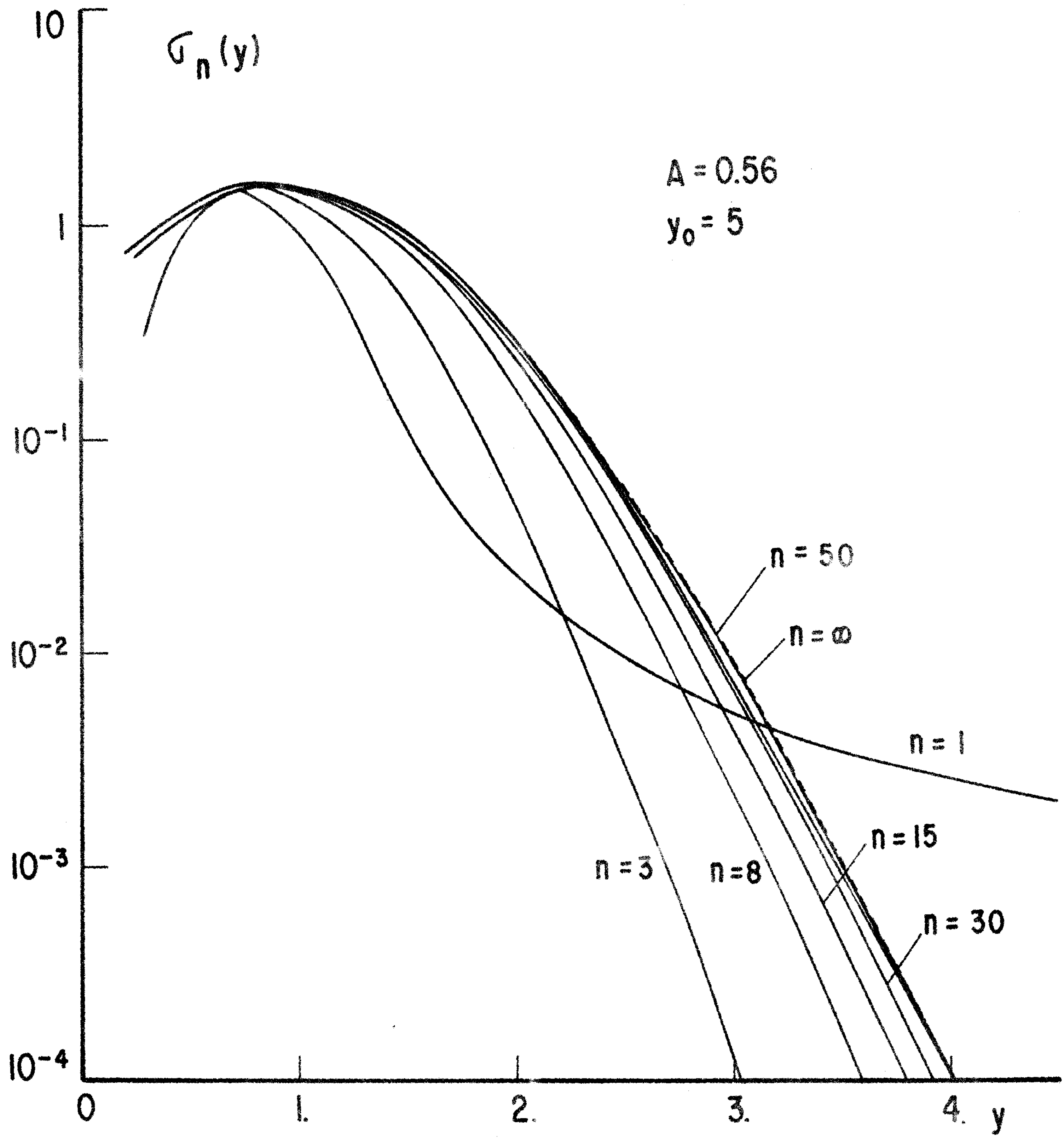


Fig.1



10
$\sigma_n(y)$
A = 0.56
y_0 = 5
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
n = 50
n = ∞
n = 1
n = 15
n = 3
n = 8
n = 30
0
1.
2.
3.
4.
y

Fig.2