NOTAS DE FÍSICA

VOLUME XVI

Nº 17

GRAVITATION AS A CONSEQUENCE OF THE SELF INTERACTION OF THE Γ-FIELDS

by

Mario Novello

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71 - Botafogo - ZC-82

Rio de Janeiro, Brazil

1971

Notas de Física - Volume XVI - Nº 17

GRAVITATION AS A CONSEQUENCE OF THE SELF INTERACTION OF THE Γ-FIELDS *

Mario Novello

Centro Brasileiro de Pesquisas Físicas

Rio de Janeiro, Brasil

Université de Genève

(Received February 10, 1971)

ABSTRACT

We investigate an algebraic-geometry approach to field theory. We try to prove that it is possible to understand Einstein's gravitational equation as a consequence of the self interaction of the fundamental Γ-fields of the assumed Clifford-algebra. Furthermore, the anti-symmetric object of the algebra obeys a set of equations that has the same structure of Maxwell's equations.

1. THE FUNDAMENTAL OBJECTS

Let us consider a set of objects e^a that can generate a universal Clifford algebra (C-algebra) over a four dimensional differentiable manifold V_4. For the well-known property of the C-algebra

$$\{e^a, e^b\} \tag{1}$$

is a multiple of the identity of the algebra, where as usually

$$\{M, N\} = M N + N M \tag{2}$$

* This work has been supported by the "Coordenação do Aperfeiçoamento de Pessoal de Nivel Superior" (CAPES) and by the "Centro Brasileiro de Pesquisas Físicas.

It is well-known [2] that the dimension of this algebra is 2^4. We will represent the elements of the basis as the set

$$\mathbb{1}$$

$$\Gamma_\alpha$$

$$\Sigma_{\alpha\beta} = \frac{1}{2}(\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha) \tag{3}$$

$\Gamma^5 = \phi(x)\, \varepsilon^{\alpha\beta\lambda\mu}\, \Gamma_\alpha \Gamma_\beta \Gamma_\lambda \Gamma_\mu$, where $\phi(x)$ is such that

$\Gamma_\alpha \Gamma^5$ normalizes Γ^5, that is $\Gamma^5 \Gamma^5 = \mathbb{1}$

$$\Gamma_\alpha \Gamma^5 = \phi(x)\, \varepsilon^{\mu\nu\rho\sigma}\, \Gamma_\alpha \Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma$$

where the indice α,β has a tensorial character. This means that if we make a transformation of coordinates

$$x^\alpha \to x'^\alpha = f^\alpha(x^\beta) \tag{4}$$

the Γ_α behave as a vector, the Γ^5 as a scalar and $\Sigma_{\alpha\beta}$ as an antisymmetrical tensor of second order. In our choice of the representation of the algebra, the elements of the basis have a two-indice property. Let us write

$$\Gamma_\alpha^{AB} \tag{5}$$

where A, B may assume the values 1, 2, 3 or 4. We assume that there is a group of internal transformation with a space-time dependence, that is, the Γ's may suffer a transformation like

$$\Gamma_\alpha^{AB}(x) \to \Gamma'^{AB}_\alpha(x) = M^A{}_C\, \Gamma_\alpha^{CD}(x)\, M^{-1}{}_D{}^B \tag{6}$$

where $M^A{}_B$ is not a constant.

The fundamental property of the C-algebra

$$\{\Gamma_\mu(x), \Gamma_\nu(x)\} = 2\, g_{\mu\nu}(x)\, \mathbb{1} \tag{7}$$

defines the symmetric metric tensor $g_{\mu\nu}(x)$.

