CBPF-NF-017/83

FOURIER ANALYSIS AND WILSON LOOPS

by

C.G. Bollini and J.J. Giambiagi

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Xavier Sigaud, 150
22290 Rio de Janeiro, RJ - BRASIL

ABSTRACT

The quantum average of the Wilson Loop is computed through Fourier analysis of the potentials and functional integration over the coefficients. Simple results are obtained in the abelian case as well as in the $N \to \infty$ limit of the non-abelian theory

INTRODUCTION

The Wilson Loop (WL) has apparently all the relevant information contained in gauge fields. The difficulty with this non local gauge invariant quantity laies in finding equations of motion, solving and interpreting them. A fruitful and simple method is the 1/N expansion introduced by t'Hooft[1][2]. Evolution equations in this approximation have been found by Makeenko and Migdal[3][4].

It seems of interest to introduce the Fourier analysis of the Gauge potentials to see what kind of operation it induces on the WL, after performing the corresponding functional integration.

In this paper we are going to compute the WL average for non-abelian gauge theories in the $N \to \infty$ limit, also for comparaison we find that average for the Abelian case, by calculating the influence of a single frequency and heuristically performing the integration, as indicated in the text

I. Abelian Case

We shall consider a single Fourier component

$$A_\mu = a_\mu \cos k \cdot x \qquad k \cdot x = \sum_{i=1}^{n} k_i x^i \tag{1}$$

$$F_{\mu\nu} = (a_\mu k_\nu - a_\nu k_\mu) \text{ sen } k \cdot x \tag{2}$$

and, in a finite box of volume V

$$\int F^2_{\mu\nu} \, d^n x = \frac{V}{2} \, (a_\mu k_\nu - a_\nu k_\mu)^2 \tag{3}$$

As the longitudinal component of a_μ does not appear in (3) we write

$$a_\mu = a_i \varepsilon^i_\mu , \quad k^\mu \varepsilon^i_\mu = 0 \qquad \varepsilon^i_\mu \varepsilon^j_\mu = \delta_{ij} \qquad i,j = 1,2,\ldots,n-1 \tag{4}$$

so that

$$\int F^2_{\mu\nu} \, d^n x = V k^2 \sum_i a^2_i \tag{5}$$

In order to compute the WL we need

$$\oint A_\mu \, dx^\mu = \int \mathrm{sen}\, k\cdot x \, (a_\mu k_\nu - a_\nu k_\mu) \, d\sigma^{\mu\nu}$$

where Stokes theorem and form(2) have been used. Taking into account (4) we have

$$\oint A_\mu \, dx^\mu = f^i a_i \tag{7}$$

where

$$f^i = \int \mathrm{sen}\, k\cdot x (\varepsilon^i_\mu k_\nu - \varepsilon^i_\nu k_\mu) \, d\sigma^{\mu\nu} \tag{8}$$

and the domain of integration is any surface whose border is the loop.

From (5) and (7)

$$-\frac{1}{4} \int F^2_{\mu\nu} \, d^n x + i g_o \oint A_\mu d^\mu x = -\frac{V}{4} k^2 a^2_i + i g_o \, a_i f^i =$$

$$= -\frac{V}{4} k^2 \left(a^2_i - \frac{4 a_i \, i f_i}{V k^2}\right) = -\frac{V k^2}{4} \left(a_i - \frac{2 i f^i}{V k^2}\right)^2 - \frac{g^2_o}{V} \frac{f^2_i}{k^2} \tag{9}$$

Defining

$$I=\int D\alpha_i \exp\{-\frac{1}{4}\int F^2_{\mu\nu}d^nx+ig_o\oint A_\mu dx^\mu\}=\int D\alpha_i \exp\{\frac{Vk^2}{4}(\alpha_i-\frac{2if_i}{Vk^2})^2-\frac{g_o^2}{V}\frac{f_i^2}{k^2}\} \tag{10}$$

We have finally

$$<e^{\oint A_\mu dx^\mu}> = W(C) = \frac{\int D\alpha_i e^{-\frac{1}{4}\int F^2_{\mu\nu}d^nx+ig_o\oint A_\mu dx^\mu}}{\int D\alpha_i e^{-\frac{1}{4}\int F^2_{\mu\nu}d^nx}} = \frac{I(f)}{I(f=0)} = e^{-\frac{g_o^2}{V}\frac{f_i^2}{k^2}} \tag{11}$$

