NOTAS DE FÍSICA

VOLUME IX

Nº 17

MULTIPLE SCATTERING MEASUREMENTS IN THE PRESENCE OF ENERGY LOSSES

by

A. Marques

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1962

Notas de Física - Volume IX - Nº 17

MULTIPLE SCATTERING MEASUREMENTS IN THE PRESENCE OF ENERGY LOSSES*

A. Marques
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro

(Received August 13, 1962)

ABSTRACT. A first order correction to the estimator of true scattering that accounts for small energy losses is derived. Energy values obtained from scattering measurements performed on pions, starting at 8mm and 12 mm residual range, show excellent agreement with values taken from range-energy tables when the original observations are treated according to the prescriptions developped throughout.

---

* Submitted for publication to Nuclear Physics.

The estimate of the average projected angle of multiple scattering of fast charged particles has been the object of many investigations; most of that work, however, has been restricted to the case in which the particles lose a negligible amount of kinetic energy along the extent of path where measurements are taken.

That condition is met approximately in a number of situations but it is not rare the event in which serious systematic errors are introduced by neglecting the energy losses; in that case, trends will be different for different laboratories since they depend ultimately on the distribution of path lengths employed.

In this paper an approximated method to correct the estimator of true scattering for small energy losses is derived; the treatment is semi-empirical and requires a knowledge of the exponent of the range-energy relationship and the residual range. The discussion is limited to Moliere-d'Espagnat's estimator but other estimators could be handled similarly.

In Section I a short account of Moliere-d'Espagnat's theory of estimation is given; Section II deals with the derivation of a corrective term to the estimator of true scattering in the presence of small energy losses. In Section III results of 150 scattering measurements taken on pions stopping in the same emulsion sheet, at 8 mm and 12 mm residual range, are presented and compared with the results of previous sections.

I. Molière-d'Espagnat's Method.

In this Section a brief presentation of Molière-d'Espagnat's method of estimation is given [1,2]; we mention only the points that were found necessary to provide the grounds upon which the arguments of Section II are developped. Further details or clarification should be sought for in the papers of references 1 to 4. We follow very closely Solntseff's [4] notation.

It can be shown that the joint distribution function of n finite differences of order r of observed ordinates of points on the trajectory of a fast charged particle can be written as (Gaussian approximation):

$$f(\underline{x})d^n x = \left\{(2\pi)^n |\underline{\Delta}|\right\}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\,\underline{x}^+ \underline{\Delta}^{-1}\,\underline{x}\right\} d^n x \qquad (1)$$

where $\underline{x}$ is the row vector $(x_1, x_2, \ldots, x_n)$, $\underline{x}^+$ its transposed and x_i is the r^{th}- difference of ordinates y_j, defined by

$$x_i \sum_j c_{ij}\, y_j \qquad (2)$$

with

$$\begin{aligned} c_{i,i+k-r/2} &= (-1)^k\, C_r^k,\ k = 0, 1, \ldots r \\ c_{i,j} &= 0 \qquad \text{if} \quad |j-i| > r/2 \end{aligned} \qquad (3)$$

if r is even, and

$$\begin{aligned} c_{i,i+k-(r+1)/2} &= (-1)^k\, C_r^k,\ \ k = 0, 1, \ldots r \\ c_{ij} &= 0 \ \text{ if } j < i-(r+1)/2 \text{ or } j > i+(r+1)/2-1 \end{aligned} \qquad (4)$$

if r is odd; C_r^k in (3) and (4) are the combinatorial numbers $r!/(r-k)!\,k!$.

Furthermore, in (1), $\underline{\Lambda}^{-1}$ is the inverse of the matrix of moments, $\underline{\Lambda}$, (the matrix of the mathematical expectations of $x_i x_j$), and $|\Lambda|$ its determinant. The matrix of moments consists of two additive, independent contributions, $\underline{\Lambda}^{(s)}$ and $\underline{\Lambda}^{(\nu)}$:

$$\underline{\Lambda} = \underline{\Lambda}^{(s)} + \underline{\Lambda}^{(\nu)} \quad ; \tag{5}$$

$\underline{\Lambda}^{(s)}$ is the matrix of moments of the r^{th}-differences in absence of noise and $\underline{\Lambda}^{(\nu)}$ is the corresponding matrix in absence of genuine scattering. The elements of $\underline{\Lambda}^{(s)}$, $\lambda_{ij}^{(s)}$, can be written as

$$\lambda_{ij}^{(s)} = (1/2) \int_0^{\ell_T} x_c'^2 \, B_r \, \gamma_i(\ell) \, \gamma_j(\ell) \, d\ell \tag{6}$$

whereas the elements of $\underline{\Lambda}^{(\nu)}$, $\lambda_{ij}^{(\nu)}$, are given by

$$\lambda_{ij}^{(\nu)} = \sum_k c_{ik} \, c_{jk} \, \sigma_N^2 \tag{7}$$

where the coupling factors, $\gamma_i(\ell)$, are defined by

$$\gamma_i(\ell) = \sum_{k=0}^{r} (-1)^k \, C_r^k \, \gamma'_{i+k-r/2}(\ell) \tag{8}$$

for even r, and

$$\gamma_i(\ell) = \sum_{k=0}^{r} (-1)^k \, C_r^k \, \gamma'_{i+k-(r+1)/2}(\ell) \tag{9}$$

for odd r; ℓ is a coordinate of position taken along an axis parallel to the trajectory and such that $\ell = 0$ and $\ell = \ell_T$ are the abscissae of the beginning of the first cell and of the end

of the last one, respectively. Finally $\gamma_j'(\ell)$ is defined by

$$\begin{aligned} \gamma_j'(\ell) &= js-\ell \quad \text{if} \quad \ell<js \\ \gamma_j(\ell) &= 0 \quad\quad \text{if} \quad \ell\geqslant js \end{aligned} \tag{10}$$

and s is the cell size.

Furthermore, in (6), χ_c' is given by

$$\chi_c' = (Zze^2/pv)(4\pi N)^{\frac{1}{2}} \tag{11}$$

for a particle with charge Ze, momentum $\underline{p}$, velocity $\underline{v}$, travelling in a medium with N atoms per cubic centimeter and atomic number z; B_r can be obtained from the set of equations:

$$x_c^2 = \int_0^{\ell_T} \chi_c'^2 \gamma_i^2(\ell)\, d\ell \tag{12}$$

$$\ell n\, x_a^2 = \int_0^{\ell_T} \chi_c'^2 \gamma_i^2(\ell)\, \ell n\left[\chi_a^2 \gamma_i^2(\ell)\right] d\ell \tag{13}$$

$$B_r - \ell n\, B_r = \ell n\, \frac{\epsilon}{\gamma^2}\, \frac{x_c^2}{x_a^2} \tag{14}$$

where $\ell n\, \epsilon = 1$ and $\ell n\, \gamma = C$ is Euler's constant. χ_a in (13) is the angle beyond which screening effects become important.

Finally, in (7), σ_N is the standard deviation of the observed ordinates when genuine scattering can be neglected

in comparison with noise; we assume that only random noise is present and that the ordinates due to noise alone are distributed according to a gaussian law with standard deviation σ_N.

