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ON VECTOR-VALUED VERSUS SCALAR-VALUED
HOLOMORPHIC CONTINUATION

by

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ON VECTOR-VALUED VERSUS SCALAR-VALUED
HOLOMORPHIC CONTINUATION*

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ABSTRACT

The purpose of this note is to show that, under mild assumptions, obstacles to holomorphic continuation of vector-valued mappings are the same as in the case of scalar-valued functions.

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We denote by E and F two separated complex locally convex spaces; by \widehat{F} a completion of F containing it; by $H(U;F)$ the vector space of all mappings defined on the non-void open subset U of E with values in F which are holomorphic when considered as having its values in \widehat{F} ; by wF the vector space F endowed with the topology $\sigma(F,F')$ defined on F by its topological dual space F' ; by \widehat{wF} the algebraic dual space of F' , where the natural vector space isomorphism $i:F \longrightarrow \widehat{wF}$ is a completion mapping when the two spaces in question are respectively endowed with the weak topologies $\sigma(F,F')$ and $\sigma(\widehat{wF},F')$.

Once E is fixed, we say that weak holomorphy plus slight holomorphy imply holomorphy on E if, for every F , we have that $f \in H(V;F)$ whenever V and W are connected non-void open subsets of E with $W \subset V$, $f \in H(V;wF)$ and $f|_W \in H(W;F)$.

Once F is fixed, we say that it is confined if, for every E , we have that $f^{-1}(F) = U$ whenever U is a connected non-void open subset of E , $f \in H(U;\widehat{F})$ and $f^{-1}(F)$ has a non-void interior. To check this requirement, it suffices to take U as the open disc of center 0 and radius 1 in $E = \mathbb{C}$, to assume that $f \in H(U;\widehat{F})$ and that 0 is interior to $f^{-1}(F)$, and to conclude that $f^{-1}(F) = U$.

LEMMA. wF is confined if and only if F is confined.

Prof. Necessity being trivial, let us prove sufficiency. Denote by U the open disc in \mathbb{C} of center 0 and radius 1, and take $f \in H(U;\widehat{wF})$. Assume that we have $f(V) \subset i(F)$, where V is the open disc in \mathbb{C} of center 0 and some radius r , $0 < r < 1$. For every $\phi \in F'$, we have $f_\phi \in H(U;\mathbb{C})$, where we define $f_\phi(x) = f(x)(\phi)$ for any $x \in U$. Set $g = i^{-1} \circ (f|_V)$. Then $\phi \circ g = f_\phi|_V \in H(V;\mathbb{C})$ for every $\phi \in F'$. Hence $g \in H(V;\widehat{F})$. Set

$$a_m = \frac{1}{m!} g^{(m)}(0) \in \widehat{F}$$

for every $m \in \mathbb{N}$. Then

$$|\widehat{\phi}(a_m \rho^m)| = \left| \frac{\rho^m}{m!} f_\phi^{(m)}(0) \right| \leq \sup_{|t| \leq \rho} |f_\phi(t)|$$

for all $\phi \in F'$, $m \in \mathbb{N}$, $\rho \in \mathbb{R}$ and $0 < \rho < 1$, where $\widehat{\phi}$ is the continuous extension of ϕ to \widehat{F} . Hence the sequence $(a_m \rho^m)$, $m \in \mathbb{N}$, is bounded in \widehat{F} for every such ρ . Thus g may be holomorphically extended to U by defining

$$g(x) = \sum_{m=0}^{+\infty} a_m x^m$$

for every $x \in U$, so that $g \in H(U, \widehat{F})$. We conclude that $g(U) \subset F$ because $g(V) \subset F$. For every $\phi \in F'$ we have $f_\phi = \phi \circ g$ since both functions agree on V . Thus $f(U) = i[g(U)] \subset i(F)$. QED

Let U , V and W be connected non-void open subsets of E , with $W \subset U \cap V$. If $F \neq 0$, we say that V is a holomorphic F -valued continuation of U via W if, for every $f \in H(U; F)$, there exists $g \in H(V; F)$ such that $f = g$ on W .

PROPOSITION. Assume that weak holomorphy plus slight holomorphy imply holomorphy on E , that F is confined and $F \neq 0$. Then V is a holomorphic F -valued continuation of U via W if and only if V is a holomorphic \mathbb{C} -valued continuation of U via W .

Proof. Necessity being easy, let us prove sufficiency. Let $f \in H(U; F)$ be given. For every $\phi \in F'$ we have that $\phi \circ f \in H(U; \mathbb{C})$.

There is a unique $g_\phi \in H(V; C)$ such that $\phi \circ f = g_\phi$ on W . For every $x \in V$, consider $g(x) \in \widehat{wF}$ defined by $g(x)(\phi) = g_\phi(x)$ for all $\phi \in F'$. Thus we get $g \in H(V; \widehat{wF})$ such that $g(W) = i[f(W)] \subset i(F)$. The lemma implies that $g(V) \subset i(F)$. We then obtain $i^{-1} \circ g \in H(V; F)$ for which $f = i^{-1} \circ g$ on W . QED

I want to thank Richard Aron for the conversation which led me to the above proposition when E and F are Banach spaces [1]. Henri Hogbe-Nlend and Martin Schottenloher called my attention to the idea of dropping an additional assumption that I had included in the statement of the proposition; the proof remained unchanged. Schottenloher [4] has found another approach to the question treated here via the ε - product of Laurent Schwartz. For further sources on holomorphic continuation, we quote [2] and [3].

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