From time series to superstatistics: Upgrading the criterion for evaluating long-time scales

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In this manuscript we introduce an additional condition to the criterion recently presented in this journal by C. Beck *et al.* [Phys. Rev. E **72**, 056133 (2005)] in order to extract the update scale of time of the intensive parameter, β in superstatistical time series. With this new condition the criterion previously presented turns out to be effectivelly capable of evaluating the actual long time scale. In addition it transforms the criterion into a valuable way to verify, or not, the superstatistical nature of the process under study.

Nesta Nota de Física introduz-se uma condição inicial ao critério recentemente apresentado por C. Beck *et al.* [Phys. Rev. E **72**, 056133 (2005)] para extracção da escala de actualização do parâmetro, β, em séries temporais superestatísticas. Com esta nova condição esse critério torna-se efectivamente capaz de determinar a verdadeira escala. Adicionalmente, transforma o critério numa ferramenta capaz de verificar, ou não a Natureza superestatística do processo em estudo.

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Recently, C. Beck and E.G.D. Cohen have introduced a theoretical framework in order to describe complex driven non-equilibrium systems. This framework which is named *superstatistics* [1] considers that such systems are described by the superposition of dynamics on different time scales. In other words, this kind of systems is assumed to be as a composition of smaller space-time cells in local equilibrium, thus obeying Boltzmann-Gibbs (BG) statistical mechanics. From one cell to another, an intensive parameter, β, changes its value and within each cell the value of β also changes according to a certain distribution, $p(\beta)$, after a time interval T [1]. This time scale is considered to be much larger than the local time scale related to local equilibrium. Hence, in the long-term, the probability density function associated to an observable, $\mathcal{O}$, of the non-equilibrium system comes from BG statistics associated with the small cells that are averaged over the various values of the intensive parameter,

$$P_{stationary}(\mathcal{O}) = \int P_{BG}(\mathcal{O})\, p(\beta)\, d\beta. \qquad (1)$$

As a paradigmatic example [2] we can indicate the case of a Langevin equation,

$$dv = -\gamma\, v\, dt + \sigma\, dW_t,$$

(W_t represents an ordinary Wiener process) where the σ, more precisely $\beta = \frac{\gamma}{\sigma^2}$, is associated with the inverse of the temperature. If we consider that $\beta(\sigma)$ value is updated, at each cell, on a time scale greater than the time scale γ^{-1} needed by the system to reach local equilibrium, then the long-term velocity distribution will be given by Eq. (1) where

$$P_{BG}(u) = \sqrt{\frac{\beta}{\pi}} \exp\left[-\beta\, u^2\right].$$

Specifically, if $p(\beta)$ is a Gamma (or χ^2-) distribution, $p(\beta) = \frac{1}{b\,\Gamma[c]}\left(\frac{\beta}{b}\right)^{c-1} e^{-\beta/b}$ then

$$P_{stationary}(u) \propto \left[1 + (1-q)\,\beta_0\, u^2\right]^{1/(1-q)},$$

where $\beta_0 = b\,c$ and $q = 1 + \frac{1}{c}$. Consequently, the long-term distribution maximises the non-extensive entropy proposed some years ago by C. Tsallis [3],

$$S_q = \frac{1 - \int [p(x)]^q\, dx}{q-1} \qquad (q \in \Re). \qquad (2)$$

The superstatistical framework has been successfully applied not only on the dynamical foundations of non-extensive statistical mechanics [2], but also on a wide broad of problems like interactions between hadrons from cosmic rays [4], fluid turbulence [2, 5, 6], electronics [7] and economics [8–11] among many others [12].