We can see from equations (5) through (7) and (11) through (14) that the whole energy dependence of the process is contained in $\underline{\Delta}^{(s)}$, through the term $\chi'^2_c B_r$. That dependence can be made more explicit with the help of the following definitions:

$$A_{ij}^{(s)} = (\chi'^2_c B_r/2)^{-1} \lambda_{ij}^{(s)} \tag{15}$$

$$a_{ij}^{(s)} = A_{ij}^{(s)}/A_{ii}^{(s)} \tag{16}$$

$$A_{ij}^{(\nu)} = {}^{(\nu)}_{ij}/\sigma_N^2 \tag{17}$$

$$a_{ij}^{(\nu)} = A_{ij}^{(\nu)}/A_{ii}^{(\nu)} \tag{18}$$

$$\sigma_s^2 = (\chi'^2_c B_r/2) A_{ii}^{(s)} \tag{19}$$

$$\sigma_\nu^2 = \sigma_N^2 A_{ii}^{(\nu)} \tag{20}$$

$$\mu_r = \sigma_\nu^2/\sigma_s^2 \tag{21}$$

Equation (1) can then be written

$$f(\underline{x})d^n x = \left\{(2\pi\sigma_s^2)^n |\underline{a}^{(s)} + \mu_r \underline{a}^{(\nu)}|\right\}^{-\frac{1}{2}} \exp\left\{-\underline{x}^+ (\underline{a}^{(s)} + \mu_r \underline{a}^{(\nu)})^{-1} \underline{x}/2\sigma_s^2\right\} d^n x \tag{22}$$

We can see that the whole energy dependence of the process is now contained in the parameter σ_s^2; the estimation of the energy of the particle can then be done by means of that parameter. Before going into this part of the problem it is worth to mention that the elements of the matrices $\underline{a}^{(s)}$ and $\underline{a}^{(\nu)}$ were calculated by authors of references 1 - 4 ; the elements of the matrix $\underline{a}^{(s)}$, however, have to be corrected for our purpose, by including their energy dependence.

The estimation of σ_s^2 can be done in the following way:

Let us choose an estimator of the form

$$R = \sum_{ij} b_{ij}\, x_i\, x_j \tag{23}$$

where the number b_{ij} are left undetermined for a while. Suppose we have an infinite sample of trajectories of identical particles and that we compute R according to (23) for each of them. The mathematical expectation of R, E(R), is given by

$$E(R) = \sum_{ij} b_{ij}\, E(x_i x_j) \tag{24}$$

Since, by definition,

$$E(x_i x_j) = \lambda_{ij} = \lambda_{ij}^{(s)} + \lambda_{ij}^{(\nu)}$$

we can write:

$$E(R) = Tr(\underline{b}\,\underline{\Lambda}) = Tr(\underline{b}\,\underline{\Lambda}^{(s)}) + Tr(\underline{b}\,\underline{\Lambda}^{(\nu)}) \tag{25}$$

where $\underline{b}$ is the matrix of the coefficients b_{ij} and Tr holds for

"trace" of the matrix.

Suppose, now, that there is a particular choice of the matrix $\underline{b}$, say, $\underline{b}^{(s)}$, such that:

$$\mathrm{Tr}(\underline{b}^{(s)} \underline{\Lambda}^{(\nu)}) = 0 \tag{26}$$

In this case (23) becomes

$$E(R) = \mathrm{Tr}(\underline{b}^{(s)} \underline{\Lambda}^{(s)}) \tag{27}$$

Now, from (15), (16) and (19)

$$\underline{\Lambda}^{(s)} = \sigma_s^2 \underline{a}^{(s)}; \tag{28}$$

introducing the result above in (27) the following expression for σ_s^2 is obtained

$$\sigma_s^2 = E(R)/\mathrm{Tr}\ (\underline{b}^{(s)} \underline{a}^{(s)}) \tag{29}$$

If, now, R* is an estimate of E(R) (R* is usually taken as the value of R obtained in a single trajectory) equation (29) provides an estimate of σ_s^2:

$$\sigma_s^{*2} = R^*/\mathrm{Tr}\ (\underline{b}^{(s)} \underline{a}^{(s)}) \tag{30}$$

provided elements of the matrix $\underline{b}^{(s)}$ are known.

The elements of $\underline{b}^{(s)}$ have to satisfy (26), being otherwise arbitrary; therefore we can choose them in such a way as to make $\underline{b}^{(s)}$ as simple as possible [3]. Such a choice is:

$$b_{ii}^{(s)} = 1,\ i = 1, 2, \ldots n$$

$$b^{(s)}_{i,j} = b, \quad \text{if} \quad |i-j| = 1$$

$$b^{(s)}_{i,j} = 0, \quad \text{if} \quad |i-j| > 1$$

Using this matrix in (26) we obtain:

$$b = -\,[n/(n-1)]/2\; a^{(\nu)}_{i,i+1} \tag{31}$$

With that choice for $\underset{\sim}{b}^{(s)}$ the estimate of σ_s^2 given in (30) becomes:

$$\sigma_s^{*2} = \left(a^{(\nu)}_{i,i+1}\, \bar{x}^2 - \bar{x}' \right) / \left(a^{(\nu)}_{i,i+1} - a^{(s)}_{i,i+1} \right) \tag{32}$$

where $\bar{x}^2 = \sum_i x_i^2/n$ and $\bar{x}' = \sum_i x_i x_{i+1}/(n-1)$.

The variance of the estimate (30) can also be found but actual computation will be omitted here for the sake of brevity. The result is

$$S(\sigma_s^*)/\sigma_s^* = \left\{ \mathrm{Tr}\left(\underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(s)}\, \underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(s)} \right) + 2\mu_r\, \mathrm{Tr}\left(\underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(s)}\, \underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(\nu)} \right) + \mu_r^2\, \mathrm{Tr}\left(\underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(\nu)}\, \underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(\nu)} \right) \right\}^{\frac{1}{2}} / \sqrt{2}\; \mathrm{Tr}\left(\underset{\sim}{b}^{(s)}\, \underset{\sim}{a}^{(s)} \right) \tag{33}$$

where the left hand side is the relative variance of σ_s^*.

The estimate of σ_s given by (32) should be used in equation (19) to yield an estimate of pv. However, since the gaussian approximation that has been used throughout is not accurate, this procedure would give rise to biased estimates.

It has been shown by Molière [1] that it is possible to write an expression for the exact frequency function of the modulus of

any given r^{th}-difference by means of a series expansion, the first term of which corresponds to the gaussian approximation. The mathematical expectation of the absolute value of any given r^{th}-difference can be computed from that expression and the result is:

$$E(|x_i|) = (2\pi)^{\frac{1}{2}} \sigma_s (1 + 0.982 B_r - 0.117 B_r^2 + \ldots) \quad (34)$$

By other hand, a cut-off value is imposed on the observed sample so that any r^{th}-difference greater than that value is removed from the sample. This procedure tends to yield samples that are more close to the gaussian than the original one, the closeness depending on the actual cut-off employed. An exact expression for the joint distribution function of r^{th}-differences in a filtered sample cannot be obtained but a fairly good approximation to the corresponding frequency function can be derived by cutting the exact Molière's expression at the appropriate value. With this truncated distribution we obtain for $E(|x_i|)$:

$$E(|x_i|) = (2/\pi)^{\frac{1}{2}} \sigma_s (1 + a/Br + b/Br^2 + \ldots) \quad (35)$$

where a and b are numbers that depend on the actual cut-off procedure.

