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STATIONARY CYLINDRICALLY SYMMETRIC ELECTROVAC FIELDS

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E R R A T A

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Equation (1.1) $R_{\mu}{}^{\nu} = -8\pi\, T_{\mu}{}^{\nu}$

Equation (2.4) $dt = dt' \cosh u - d\phi' \sinh u, \quad d\phi = d\phi' \cosh u - dt' \sinh u$

Equation (2.5) $m = \frac{1}{2}\,(L-F) \sinh 2u$

Equation (2.11) $r\,\psi_{11} - \psi_1 - 2\alpha_1(1-r\alpha_1) =$

Equation (2.14) $= \sqrt{-g}\,(F^{01}\, F_{01} - F^{31}\, F_{31})$

Equation (2.15) $-\frac{1}{2}\,\gamma^2\, \omega\, \frac{\partial}{\partial r} \left[1 - 2r\alpha_1\right] =$

Equation (2.16) $\frac{1}{2}\,\gamma^2\, \omega\, \frac{\partial}{\partial r} \left[1-2r\alpha_1\right] =$

Equation (2.18) $-\frac{F^{31}}{F^{01}} = \frac{F_{31}}{F_{01}} = \beta$

before equation (2.21) $e^{2\alpha}(\alpha_{11} + \frac{\alpha_1}{r}) = \frac{A^2}{\gamma^2}$ Equation (2.21) $e^{2\alpha} = (r^c + a\, r^{2-c})^2$

Equation (2.22) $a = \frac{A^2}{4(1-c)^2\, \gamma^2}$

Equation (2.56)

$$f = \gamma^2 \left[(r^c + a\, r^{2-c})^2 - \omega^2\, r^2 (r^c + a r^{2-c})^{-2}\right]$$

$$\ell = \gamma^2 \left[r^2 (r^c + a\, r^{2-c})^{-2} - \omega^2 (r^c + a\, r^{2-c})^2\right]$$

$$m = \gamma^2 \omega \left[r^2 (r^c + a\, r^{2-c})^{-2} - (r^c + a\, r^{2-c})^2\right]$$

$$e^{2\psi} = K\, r^{2c(c-2)} \left[r^c + a\, r^{2-c}\right]^2$$

Equation (2.25) $\sqrt{-g}\, F^{01} = B$

after equation (2.27) $b = -\frac{B^2}{4c^2\, \gamma^2}$

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ABSTRACT

A particular type of exact solutions of Einstein-Maxwell's equations corresponding to stationary cylindrically symmetric electrovac fields is presented here. The solutions are linear combinations of static fields with constant coefficients.

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1. INTRODUCTION

In general relativity for regions in which there is only an electromagnetic fields,

$$R_{\mu}^{\ \ \nu} = 8\pi\, T_{\mu}^{\ \ \nu} \qquad (1.1)$$

with $$4\pi\, T_{\mu}^{\ \ \nu} = - F_{\mu\alpha} F^{\nu\alpha} + \frac{1}{4} \delta_{\mu}^{\ \ \nu} F^{\rho\lambda} F_{\rho\lambda} \qquad (1.2)$$

where $F^{\mu\nu}$, the electromagnetic field tensor satisfies the Maxwell's equations

$$F^{\mu\nu}_{;\nu} = 0 \qquad (1.3)$$

$$F_{[\alpha\beta;\gamma]} = 0 \qquad (1.4)$$

Perjés[1] discussed some properties of stationary electrovac fields and obtained the field equations in a special coordinates. The only possible solution of the equations is obtained by gravitational radiation theory.

In this work we study the solution of stationary electrovac fields following the method first introduced by lewis[2] to obtain the solutions for the axially symmetric gravitational fields. Using Weyl-like canonical coordinates we give here a special class solutions obtained from the linear combinations of Weyl's[3] static fields. The solution admits a very simple interpretation that an observer in canonical space (r,θ,z) describes the static fields of the canonical space (r',θ',z') using a reference system which rotates with constant angular speed. For vanishing rotation one gets the static field only.