$\mathbb{1}$ is the element identity of the algebra. We have furthermore

$$\{\Gamma_\mu(x), \Gamma^5(x)\} = 0 \tag{8}$$

2. THE SELF INTERACTION

We will admit [1] that the Γ's satisfy an equation of the type

$$\Gamma_{\alpha\|\beta}(x) = [U_\beta(x), \Gamma_\alpha(x)] \tag{9}$$

where the brackets, as usually, means the commutator and the symbol $\|$ means the covariant derivative defined by

$$\Gamma_{\alpha\|\beta} = \Gamma_{\alpha|\beta} - \left\{ {\varepsilon \atop \alpha\beta} \right\} \Gamma_\varepsilon + [\tau_\beta, \Gamma_\alpha] \tag{10}$$

$\Gamma_{\alpha|\beta}$ means $\dfrac{\partial \Gamma_\alpha}{\partial X^\beta}$

$\left\{ {\varepsilon \atop \alpha\beta} \right\}$ is the Christoffel symbol

τ_β is the internal affinity.

The origin of the expression (9) rests on the assumption that the co-relation, at separate points, between the Γ's, even in the existence of the group of transformation (6), does not introduce any new field. This is equivalent to assume that the internal affinity may be expressed as a function of the objects of the C-algebra only.

The object $U_\nu^{AB}(x)$ as obtained in [1] has the form

$$U_\nu^{AB}(x) = \{\Gamma_\nu(x)\ (\mathbb{1} + \Gamma^5(x))\}^{AB} \tag{11}$$

A straightforward calculation shows that the covariant derivative is not commutative and that we may write

$$\Gamma_{\alpha\|\beta\|\lambda} - \Gamma_{\alpha\|\lambda\|\beta} = R_{\alpha\varepsilon\beta\lambda}\,\Gamma^{\varepsilon} + \left[\mathbb{R}_{\beta\lambda}, \Gamma_{\alpha}\right] \qquad (12)$$

where

$R_{\alpha\varepsilon\beta\lambda}$ is the Riemann (curvature) tensor

$\mathbb{R}_{\alpha\beta}$ is the internal curvature.

3. THE GRAVITATIONAL FIELD

From equations (9) and (11) we obtain

$$(\Gamma_{\alpha\|\beta\|\lambda} - \Gamma_{\alpha\|\lambda\|\beta})\; g^{\alpha\beta} = 0 \qquad (13)$$

So, we have

$$R_{\alpha\varepsilon\beta\lambda}\, g^{\alpha\beta}\, \Gamma^{\varepsilon} + \left[\mathbb{R}_{\beta\lambda}, \Gamma_{\alpha}\right] g^{\alpha\beta} = 0 \qquad (14)$$

or

$$R_{\varepsilon\lambda}\, \Gamma^{\varepsilon} + \left[\mathbb{R}_{\varepsilon\lambda}, \Gamma^{\varepsilon}\right] = 0 \qquad (15)$$

What can we say about the form of the internal curvature ? It is an easy matter to prove that

$$\Gamma^{5}{}_{\|\alpha\|\beta} = 0 \qquad (16)$$

From this and from the consideration that

$$\Gamma^{5}{}_{\|\alpha\|\beta} - \Gamma^{5}{}_{\|\beta\|\alpha} = \left[\mathbb{R}_{\alpha\beta}, \Gamma^{5}\right] \qquad (17)$$

we obtain

$$\left[\mathbb{R}_{\alpha\beta}, \Gamma^{5}\right] = 0 \qquad (18)$$

The most general expression of the internal curvature obtained as an element of the Clifford algebra and satisfying equation(18) has the form

$$\mathbb{R}_{\alpha\beta} = S_{\alpha\beta}\;\mathbb{1} + P_{\alpha\beta}\,\Gamma^{5} + B_{\alpha\varepsilon}\,\Gamma^{\varepsilon}\,\Gamma_{\beta} - B_{\beta\varepsilon}\,\Gamma^{\varepsilon}\,\Gamma_{\alpha} \qquad (19)$$

where $S_{\alpha\beta}$, $P_{\alpha\beta}$ and $B_{\alpha\beta}$ are pure tensors that satisfies the symmetry conditions