Therefore we take $(2\pi)^{\frac{1}{2}} \sigma_s^*$ as the estimate of the left hand side of (35) and use the resulting equation to estimate pv. The equation of estimation can be written as:

$$(2/\pi)^{\frac{1}{2}} \sigma_s^* = K_r \, s^{3/2}/pv \quad (36)$$

where

$$K_r = 2Zze^2 \left[(A_{ii}^{(s)}/s^3)\ N\ B_r \right]^{\frac{1}{2}} \cdot M_r \qquad (37)$$

with

$$M_r = 1 + a/B_r + b/B_r^2 + \ldots \qquad (38)$$

and s is the cell length.

K_r in (37) is the "scattering constant" for r^{th}-differences. It depends on the cut-off value chosen and is a slowly varying function of the cell length and the energy. K_o has been calculated by different authors [5] and plotted as a function of the cell length and v/c for different cut-off criteria. Solntseff [4] shows how to connect K_o with K_r for different orders of finite difference.

II. Multiple Scattering in the Presence of Energy Losses.

The results of Section I are valid only if energy losses are negligible; in the opposite case, say, when losses are heavy compared to the kinetic energy, the whole problem is meaningless. However there is the intermediate situation characterized by small but non negligible energy losses, upon which we focus attention in this Section. We will assume throughout that those losses are not great enough to invalidate the small angle approximation, or else, that energy losses and accumulation of small angles can be considered as independent stochastic processes.

In the presence of such energy losses, χ_c' and B_r cannot be considered as constants; they will vary slightly along the track interval used in the measurement. It is possible to account for that if an expression for that variation as a function of a position coordinate along the trajectory is available.

The dependence of χ_c' with a position coordinate along the trajectory can be derived easily with the help of the range--energy relationship:

$$E = aR^k = a(R_o - \ell)^k = E_o(1 - \ell/R_o)^k \qquad (1)$$

where R_o is the residual range, E_o the kinetic energy at R_o and ℓ is a coordinate of position taken along the trajectory whose zero is at R_o.

By using equations (I.11) and (1) it is easy to show that

$$\chi_c'(\ell) = \chi_c'(0)(1 - \ell/R_o)^{-k} \qquad (2)$$

where $\chi_c'(\ell)$ is the value of χ_c' at any position ℓ and $\chi_c'(0)$ is the corresponding value at $\ell = 0$.

Equation (2) provides the required expression for the variation of χ_c' with a position coordinate along the track; the variation of B_r along the track can also be described by means of equation (2) since all energy dependence of that parameter comes from the presence of a term in $\chi_c'^2$ in the expressions for x_o^2 and x_a^2. Since, however, B_r varies rather slowly with energy [5] we prefer to replace it by an averaged value, $\bar{B}_r$, taken over the track interval used, which will be discussed

later on.

We can now introduce $\chi'_c(\ell)$ instead of χ'_c in the definition of the elements of the matrix of moments, $\underline{\Lambda}^{(s)}$, to obtain

$$\lambda_{ij}^{(s)} = (\bar{B}_r/2) \int_0^{\ell_T} \chi'_c(\ell)\, \gamma_i(\ell)\, \gamma_j(\ell)\, d\ell \qquad (3)$$

The expression above can also be written

$$\lambda'^{(s)}_{ij} = (\chi'^2_c\, \bar{B}_r/2) \int_0^{\ell_T} (1-\ell/R_o)^{-2k}\, \gamma_i(\ell)\, \gamma_j(\ell)\, d\ell \qquad (4)$$

where, for simplicity, χ'^2_c holds for $\chi'^2_c(0)$.

We now define the elements of the matrix $\underline{A}'^{(s)}$ as follows

$$\lambda'^{(s)}_{ij} = (\chi'^2_c\, \bar{B}_r/2)\, A'^{(s)}_{ij} \qquad (5)$$

It follows from (4) and (5) that, up to terms in ℓ/R_o,

$$A'^{(s)}_{ij} = A^{(s)}_{ij} + (2k/R_o) \int_0^{\ell_T} \ell\, \gamma_i(\ell)\, \gamma_j(\ell)\, d\ell \qquad (6)$$

where the $A^{(s)}_{ij}$ are the same as in (I.15).

The integration in (6) can be performed with the help of the coupling factors $\gamma_i(\ell)$ given in (I.8) and (I.9); explicit representation of those factors can be obtained by performing the summations indicated in those equations and using the definition of $\gamma'_i(\ell)$ given in (I.10). The expressions for the

coupling factors thus obtained are:

i) Second Differences:

$$\begin{aligned} \gamma_i(\ell) &= \ell - (i-1)s \quad \text{for} \quad (i-1)\,s \leq \ell < is \\ \gamma_i(\ell) &= (i+1)\,s - \ell \quad \text{for} \quad is \leq \ell < (i+1)\,s \\ \gamma_i(\ell) &= 0 \quad \text{elsewhere.} \end{aligned} \tag{7}$$

ii) Third Differences:

$$\begin{aligned} \gamma_i(\ell) &= \ell - (i-2)s \quad \text{for} \quad (i-2)s \leq \ell < (i-1)s \\ \gamma_i(\ell) &= (2i-1)s - 2\ell \quad \text{for} \quad (i-1)s \leq \ell < is \\ \gamma_i(\ell) &= \ell - (i+1)s \quad \text{for} \quad is \leq \ell < (i+1)s \\ \gamma_i(\ell) &= 0 \quad \text{elsewhere.} \end{aligned} \tag{8}$$

iv) Fourth Differences:

$$\begin{aligned} \gamma_i(\ell) &= \ell - (i-2)s \quad \text{for} \quad (i-2)s \leq \ell < (i-1)s \\ \gamma_i(\ell) &= (3i-2)s - 3\ell \quad \text{for} \quad (i-1)s \leq \ell < is \\ \gamma_i(\ell) &= -(3i+2)s + 3\ell \quad \text{for} \quad is \leq \ell < (i+1)s \\ \gamma_i(\ell) &= (i+2)s - \ell \quad \text{for} \quad (i+1)s \leq \ell < (i+2)s \\ \gamma_i(\ell) &= 0 \quad \text{elsewhere} \end{aligned} \tag{9}$$

where $\underline{s}$ is the cell length.

Here-to-after, unless explicit mention is done, we will consider only second differences; higher order differences can

be handled much in the same way.