2. THE SOLUTIONS

We consider a stationary cylindrically symmetric line element

$$ds^2 = fdt^2 - e^{2\psi}(dr^2 + dz^2) - ld\phi^2 + 2md\phi\, dt \qquad (2.1)$$

where f, ψ, l, and m are functions of r only. We shall number the coordinates t,r,z,ϕ as 0,1,2,3 respectively. We assume that only surving components of $F^{\mu\nu}$ are $F^{31}(= - F^{13})$ and $F^{01}(= - F^{10})$. Then from equations (1.1) and (1.2) it follows that.

$$R^0_0 + R^3_3 = 0 \qquad (2.2)$$

One can now introduce Weyl-like canonical coordinate system in this stationary case such that Van Stockum[4]

$$fl + m^2 = r^2 \qquad (2.3)$$

If one makes a linear transformations (Lewis) of the coordinate differentials such as

$$dt = dt^1 \cosh u - d\theta \sinh u,\quad d\theta = d\theta^1 \cosh u - dt^1 \sinh u \qquad (2.4)$$

$$\text{with } f = F\cosh^2 u - L\sinh^2 u, \qquad l = L\cosh^2 u - F\sinh^2 u$$

$$M = \frac{1}{2}(L - F)\sinh 2u \qquad (2.5)$$

the fundamental quadratic form (2.1) transforms into

$$ds^2 = F\, dt'^2 - e^{2\psi}(dr^2 + dz^2) - L\, d\phi'^2 \qquad (2.6)$$

in Weyl's canonical system.

In general the transformation (2.4) is purely local. In our case, we choose u as constant. Now let

$$\cosh u = \gamma \quad \text{and} \quad \sinh u = \gamma\omega \qquad (2.7)$$

where γ and ω are constantes such that

$$\gamma = (1-\omega^2)^{-1/2} \tag{2.8}$$

From equations (2.3) and (2.5) we have

$$fl + m^2 = r^2 = FL \tag{2.9}$$

We now choose

$$F = e^{2\alpha} \text{ and } L = r^2 e^{-2\alpha} \tag{2.10}$$

where α is a function of r only. Equation (2.5) takes the form

$$f = \gamma^2(e^{2\alpha} - \omega^2 r^2 e^{-2\alpha}), \quad l = \gamma^2(r^2 e^{-2\alpha} - \omega^2 e^{2\alpha}), \quad m = \gamma^2\omega(r^2 e^{-2\alpha} - e^{2\alpha}) \tag{2.5a}$$

The field equations may now be explicitly written as:

$$r\,\psi_{11} - \psi_1 - 2(1 - r\alpha_1) = \sqrt{-g}\,(F^{01} F_{01} + F^{31} F_{31}) \tag{2.11}$$

$$r\,\psi_{11} + \psi_1 = -\sqrt{-g}\,(F^{01} F_{01} + F^{31} F_{31}) \tag{2.12}$$

$$\gamma^2 \frac{\partial}{\partial r}\left[1 - r\alpha_1(1+\omega^2)\right] = -\sqrt{-g}\,(F^{01} F_{01} - F^{31} F_{31}) \tag{2.13}$$

$$\gamma^2 \frac{\partial}{\partial r}\left[r\,\alpha_1(1+\omega^2)\right] = \sqrt{-g}\,(F^{01} F_{31} - F^{31} F_{31}) \tag{2.14}$$

$$\frac{1}{2}\gamma^2\,\omega\frac{\partial}{\partial r}\left[(1 - 2r\alpha_1)\right] = \sqrt{-g}\,F^{01} F_{31} \tag{2.15}$$

$$-\frac{1}{2}\gamma^2\omega\frac{\partial}{\partial r}\left[(1 - 2r\alpha_1)\right] = \sqrt{-g}\,F^{31} F_{01} \tag{2.16}$$

From equations (2.15) and (2.16) one obtains

$$F^{01} F_{31} + F^{31} F_{01} = 0 \tag{2.17}$$

$$\text{or} \quad \frac{F^{31}}{F^{01}} = -\frac{F_{31}}{F_{01}} = \beta \tag{2.18}$$

where β is a constant.