$$S_{\alpha\beta} + S_{\beta\alpha} \neq 0 \qquad (20)$$

$$P_{\alpha\beta} + P_{\beta\alpha} = 0 \qquad (21)$$

If we put expression (19) into (15) we obtain two separate equations

$$P_{\alpha\beta} = 0 \qquad (22)$$

$$R_{\alpha\beta} - 2\,B\,g_{\alpha\beta} - 4\,B_{\alpha\beta} = 0 \qquad (23)$$

where

$$B = B_{\alpha\beta}\,g^{\alpha\beta} \qquad (24)$$

From these considerations we see that the expression (9) induces a relation between the contracted riemannian curvature (Rici tensor) and a tensorial field. If we assume that the divergence of the tensor field $B_{\alpha\beta}(x)$ is null then we arrive at a contradiction. So, we cannot identify $B_{\alpha\beta}$ directly as a conserved energy-momentum tensor. If we assume otherwise that

$$B_{\alpha\beta} = M\,g_{\alpha\beta} - N\,T_{\alpha\beta} \qquad (25)$$

where

$$T^{\alpha}{}_{\beta||\alpha} = 0 \qquad (26)$$

then, expression (23) assumes the form

$$R_{\alpha\beta} - \frac{1}{2}\,R\,g_{\alpha\beta} = -\,k\,T_{\alpha\beta} \qquad (27)$$

for a particular choice of the functions M and N.

We see that relation (25) (where $T_{\alpha\beta}$ is the energy-momentum tensor) implies that equation (23) is just Einstein's equation of gravitational theory.

4. THE ANTISYMMETRICAL OBJECT

Let us see what are the relation that equation (9) gives for the antisymmetrical product

$$\Sigma_{\alpha\beta} = \frac{1}{2} (\Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha) \tag{28}$$

A straightforward calculation can show that

$$\Sigma_{\alpha\beta\|\lambda} = 2 \{g_{\alpha\lambda} \Gamma_\beta - g_{\beta\lambda} \Gamma_\alpha\} \quad (\mathbb{1} + \Gamma^5) \tag{29}$$

If we define

$$\Sigma_{\{\alpha\beta\|\lambda\}} = \frac{1}{3!} \{\Sigma_{\alpha\beta\|\lambda} - \Sigma_{\alpha\lambda\|\beta} + \Sigma_{\beta\lambda\|\alpha} - \Sigma_{\beta\alpha\|\lambda} + \Sigma_{\lambda\alpha\|\beta} - \Sigma_{\lambda\beta\|\alpha}\} \tag{30}$$

then (29) gives

$$\Sigma_{\{\alpha\beta\|\lambda\}} = 0 \tag{31}$$

Let us evaluate now the divergence of the $\Sigma^{AB}_{\alpha\beta}$. We obtain

$$\Sigma^{\alpha}{}_{\beta\|\alpha} = 6 \Gamma_\beta (\mathbb{1} + \Gamma^5) \tag{32}$$

If we define the current J^{AB}_{β} as

$$J_\beta = 6 \Gamma_\beta (\mathbb{1} + \Gamma^5) \tag{33}$$

we obtain the continuity equation

$$J^{B}{}_{\|\beta} = 0 \tag{34}$$

Equation (31) shows that we may introduce a potential ϕ^{AB}_{α} such that

$$\Sigma^{AB}_{\alpha\beta} = \phi^{AB}_{\alpha\|\beta} - \phi^{AB}_{\beta\|\alpha} \tag{35}$$

A choice for this potential may be

$$\phi_\alpha = (\text{constant})\, \Gamma_\alpha \tag{36}$$

We see that these relations are just Maxwell's equations applied object that has an internal structure besides the tensor character of the usual electromagnetic field.

ACKNOWLEDGEMENTS

It is a pleasure to thank Professor J. M. Jauch for many very stimulating conversations.

REFERENCES

1. For a general review of the literature see W. L. Bade - H. Jehle - Rews. Mod. Phys. 25, 714 (1953).
2. Brill, D. R. - Wheeler, J. A. - Rews. Mod. Phys. 29, 465 (1957).
3. See reference in Pirani/Trautmann) Bondi Brandeis - Lectures on general relativity (1964).
4. Novello, M. - N. Cim. 64, 954 (1969).
5. Porteous, I. R. - Topological Geometry (1969). Van Nostrand Reinhold Co. London.
6. Novello, M. - Vers une Théorie Unifiée les Interactions Faibles et Gravitationnelles (unpublished).