The integration in the right hand side of (6) can be done without difficulty by means of (7) and the result is:

$$A'^{(s)}_{ii} = A^{(s)}_{ii}\left[1 + 2iks/R_o\right]$$

$$A'^{(s)}_{ii} = A^{(s)}_{i,i+1}\left[1 + (2i+1)ks/R_o\right] \qquad (10)$$

$$A'^{(s)}_{ij} = 0 \qquad \text{if} \qquad |j-i| > 1$$

Now if we define σ'^2_s by

$$\sigma'^2_s = \left(\chi'^2_c \bar{B}_r/2\right) A^{(s)}_{ii} \qquad (11)$$

we obtain

$$\lambda'^{(s)}_{ij} = \sigma'^2_s \left(A'^{(s)}_{ij}/A^{(s)}_{ii}\right) = \sigma'^2_s \, a'^{(s)}_{ij} \qquad (12)$$

which corresponds to equation (I.28).

The parameter σ'^2_s in (11) contains the whole energy dependence of the process and is identical with the parameter defined by (I.19) except for having $\bar{B}_r$ replacing B_r; it depends on the energy through χ'^2_c which is the value of $\chi'^2_c(\ell)$ for $\ell = 0$ and through $\bar{B}_r$ which is a sort of average value of B_r taken over the track length used and is, therefore, independent of ℓ. Therefore, once an estimate of σ'^2_s is available, equation (11) will provide an approximately unbiased estimate of the energy of the particle at $\ell = 0$, in gaussian approximation.

The estimate of σ'^2_s can be obtained by taking the same

steps as in Section I; thus we can show that

$$\sigma'^{*2}_s = \left(\sum_i x_i^2 + 2b\sum_i x_i x_{i+1}\right)/T_r\left(\underline{b}^{(s)}\ \underline{a}'^{(s)}\right) \qquad (13)$$

where the matrix $\underline{a}'^{(s)}$ can be obtained from (10) and (12) and $\underline{b}^{(s)}$ is the same as in Section I, since it is defined by an equation that does not contain any energy dependent parameter.

On the other hand

$$T_r\left(\underline{b}^{(s)}\ \underline{a}'^{(s)}\right) = \left[n+2b(n-1)\right]\left[1+(n+1)\ ks/R_o\right] \qquad (14)$$

where n is the number of second differences and use has been made of the expressions for $a'^{(s)}_{ij}$ as obtained from (10) and (12) and for $b^{(s)}_{ij}$ as given in Section I.

By using (14), (13) and (I.31) we obtain finally:

$$\sigma'^{*2}_s = \left(a^{(\nu)}_{12}\ \overline{x_i}^2 - \overline{x_i x_{i+1}}\right)\left[\left(a^{(\nu)}_{12} - a^{(s)}_{12}\right)\left(1+(n+1)ks/R_o\right)\right] \qquad (15)$$

which is the required estimate of σ'^2_s.

The relative variance of σ'^*_s is given by a formula similar to (I.33) with $\underline{a}'^{(s)}$ replacing $\underline{a}^{(s)}$ and μ'_r replacing μ_r, where

$$\mu'_r = \mu_r\left[1+(n+1)ks/R_o\right] \qquad (16)$$

The traces of matrices appearing in that formula are calculated for large n by using the same method as in reference 3) and the result is

$$S(\sigma'^*_s)/\sigma'^*_s = (1/2n)^{\frac{1}{2}}\left\{2.10+0.31\mu_2(1+nks/R_o)+0.18\mu_2^2(1+2nks/R_o)\right\}^{\frac{1}{2}} \qquad (17)$$

Strictly speaking the estimate of energy should be obtained by using the estimate of σ'_s, equation (15), together with equation (11); however the arguments of the end of Section I still apply and a better estimate is obtained by means of equations (I.36), (I.37) and (I.38), where B_r is to be replaced by $\bar{B}_r$.

It remains the problem of determining $\bar{B}_r$. The first step for that purpose is to evaluate x_c^2 and $\ell n\, x_a^2$ given by (I.12) and (I.13) where the constant value x'^2_c is to be replaced by $x'^2_c(\ell)$ given by (2); let x'^2_c and $\ell n\, x'^2_a$ represent the new values of those parameters. We obtain:

$$x'^2_c = x_c^2\,(1 + 2iks/R_o) \tag{18}$$

$$\ell n\, x'^2_a = \ell n\, x_a^2 \tag{19}$$

Equation (18) shows that x'^2_c depends on the position of a particular cell on the trajectory; this is not the case with x_c^2 given by (I.12) owing to the property of the coupling factors: $\gamma_i(\ell) = \gamma_{i+k}(\ell+ks)$. Thus the parameter B'_r that could be obtained by replacing x_c^2 and $\ell n\, x_a^2$ in equation (I.14) by the corresponding values (18) and (19) would also depend upon the position of a particular cell on the trajectory. We can avoid that difficulty by defining $\bar{B}_r$ as being the solution of an equation similar to (I.14) but with x_c^2 replaced by

$$\langle x_c^2 \rangle = \sum_i x'^2_c/n = x_c^2 \left[1 + (n+1)\, ks/R_o\right] \tag{20}$$

x_a^2 remaining unchanged. Since equation (I.14) is thus only

slightly modified it is easier to compute the corresponding fractional change in B_r rather than solving the equation for $\bar{B}_r$.

Let X represent the right hand side of equation (I.14); then

$$\delta B_r = \left[(B_r - \ell n B_r)/(B_r - 1)\right]\delta X \tag{21}$$

where $\delta B_r = dB_r/B_r$ and $\delta X = dX/X$.

On the other hand, since x_a remains fixed,

$$\delta X = (2/X)\,\delta x_c \tag{22}$$

Now, taking

$$\delta x_c = (\bar{x}_c^2 - x_c^2)^{\frac{1}{2}}/x_c = (n+1)ks/(2\,R_o) \tag{23}$$

we obtain from (21), (22) and (23):

$$\delta B_r = \left[1/(B_r - 1)\right](n+1)ks/R_o \tag{24}$$

and, therefore

$$\bar{B}_r = B_r\left\{1 + \left[1/(B_r - 1)\right](n+1)ks/R_o\right\} \tag{25}$$

Further, the fractional change in the scattering constant, K_r, can be obtained from (I.37) and (I.38):

$$\delta K_r = \delta B_r/2 + \delta M_r = (1/M_r - 1/2)\,\delta B_r \tag{26}$$

The new value of the scattering constant is then given by

$$\bar{K}_r \simeq K_r(1 + \delta K_r) \tag{27}$$

From (25), (26) and (27) we obtain, finally

$$\bar{K}_r \simeq K_r \left\{ 1 + \frac{1}{2M_r} \frac{2-M_r}{B_r-1} (n+1)ks/R_o \right\} \tag{28}$$

The value of the scattering constant, K_r, appearing in (I.36) should be replaced by $\bar{K}_r$ given above to account for energy variation along the track. We see that $\bar{K}_r \geqslant K_r$ a result that agrees with the observation made by different authors about discrepancies among measured and calculated values of the scattering constant [6].