We consider now too cases:

$$\text{A)}\ \beta = \frac{1}{\omega} \qquad \text{and B)}\ \beta = \omega \tag{2.19}$$

Case A) $\beta = \frac{1}{\omega}$ In this case the observer in canonical space (r,ϕ,z) describes the pure static magnetic field in canonical space (r',ϕ',z') using a reference frame which rotates with angular speed w.

To obtain the stationary field we consider equations (1.3):

$$\sqrt{-g}\, F^{31} = A \tag{2.20}$$

where A is the constant of integration.

From equations (2.15), (2.16), (2.18) and (2.20) we have

$$e^{2\alpha} \left(\alpha_{11} + \frac{\alpha_1}{r}\right) = A^2$$

which on integration gives

$$e^{2\alpha} = (r^c + d \text{ [illegible]})^2 \tag{2.21}$$

where a and c are constants of integration satisfying the relation

$$a = \frac{A^2}{4(1-c)^2} \tag{2.22}$$

From equations (2.12), (2.15), (2.16) and (2.20) one obtains

$$r\psi_{11} + \psi_1 = \frac{1}{2} \frac{\partial}{\partial r} (2r\alpha_1 - 1) = \frac{\partial}{\partial r} (r\alpha_1)$$

wich on integration gives

$$\psi = \alpha + B \log r + D \tag{2.23}$$

where B and D are constant of integration.

Substituting the value of α from equations (2.21) we get

$$B = c(c-2) \tag{2.24}$$

Substituting α in expression (2.52) we get

$$F = \gamma^2 \left[(r^2 + br^{2-c})^2 - \omega^2 r^2 (r^c + br^{2-c})^{-2}\right]$$

$$L = \gamma^2 \left[r^2 (r^c + br^{2-c})^{-2} - \omega^2 \,(r^c + br^{2-c})^2\right] \tag{2.5b}$$

$$m = \gamma^2 \,\omega \left[r^2 (r^c + br^{2-c})^{-2} - (r^c + br^{2-c})^2\right]$$

and from (2.21), (2.23) and (2.29)

$$\psi = K \;\; r^{2c(c-2)} \;\; \left[r^c + br^{2-c}\right]^2$$

If we take A=0, the solution immediately goes to the solution given by Lewis for gravitational fields. The constant c then may be interpreted as the line density of mass distribution along the Z-axis. If the constant $\omega = 0$, the solution goes over to the solution of axially symmetric magnetic field given by Ghosh and Sengupta[5].

Case B) $\beta = \omega$, In this case the static field in the canonical space (r', ϕ', z') is purely radial electrotatic field. From the equation(1.3) one gets.

$$\sqrt{-g}\, F^{31} = B \tag{2.25}$$

Equ. (2.15), (2.16), (2.18) and (2.25) Yield

$$e^{-2\alpha} \frac{\partial}{\partial r} (1 - 2r\alpha_1) = - 2 \,\frac{B^2}{r} \tag{2.26}$$

wich on integration gives

$$e^{2\alpha} = \left[r^{-c} + br^{c} \right]^{-2} \qquad (2.27)$$

where b and c are integration constants satisfying the relation.

$$b = \frac{B^2}{4c}$$

For b = 0 i.e. B = 0, the solution again goes over to Lewis solution. When $\omega = 0$, the solution is equivalent to a number of already Known solutions for a static, cylindrically symmetric radial electrostatic field Mukherjee[6], Bonnor[7], Raychaudhuri[8].

3. CONCLUSION

We have considered only two classes of exact solutions of the stationary electrovac fields corresponding to the observer's two modes of description of the static fields - either the static axially symmetric magnetic field (2.5b) or the static cylindrically symmetric radical electrostatic field (2.27). Of course another class of solutions may be obtained when β is different form ω. In this case the solution would correspond to the observer's descriptions of static axial magnetic field as well as radial electrostatic field.

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