Taking into account its construction, two obvious questions arise in the context of superstatistics: Which is the distribution $p(\beta)$ associated with the intensive parameter? Which is the time scale, T, of evolution of β? Attempts to answer the latter question (which leads into a correct answer for the former) have been lately presented, one by analysing the self-correlation function of traded volume in financial markets [11] and other, in fact previously presented, for the case of velocity differences time series, $\Delta v_{\vec{r}}(t) = v_{\vec{r}}(t') - v_{\vec{r}}(t)$ ($\vec{r}$ represents a certain position) in a turbulent fluid which is the proposal we upgrade herein. In their proposal [5], Beck, Cohen, and Swinney (BCS) have introduced the following procedure: *(i)* Divide the time series into N equal time intervals of

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size τ (the length of the signal is $N\tau$); *(ii)* Define the local kurtosis $\kappa(\tau)$ as

$$\kappa(\tau) = \frac{1}{N\tau - \tau}\int_0^{N\tau-\tau} \frac{\left\langle (u - \langle u\rangle)^4 \right\rangle_{t_0,\tau}}{\left(\left\langle (u - \langle u\rangle)^2 \right\rangle_{t_0,\tau}\right)^2} dt_0, \quad (3)$$

where $\langle \ldots \rangle_{t_0,\tau}$ represents the average over an interval of length τ starting at t_0; *(iii)* Compute $\kappa(\tau)$ for several interval lengths τ; *(iv)* **The time scale of evolution T corresponds to the interval length τ for which $\kappa(\tau)$ corresponds to the kurtosis of the Gaussian, i.e., $\kappa(T) = 3$.**

In the sequel of this manuscript we show that the condition *(iv)*, although necessary, is not *sufficient* by itself. In addition we introduce an extra condition to turn out the BCS procedure *closed*.

Consider a time series, $x(t)$ ($\langle x\rangle = 0$ for simplicity), with a fluctuating parameter β updated at each time interval, T. Inside each interval of length T the distribution of x is Gaussian. For example, $x(t)$ might be the position at time t and β a quantity related with the variance. Another case might be the velocity difference at a certain point and the inverse temperature, respectively. For this case, the local kurtosis $\kappa(\tau)$ given by Eq. (3), evolves from $\kappa(\tau = 1) = 1$ to the value which corresponds to the kurtosis of x long-term probability density function,

$$P(x) = \int p(x|\beta)\; p'(\beta\;)\; d\beta.$$

For a satisfactory long time series of length L we have,

$$\kappa(L) \approx \frac{\int x^4\, P(x)\; dx}{\left(\int x^2\, P(x)\; dx\right)^2}.$$

Applying the local kurtosis method, $\kappa(\tau)$ increases towards the value of $\kappa(L)$, since as we augment τ, we improve the local statistics. Nevertheless, an interesting feature emergences, due to the superstatistical nature of the time series, if we change Eq. (3) by

$$\kappa(\tau) = \frac{1}{N}\sum_{i=1}^{N} \frac{\left\langle (u - \langle u\rangle)^4 \right\rangle_{i,\tau}}{\left\langle (u - \langle u\rangle)^2 \right\rangle_{i,\tau}^2}. \quad (4)$$

When we are able to divide the time series into a set of intervals which are pure Gaussians, *i.e.*, a sole β, we obtain a singularity with value 3 for the local kurtosis. However, that does not simply occur at $\tau = T$ as condition *(iv)* indicates. The equality $\kappa(\tau) = 3$ happens whenever T/τ is a integer greater than or equal to 1. *Actually, $\tau = T$, corresponds to last interval length value for which the local kurtosis has a singularity equal to* 3. This occurs because for intervals that verify

$$T/\tau = Integer\,(T/\tau) \geq 1, \quad (5)$$

we only have a replication of the number of intervals which are associated with the a intensive parameter β. For $T/\tau = 1$ we stop having this replication and thus we obtain the true long time scale T.

For $\tau = 2T, 3T\ldots, nT \leq L$ we still have singularities, but larger than 3 since they correspond to an average kurtosis of 2, 3, etc. Gaussians related to different β's. Moreover, singularities, whose value is greater than 3, also occur for other multiples and submultiples of T. Regarding that, we have a succession of singularities for the local kurtosis, which are obtained when a intervals verifies the condition

$$\tau \times r = T, \quad (6)$$

where r is a positive rational number.