The preceding arguments can be easily adapted with trivial changes, to finite differences of order greater than second. The relevant results for r^{th}-differences are summarized in the following formulae:

$$\sigma'^{*2}_s = 1/(a_{12}^{(\nu)} - a_{12}^{(s)})(a_{12}^{(\nu)} \overline{x_i^2} - \overline{x_i x_{i+1}}) \left[1 + (n+r-1)ks/R_o \right]^{-1} \tag{29}$$

$$(2/\pi)^{\frac{1}{2}} \sigma'^{*}_s = \bar{K}_r \, s^{3/2}/pv \tag{30}$$

$$\bar{K}_r = K_r \left\{ 1 + \frac{1}{2M_r} \cdot \frac{2-M_r}{B_r-1} (n+r-1)ks/R_o \right\} \tag{31}$$

The relative variance of σ^*_s for second, third and fourth differences has been tabulated by Solntseff [4], for the case in which energy losses are negligible. It can be shown that the effect of energy losses on the variances can be taken into account by replacing μ_r in those formulae by $\mu_r(1+nks/R_o)$, a correction that holds for $r-1 \ll n \ll R_o/ks$.

Higher order corrections to σ_s and K_r could be derived

similarly but should be used with reserve because the method outlined breaks down whenever energy losses are heavy enough to couple the processes of energy loss and accumulation of small angles.

III. Comparison With Experiment.

Scattering measurements were performed on pions stopping in the same emulsion sheet (Ilford, G-5) to provide material for a check on the results of the previous sections.

Measurements were taken with an ordinary Leitz-Ortholux research microscope with a 12,5x filar micrometer eyepiece and 100 x objective.

The cell size was chosen equal to 37,9 micron for all measurements; the estimate was based upon second differences with cut-off without replacement at four times the arithmetic mean. A computer program, suitable for UNIVAC 1105 systems, was produced to handle the data. The value of the scattering constant with cut-off, as obtained from formulae given by Barkas [7] plus a 2% correction mentioned by Solnseff [4], was 24,43 MeV-degree/$(100\mu)^{\frac{1}{2}}$; the exponent of the range energy relationship was taken from Powell et al [8], and the range measurements were done by rectifying the trajectories.

Two groups of measurements were considered:

i) 103 measurements starting at 8 mm residual range;

ii) 47 measurements starting at 12 mm residual range. In both cases the measurements progressed in the sense of decreasing pion energy.

The distribution of path lengths used for each set of measurements is shown in Fig. 1 and Fig. 2, for residual ranges of 8 mm and 12 mm respectively. The distribution of energy values obtained for each group is shown in Fig. 3 and Fig. 4, for the measurements taken at 8 mm residual range and in Fig. 5 and Fig. 6 for the set at 12 mm residual range. The data shown in Figs. 3 and 5 are the uncorrected ones; in Figs. 4 and 6 appear the data treated according to the procedure developped in Section II.

The average values of energy, obtained from the different groups are:

i) at 8 mm residual range:

$$E = 18.26 \pm 0.39 \text{ MeV} \quad (1)$$

$$E_c = 20.16 \pm 0.43 \text{ MeV} \quad (2)$$

ii) at 12 mm residual range:

$$E = 22.84 \pm 0.50 \text{ MeV} \quad (3)$$

$$E_c = 24.77 \pm 0.57 \text{ MeV} \quad (4)$$

where E and E_c hold for uncorrected and corrected energy respectively.

Those values are to be compared with the corresponding ones

given by range-energy tables; we have used for that purpose the formulae given by Powell et al [8]. We normalized our ranges to values in an emulsion with the same stopping power as in those formulae by means of range measurements in muons. The range of muons in our emulsions is 601.2 ± 8.3 micron.

We thus obtain:

i) at R_o = 8 mm, E_o = 20.13 MeV (5)

ii) at R_o = 12 mm, E_o = 25.02 MeV (6)

We see that the corrected values obtained from scattering measurements are in excellent agreement with the results above.

We conclude that the estimator of true scattering must be corrected to account for the energy losses and that the method presented is this paper can be used successfuly for this purpose.

Acknowledgements.

The motivations for this investigation arose during the execution of a work sugested by Professor C. M. G. Lattes. The author is indebted to him for his support and interest.

Computations were performed in a UNIVAC 1105 electronic computer that belongs to the National Service of Census (S.N.R.). The author is grateful to the Directors of that Institute for all facilities offered and to the computer crew for continuous assistence.

Finally the author is indebted to Miss L.R.Gil for doing all measurements.

Bibliography

1. G. Molière, Zeits. fur Naturf. 2a (1947) 133; 3a (1948); 10a (1955) 177.

2. B. d'Espagnat, Comptes Rendus 232 (1951) 800; Journ. Phys. Rad. 13 (1952) 74.

3. M. Huybrechts, Thèse de Doctorat, Bruxelles (1955).

4. N. Solntseff, Nucl. Phys. 4 (1957) 337; 6 (1958) 222.

5. K. Gottstein, M. G. K. Menon, J. H. Mulvey, C. O'Ceallaigh and O. Rochat, Phil. Mag. 42 (1951) 708; L. Voyvodic and E. Pickup, Phys. Rev. 85 (1952) 91.

6. M. Di Corato, B. Locatelli and D. Hirschberg, Nuovo Cim. Suppl. 4 (1956) 448; B. Depaux, Annales de Phys. 6 (1961) 1335; A.O. Vaisenberg, V. A. Smirnits-kii and E. D.Kolganova, JETP 13 (1961) 734.

7. W. Barkas, UCRL 2579 (1954).

8. C. F. Powell, P. H. Fowler and D. H. Perkins, The Study of Elementary particles by the Photographic Method, Pergamon Press (1959).