When passing to the limit for an infinite box and proceeding heuristically (see below), we have, summing up for all frequencies ($\frac{1}{V} \to \int \frac{d^nk}{(2\Pi)^n}$)

$$W = e^{-g_o^2\int \frac{d^nk}{(2\Pi)^n}\frac{f_i^2}{k^2}} \tag{12}$$

Defining

$$f_{\mu\nu} = \int \text{sen}\ k\cdot x\ d\sigma_{\mu\nu} \tag{13}$$

from (8), we have:

$$(f^i)^2 = f^{\mu\nu}f^{\rho\sigma}(\varepsilon^i_\mu k_\nu-\varepsilon^i_\nu k_\mu)(\varepsilon^i_\rho k_\sigma-\varepsilon^i_\sigma k_\rho) = 4f^{\mu\nu}\ f^{\rho\sigma}\delta_{\mu\rho}\ k_\nu k_\ell$$

Assuming that the loop is in the (1,2) plane the only non zero component of $f^{\mu\nu}$ is

$$f^{12} = f \tag{14}$$

so that

$$(f^i)^2 = 4f^2 (k_1^2 + k_2^2) \tag{15}$$

So, (12) becomes

$$W = \frac{I(f)}{I(o)} = e^{-4g_o^2 \int \frac{d^n k}{(2\Pi)^n} \frac{(k_1^2 + k_2^2)}{k^2} f^2(k)} \tag{16}$$

integrating now over all frequencies, as was done in ref [6] it is easy to obtain for example the static potential for any number of dimensions, i.e.:

$$V(r) \approx \Gamma\left(\frac{n-2}{2}\right) r^{(2-n)}$$

II. Non Abelian Case

Now, we shall start from the beginning with the complete Fourier representation for the potentials

$$A_\mu = \int a_\mu \, e^{ik\cdot x} \, d^n x \tag{17}$$

where

$$a_\mu = a_\mu^\alpha(k) T_\alpha \; ; \; T_\alpha \text{ are the } N^2-1 \text{ generators of } SU(N) \tag{18}$$

Due to the reality condition

$$a_\mu^\alpha(k) = a_\mu^{\alpha *}(-k) \tag{19}$$

For the field strength we have:

$$F_{\mu\nu} = i\int (k_\mu a_\nu - k_\nu a_\mu) e^{ik\cdot x}\, d^n k + g[A_\mu, A_\nu] \qquad (20)$$

where the commutators terms will be neglected, as $g^2 = \frac{g_o^2}{N}$

From (20) we have, then

$$\mathrm{Tr}\int F^2_{\mu\nu}\, d^n x = (2\pi)^n \int (k_\mu a^\alpha_\nu - k_\nu a^\alpha_\mu)(k^\mu a^{\nu *}_\alpha - k^\nu a^{\mu *}_\alpha) d^n k \qquad (21)$$

using (4), (21) can be written

$$-\frac{1}{4}\,\mathrm{Tr}\int F^2_{\mu\nu}\, d^n x = -\frac{(2\pi)^n}{2}\int d^n k\; a^{*\alpha}_i a_{i\alpha}\, k^2 \qquad (22)$$

To compute the loop we need Haussdorf-Campbell theorem[7]

$$e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]+[[\,,\,]]+\dots}$$

So

$$\frac{\mathrm{Tr}}{N}\, P e^{ig\oint A_\mu dx^\mu} = \frac{\mathrm{Tr}}{N}\exp[ig\oint A_\mu dx^\mu + g^2[\;,\;]+\dots] \qquad (23)$$

Taking into account that $\mathrm{Tr}A_\mu = 0$ and $\mathrm{Tr}[A,B] \equiv 0$ we have, for the dominant contribution in the $N\to\infty$ limit

$$\frac{\mathrm{Tr}}{N}\exp[i\oint A_\mu dx^\mu] = \exp\{-\frac{\mathrm{Tr}}{2N}\, g^2 (\int A_\mu dx^\mu)^2\} \qquad (24)$$

Using now Stokes theorem and (20) we get

$$\mathrm{Tr}(\oint A_\mu dx^\mu)^2 = -\int d^n k \int d^n k' (k_\mu a^\alpha_\nu - k_\nu a^\alpha_\mu)(k'_\rho a'^\alpha_\sigma - k'_\sigma a'^\alpha_\rho) f^{(k)}_{\mu\nu} f^{(k')}_{\rho\sigma} \qquad (25)$$

where now

$$f_{\mu\nu}(k) = \int e^{ik\cdot x}\, d\sigma_{\mu\nu} \tag{26}$$

Again, from (4) (and also $f_{\mu\nu}(-k) = f^{*}_{\mu\nu}(k)$).