In Fig. 1, we present an excerpt (upper panel) of a superstatistical time series of a random variable locally associated to a Gaussian whose variance follows the distribution

$$(\sigma) = \sqrt{\frac{2}{\pi}}\frac{\sigma}{5}\exp\left[-\frac{\sigma^4}{200}\right],$$

and $T = 250$ (σ plays the role of β for this case). As it can be seen in the lower panel, the local kurtosis presents a succession of singularities at τ verifying condition (6) and particularly the singularities related to condition (5) which equal 3 (Gaussian kurtosis). This includes the last singularity with value 3 which concurs to T. Applying the BCS criterion we would have $T = 48$ [14]. The utilisation of Eq. (4) instead of Eq. (3) is fundamental for this behaviour. Using the latter expression, since the time intervals (or windows) move continuously (thus overlapping), we always have a mixture Gaussians within the windows even for $\tau = T$. For the former expression only non-overlapping time intervals, allowing only windows of pure Gaussians when we are on the superstatistical scale of time.

The use of BCS criterion by itself, without taking into account the singularity structure of $\kappa(\tau)$ *vs.* τ, might also cause misleading conclusions about the superstatistical character of a time series. In Fig. 2, we present, a time series which is both locally and in the long-term (*inexistence of any fluctuating parameter*) associated to a q-Gaussian distribution [13]

$$P(x) = \frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2(q-1)}\right]}\sqrt{\frac{q-1}{\sigma^2\pi(5-3\,q)}}\times \left[1 + \frac{q-1}{(5-3\,q)\,\sigma^2}x^2\right]^{\frac{1}{1-q}}, \quad (7)$$

with $q = 1.3$ and $\sigma = 1$, *i.e.*, a fat tailed distribution with finite and constant variance which optimises Tsallis entropy (2) for $q = 1.3$. It is visible in Fig. 2 (lower panel) that if we apply the local kurtosis method we do not obtain the singularities structure of a superstatistical time series verified in Fig. 1 (lower panel), simply because this time series is not of superstatistical kind. The application

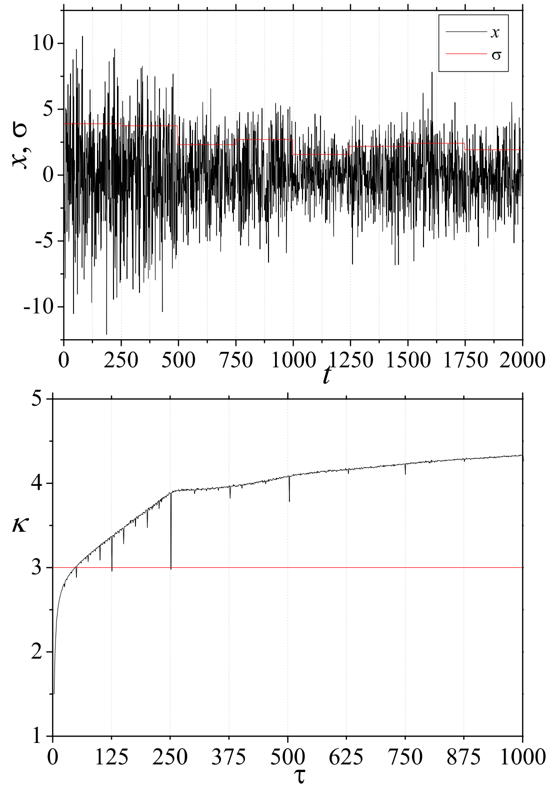

FIG. 1: (color online) Upper panel: Excerpt of a time series where the elements, x, are associated with same Gaussian distribution within a interval of length $T = 250$. The variance σ, related with intensive parameter β, varies according to the probability density function referred on the text. Lower panel: $\kappa(\tau)$ *vs.* τ. It is visible the succession of singularities in accordance with condition (6) and that the last singularity whose value is 3 coincides with the real length of the interval contrarly to first value of τ which equals 48. The singularities for small values of τ which verify condition (5) are not visible, because they are masked by the statistical effect of small τ. This effect also introduces error in the kurtosis of the singularities with $\kappa(\tau) = 3$ which are sligthly below that value.The full time series $x(t)$ has 10^7 elements.