* * *

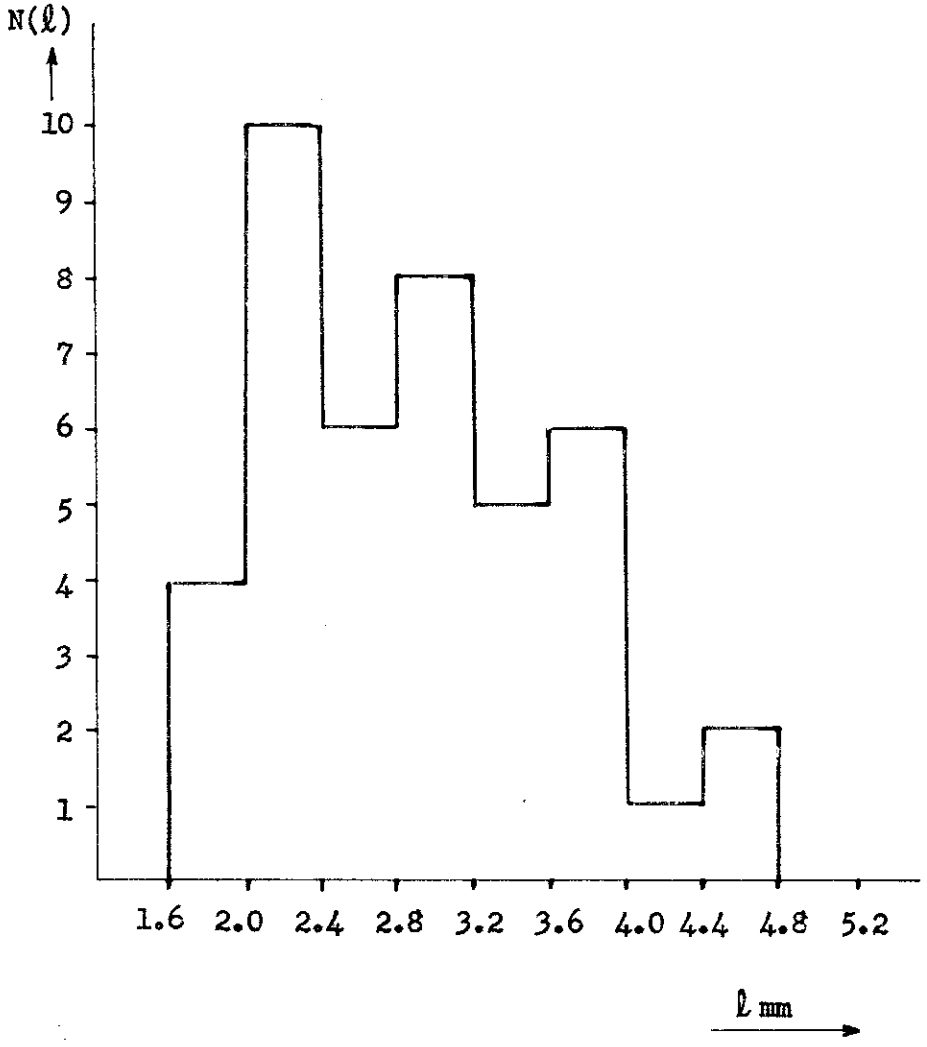

Fig. 1

The distribution of path lengths used for the measurements starting at 8 mm residual range.

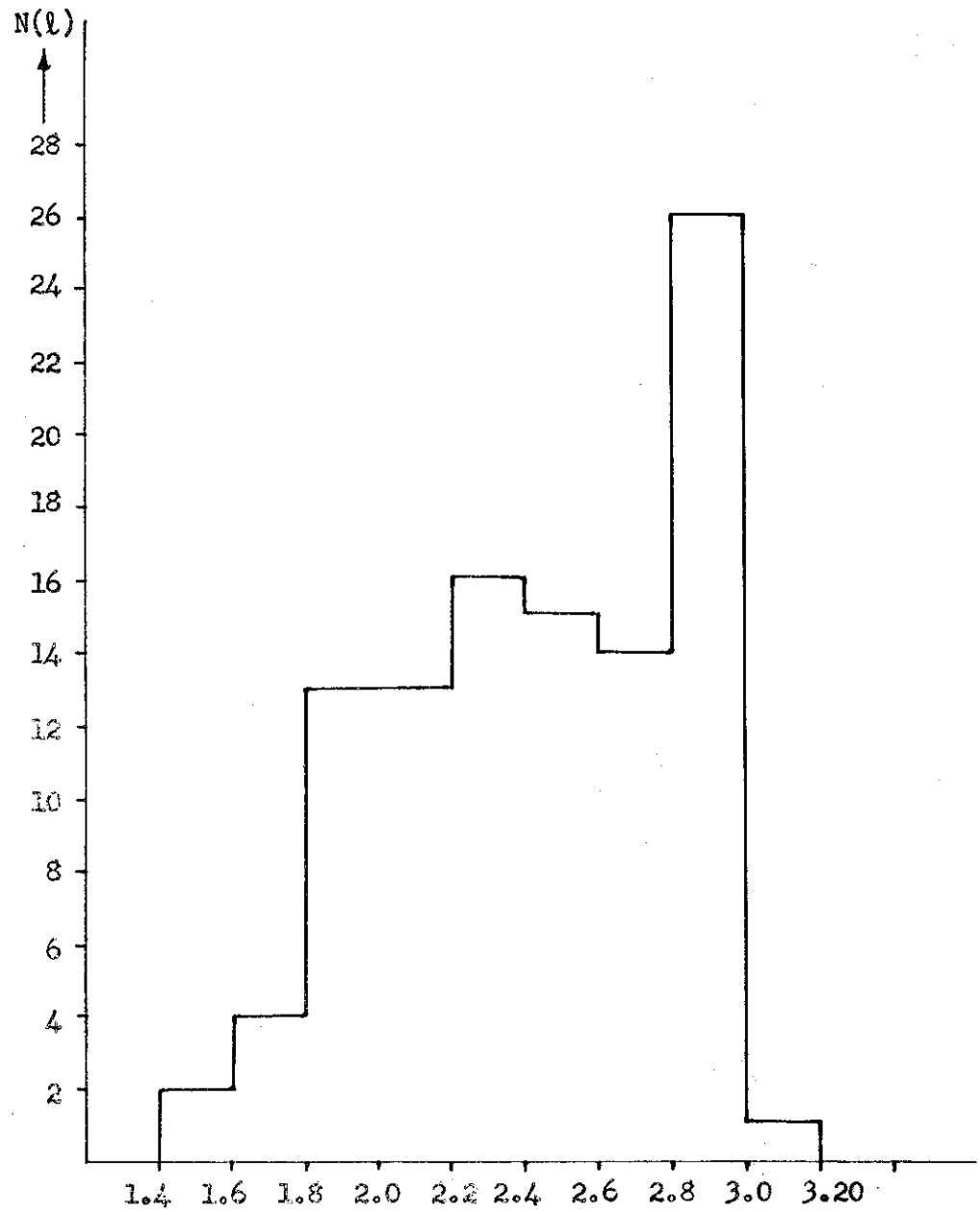

Fig. 2

The distribution of path lengths used for the measurements starting at 12 mm residual range.