$$\mathrm{Tr}\,\left(\oint A_{\mu}\, dx^{\mu}\right)^2 = \int d^n k \int d^n k'\, a^{*\alpha}_{i}(k)\; a^{\alpha}_{j}(k')\, f^{*}_{i}(k)\, f_{j}(k') \tag{27}$$

with

$$f_i(k) = (\varepsilon^{i}_{\mu} k_{\nu} - \varepsilon^{i}_{\nu} k_{\mu})\; f_{\mu\nu} \tag{28}$$

Eqs. (22), (23) and (27) allow us to write, for the loop average

$$W = \frac{I(f)}{I(o)} \tag{29}$$

where now

$$I(f) = \int Da^{\alpha}_{i}\, Da^{\alpha}_{i}\, \exp\left[-\frac{(2\pi)}{2}^{n}\int d^n k\, k^2\, a^{\alpha *}_{i}\, a^{\alpha}_{i}\right]\, \exp\left[-\frac{g^2}{2N}\int d^n k\, d^n k'\; a^{\alpha *}_{i}(k)\; a^{\alpha}_{j}(k')\, f^{*}_{i}(k)\, f_{j}(k')\right. \tag{30}$$

Formulae (29) and (30) deserve some comments, as in (21) we neglected the commutator terms which are of order g, ($g^2 = \frac{g_o^2}{N}$) while for the loop, we kept a term of order $\frac{1}{N^2}$. However, this is a consistent procedure, as if we want to compute the normalized average of a function of the form $e^{-\phi(\chi)\varepsilon^n}$ with a weight function of the form $\exp(-\sum_m f_m \varepsilon^m)$ with an aproximation ε^n, all functions f_m are irrelevant, except f_o, i.e.:

$$\frac{\int e^{-\sum_m f_m(\chi)\varepsilon^m}\quad e^{-\phi(\chi)\varepsilon^n}\, d\chi}{\int e^{-\sum_m f_m(\chi)\varepsilon^m}} = \frac{\int e^{-f_o(\chi)}\quad e^{-\phi(\chi)\varepsilon^n}}{\int e^{-f_o(\chi)}\, d\chi} + O(\varepsilon^{n+1}) \tag{31}$$

going back now to eq. (30) we write

$$I(f) = \int Da^* \, Da \; e^{-\frac{(2\Pi)^n}{2} \int d^n k \int d^n k' \, a_i^{*\alpha}(k) \, a_j^{\alpha}(k') \, M_{ij}(k,k')} \tag{32}$$

where

$$M_{ij}(kk') = k^2 \, \delta_{ij} \, \delta(k-k') + \frac{g^2}{(2\Pi)^n N} \, f_i^*(k) \, f_j(k') \tag{33}$$

The integrals in eq (32) are gaussian so that

$$I(f) = [\Delta^{-1} \, (M)]^{N^2-1} \tag{34}$$

where

$$\Delta = \mathrm{Det}\{M_{ij}(k,k')\} \tag{35}$$

In order to compute this determinant, we shall first simplify the notation:

$$M_{rs} = M_{ij}(k,k') \, \delta^n k \, \delta^n k'$$

Where r, s cover in a discrete notation the variable (k,i) (k',j) resp. The determinant of (36) is

$$\Delta(M) = \varepsilon_{s_1 s_2 s_3 \cdots} \; M_{1s_1}, M_{2s_2} \; \cdots\cdots \; M_{ns_n} \cdots\cdots$$

where $\varepsilon_{s_1 s_2 \cdots}$ is the usual complete antisymmetric symbol. We note now that the matrix elements are linear functions of g^2, i.e., (see (33))

$$\frac{d^2 M_{rs}}{(dg^2)^2} = 0 \qquad \frac{d\, M_{rs}}{dg^2} = \frac{1}{(2\Pi)^n N}\, f_r^* \, f_s \, \delta^n k \, \delta^n k' \tag{38}$$

As a consequence, we have the identity

$$f_m^* \frac{d\, M_{rs}}{dg^2} - f_r^* \frac{d\, M_{ms}}{dg^2} = 0 \tag{39}$$

From (38) and (39) it also follows that

$$\frac{d^2 \Lambda}{(dg^2)^2} \equiv 0 \tag{40}$$

i.e.

$$\Lambda = A + g^2 B \tag{41}$$

The value of A is obtained from (37), setting $g^2 = 0$, (from (33) and (36)):

$$A = \prod_k (k^2\, \delta k)^3 \tag{41}$$

We will see below that this factor drops out from the final result.