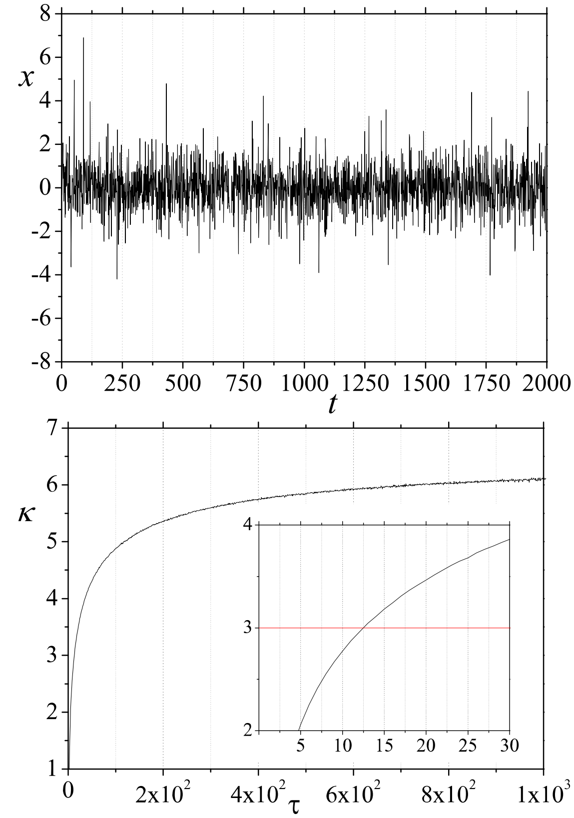

FIG. 2: (color online) Upper panel: Excerpt of a time series with 10^7 where the elements, x, associated to a non-superstatistical time series with a probability density function which is a q-Gaussian distribution with $q = 1.3$ and unitary variance. Lower panel: $\kappa(\tau)$ *vs.* τ. For this case we do not have the succesion of singularities like it appears in a superstatistical time series. The BCS condition $\kappa(T) = 3$ not only leads to a specious value of T, but also to a classification of the stochastic process as superstatistical. The inset presents the region where $\kappa(\tau)$ equals 3, $\tau \simeq 12.5$. The saturation of $\kappa(\tau = \infty, L = \infty)$ occur at $\left(\frac{15-9q}{7-5q}\right)_{q=1.3} = 6.6$.

of the BCS criterion as it stands might lead into an erroneous classification of the process as superstatical with a β-scale of 12.5 (approximately). This example emphasises the importance of our additional condition. Despite the fact that Eq. (7) is the Lagrange Transform some other function $f(\bar{\sigma})$, that does not mean that the underlying dynamics related with distribution (7) presents a superstatistical character, as we have shown.

To conclude, in this manuscript we have introduced a complementary condition, based on geometrical arguments, in Beck-Cohen-Swinney criterion to determine the intensive parameter evolution time-scale, T, for superstatistical processes. This additional condition, which closes the BCS criterion, connects the last singularity with value 3 in $\kappa(\tau)$ with the time scale T. Furthermore, we have shown that the presence/absence of a succession of singularities in the local kurtosis of a time series is fundamental to the classification of a process as superstatistical, since it represents a coarse-grained effect and hence relevant on the construction of a dynamical scenario departing from statistical aspects of a system. Last of all, let us refer that the complimentary condition introduced herein also has clear consequences in the question, "Which is the distribution associated with the intensive parameter β?". Since it brings an accurate answer to the typical time scale T, it will certainly lead to correct experimental measurements or numerical evaluations of β for natural systems and consequently to its probability density function.

Before ending, let us emphasise two points which are important in experimental applications of the method. The first one deals with the fact that, generally, the initial time for measurements does not coincide with an update of the intensive parameter. Regarding that, the method must be applied successively. At each time, the previous initial time must be neglected until the singularities profile is obtained. Last of all, it is important to refer that difficulties in find the scale of time T can happen either when this value is smaller than the minimum precision of the measurement instruments or T is smaller enough to be covered by the statistical error effect on the kurtosis computation. For these cases new methods must be, in principle, introduced.

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