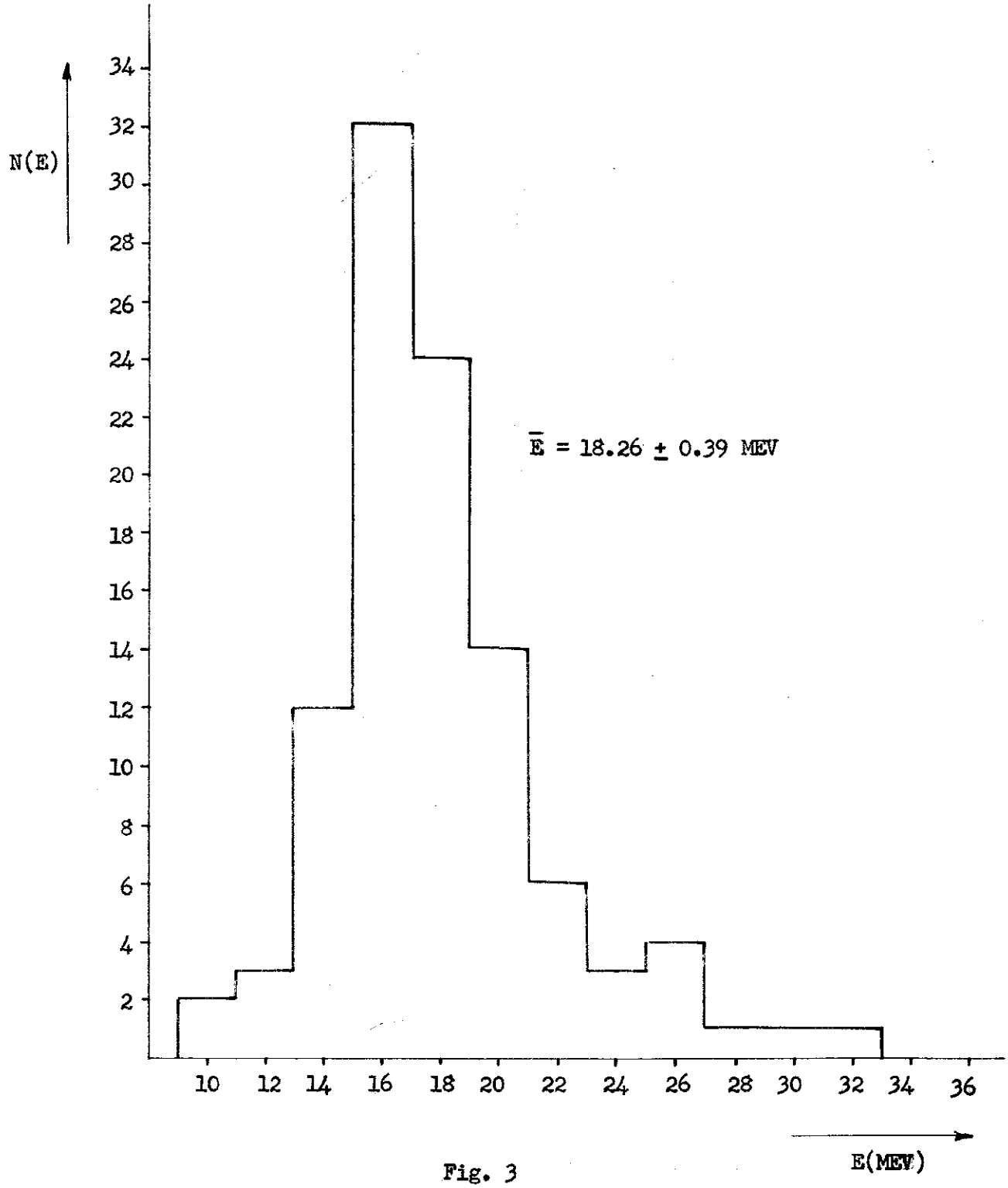

Fig. 3

The distribution in energy obtained from uncorrected observations starting at 8 mm residual range.

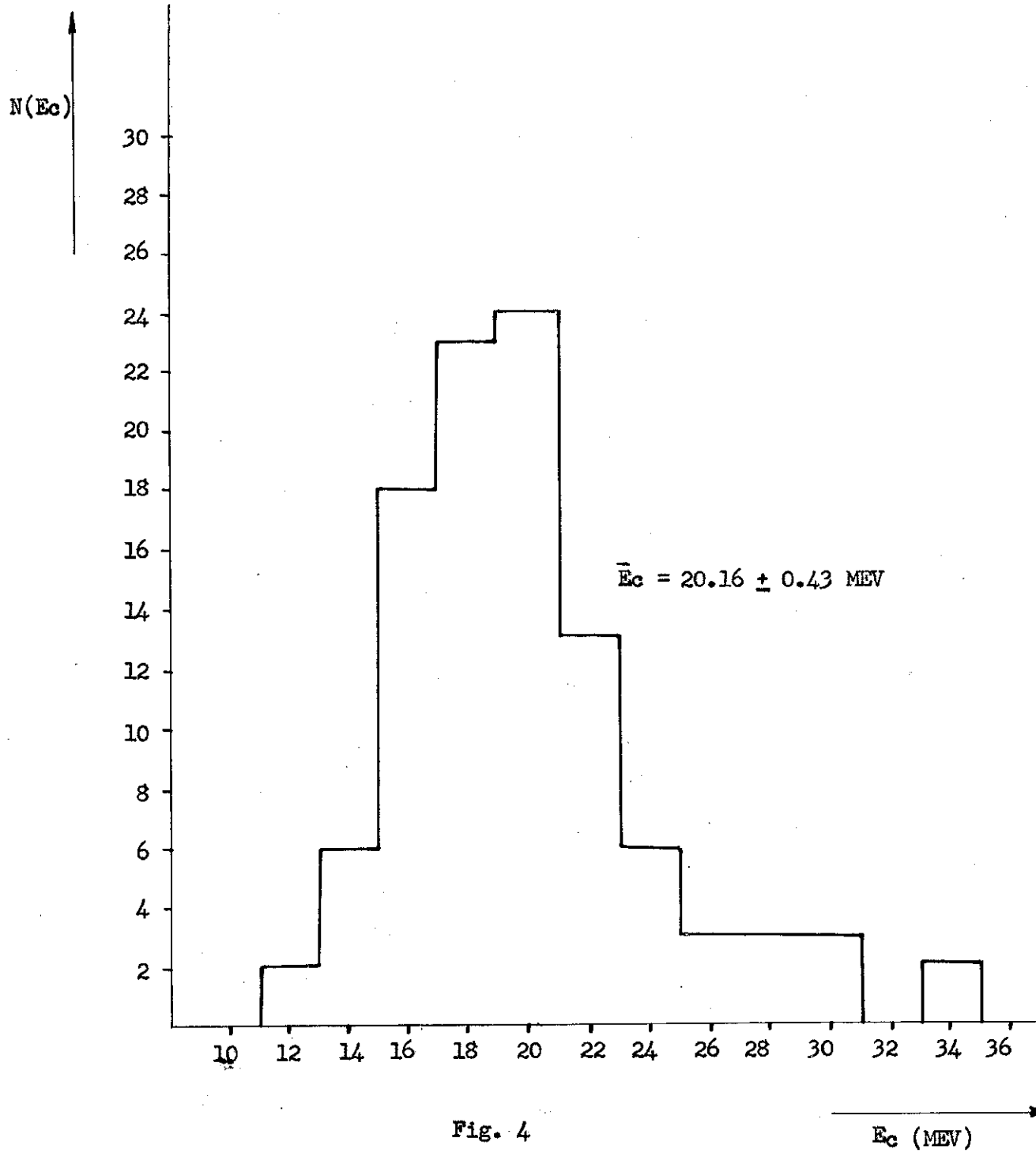

Fig. 4

The distribution in energy obtained from corrected observation starting at 8 mm residual range.