To compute B, we set $g^2 = 0$ in $\frac{d\Lambda}{dg^2}$. Now $\frac{d\Lambda}{dg^2}$ is a series of determinants

$$\frac{d\Lambda}{dg^2} = \sum_{n=1}^{\infty} \epsilon_{s_1 s_2 \ldots s_m} \ldots M_{1 s_1} \, M_{2 s_2} \ldots \frac{dM_{m s_m}}{dg^2} \ldots$$

In $g^2=0$ all rows reduce to the diagonal element except $\frac{d\, M_{ns_n}}{dg^2}$ which is given by (38)

So,

$$B = \frac{d\Lambda}{dg^2}\bigg|_{g^2=0} = \sum_m \prod_{k^2\neq m} (k^2\,\delta^n k)^3 \frac{1}{(2\Pi)^n N} f_m^* f_m (\delta^n k)^2$$

$$B = \prod_{k^2} (k^2\,\delta^n k)^3 \sum_m \frac{1}{(2\Pi)^n N} \frac{f_m^* f_m\ (\delta^n k)^2}{k_m^2\ \delta^n k}$$

Finally

$$B = \frac{\Lambda}{(2\Pi)^n N} \int \frac{d^n k}{k^2} f^*(k)\ f(k) \tag{43}$$

and

$$\Lambda = \Lambda(1+ \frac{g^2}{(2\Pi)^n N} \int \frac{d^n k}{k^2} \left| f^*(k) \right|^2) \tag{44}$$

From (29) and (34)

$$W_N = \frac{I(f)}{I(o)} = [1+ \frac{g^2}{(2\Pi)^n N} \int \frac{d^n k}{k^2} \left| f_i(k) \right|^2]^{-(N^2-1)} \tag{45}$$

Putting $g^2 = g_o^2/N$ and taking limit $N \to \infty$

$$W = e^{- \frac{g_o^2}{(2\Pi)^n} \int d^n k \frac{|f_i|^2}{k^2}} \tag{46}$$

which coincides exactly with (12) for the Abelian case .

DISCUSSION

Form (46), (as well as (12)) shows the result of Fourier analising the gauge field in the $N \to \infty$ limit. One integrates in the exponential the contribution of each frequency, coming from the loop form factor $|f|^2$ divided by k^2. However, some remarks must be made about the approximations used in the non-abelian case. The abelian case is exact. In form (19) we have neglected the commutator term, which may be a good approximation in the $N \to \infty$ limit. But for finite N this clearly is not true for k values near the origin in which case the commutator term becomes dominant over the curl term. A similar statement must be made about (23) and (24) nevertheless it is not true that the calculation is completely U(1) in character, as the evaluation of (32) requires the computation of a determinant for an infinite matrix. Further, for any finite N, the result form (45) is different from that of the abelian case (form (16)).

With an uniform treatment of all frequencies the non abelian case turns out to give the same answer as in the abelian one. (Which implies Coulomb-like potentials for n-dimensional spaces). For a more exact calculation, the low frequencies should be separated and treated differently [8] i.e.: the order N of the group should be considered a big, but finite number. Then we have to consider two regions $k < \frac{1}{NL}$ and $k > \frac{1}{NL}$ (L being a length characterizing the loop) and a different calculation should proceed in the two regions. In other words, one should be careful with the order of the limits $k \to 0$ and $N \to \infty$, as they do not commute.

BIBLIOGRAPHY

[1] G t'Hooft: Nucl. Phys. B72 (1974) 461

[2] E. Witten Nucl. Phys. B160 (1979) 57

[3] Y.M. Makeenko, A.A. Migdal: Physics Letters 88B (1979) 135

[4] A.M. Polyakov: Nucl. Phys. B164 (1980) 171

[5] R.A. Brandt, A. Gocksh, M.A. Sato, F.Neri: Phys. Rev. D26, (1982) 3611

[6] M. Abud, C.G. Bollini, J.J. Giambiagi: Nucl. Phys. B204 (1982) 109

[7] Jean Marie Normand, A Lie Group North Holland Publ/C_o 1980

[8] C.G. Bollini, J.J. Giambiagi: CBPF-NF-043/82 (1982)