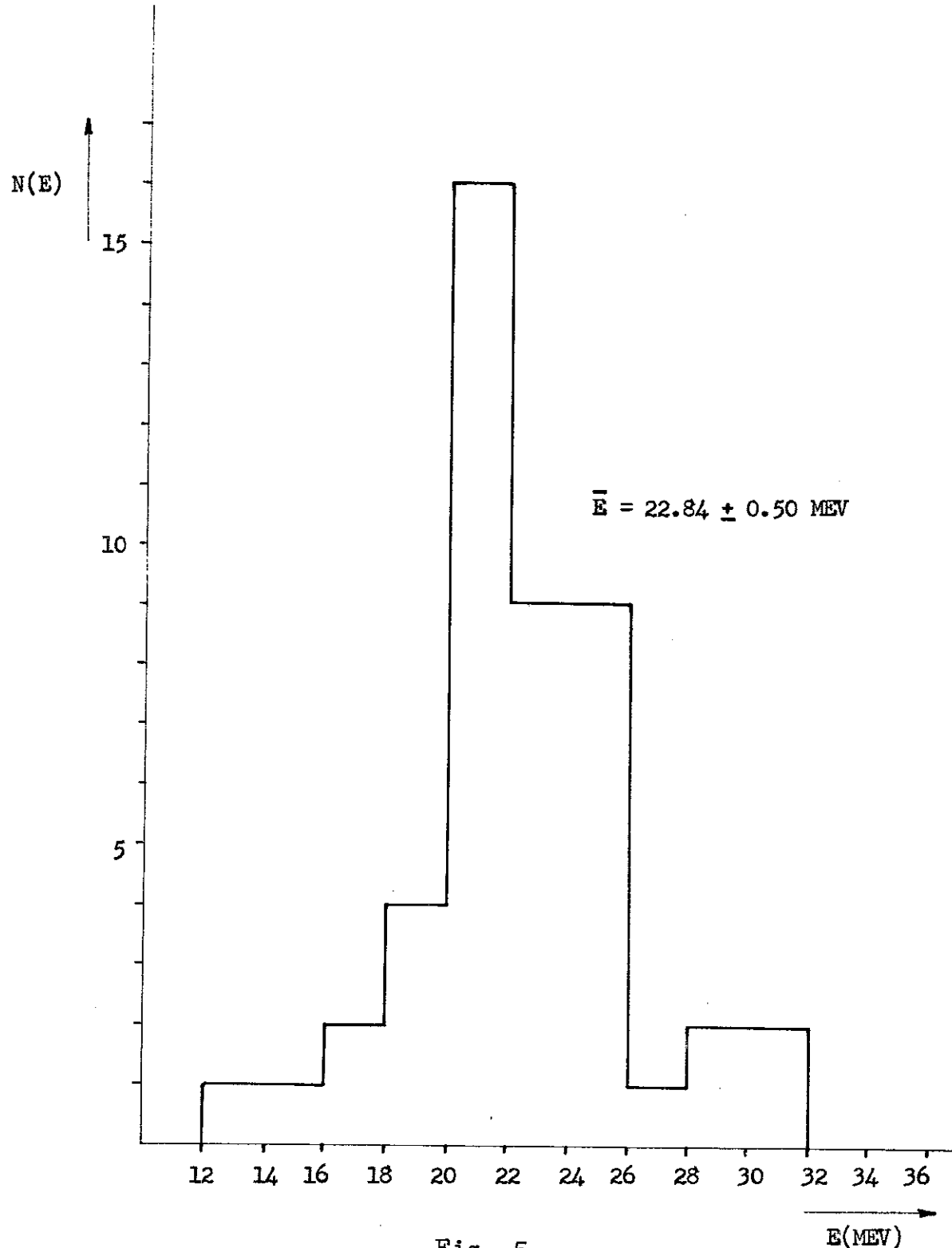

Fig. 5

The distribution in energy obtained from uncorrected observations starting at 12 mm residual range.

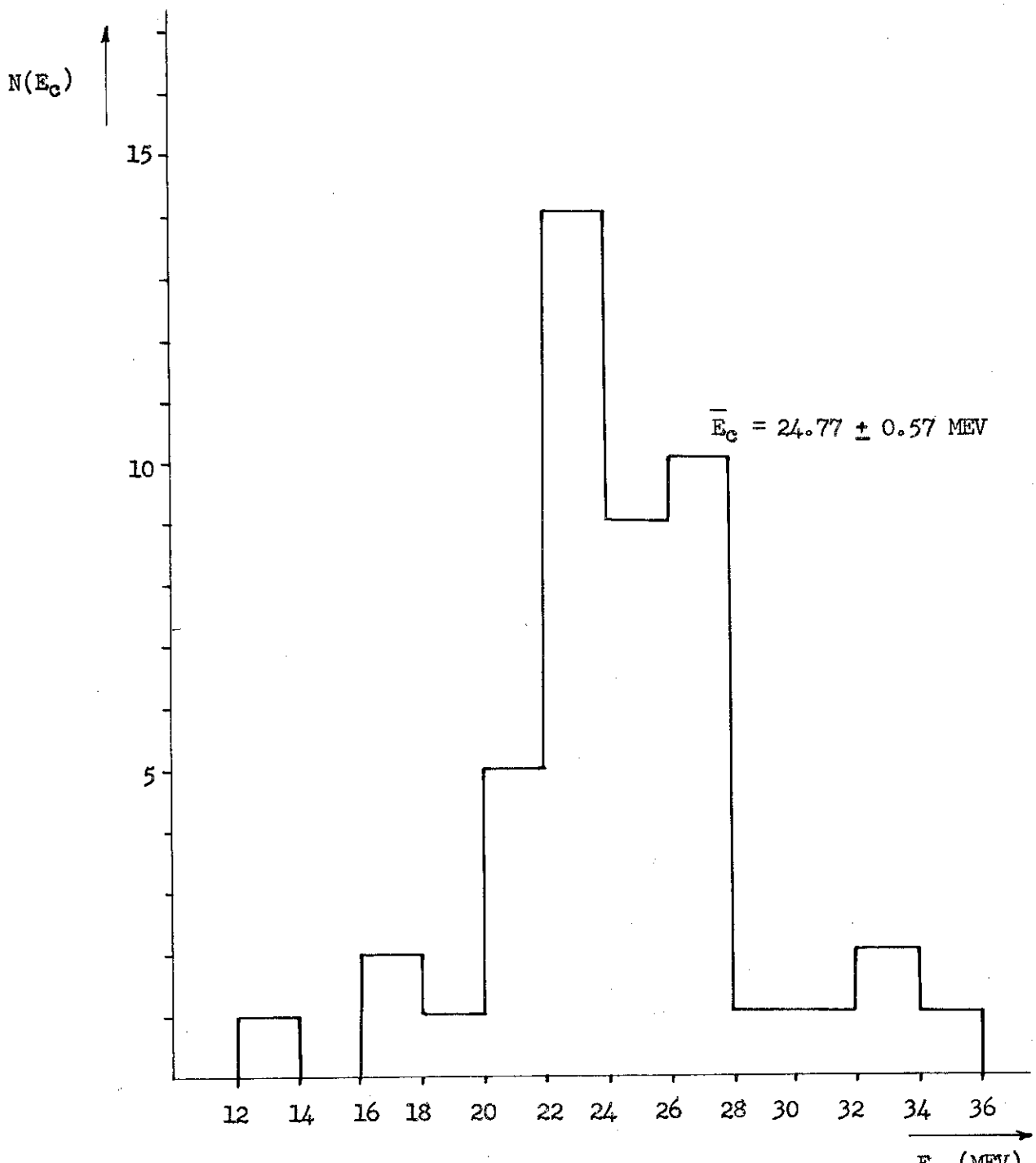

Fig. 6

The distribution in energy obtained from corrected observations starting at 12 mm residual range.