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## Notas de Física

CBPF-NF-016/94

January 1994

### Solutions of the Spherically Symmetric Wave Equation in $p + q$ Dimensions

W. Bietenholz and J.J. Giambiagi

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, 22290-180, Rio de Janeiro/RJ,  
Brasil

Work supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)

### Abstract

We discuss solutions of the spherically symmetric wave equation and Klein Gordon equation in an arbitrary number of spatial and temporal dimensions. Starting from a given solution, we present various procedures to generate further solutions in the same or in different dimensions. The transition from odd to even and non integer dimensions can be performed by fractional derivation or integration. The dimensional shift, however, can also be interpreted simply as a modification of the dynamics. We also discuss the analytic continuation to arbitrary real powers of the D'Alembert operator. There, particular peculiarities in the pole structure show up when  $p$  and  $q$  are both even. Finally, we give operators which transform a space into a time coordinate and v.v. and comment on their possible relation to black holes. In this context, we describe a few aspects of the extension of our discussion to a curved metrics.

Key-words: Wave equation; Klein Gordon equation; Field theory; Black holes; Distribution theory.

## 1 Introduction

The wave equation plays a very important role in practically all branches of physics. It has a fundamental meaning in classical as well as quantum physics, including field theory. This refers to both, the non relativistic as well as the relativistic description. Hence it is strongly motivated to discuss solutions of the wave eq., or variants thereof such as the Klein Gordon eq., in all possible situations.

Also the use of space-time dimensions different from  $3 + 1$  is well established nowadays in most parts of physics. In this paper we provide a background of solutions for the spherically symmetric wave eq., which may turn out to be useful anywhere. But we do not discuss particular applications here.

Recently, Bollini and Giambiagi described fractional powers of the D'Alembert operator and of the Laplace operator in one temporal and an arbitrary number of spatial dimensions and discussed Huygens' principle and causality in this framework [1]. In a subsequent paper, Giambiagi referred to the standard D'Alembert operator and discussed relations among solutions of the wave eq. again in one temporal and different spatial dimensions. He also considered the Klein Gordon eq. and Green's functions.

Now we want to generalize both of these considerations to  $p + q$  dimensions. We are particularly interested in operations shifting the dimensions in which a solution is valid. As we will see, interesting and qualitatively new properties arise, particularly in the pole structure for even  $p$  and  $q$ . The mathematical background for this is beautifully outlined in the classical book of Gelfand and Shilov [3]. At the end, we add some remarks about a generalization to curved metrics. This is motivated from the fact that transitions of a spatial to a temporal dimension and vice versa are known in general relativity: if we cross the boundary of a black hole - described e.g. by the Schwarzschild metrics - time becomes spatial and the radial component becomes temporal.

## 2 The spherical wave equation in $d + 1$ dimension

The case of one time and an arbitrary number of spatial dimensions has been discussed extensively in [2]. We start by adding some complementary observations to this case.

We consider solutions of the spherically symmetric wave eq.

$$\left[ \partial_r^2 + \frac{d-1}{r} \partial_r - \partial_t^2 \right] \phi_d(t, r) = 0 \quad (1)$$

where  $\partial_r \doteq \partial/\partial r$  etc. We are not interested in constant factors or additive constants in  $\phi_d$ .

With the well known substitution:

$$\Omega_d(t, r) \doteq r^{(1-d)/2} \phi_d(t, r) \quad (2)$$

we can absorb the linear derivative. Eq. (1) takes the form:

$$\left[ \partial_r^2 + \frac{(d-1)(d-3)}{4r^2} - \partial_t^2 \right] \Omega_d(t, r) = 0 \quad (3)$$

and we can immediately read off general solutions for  $d \in \{1, 3\}$ :

$$\begin{aligned} \Omega_1(t, r) &= F_1(t - r) + F_2(t + r) ; & \Omega_3(t, r) &= f_1(t - r) + f_2(t + r) \\ \rightarrow \phi_1(t, r) &= F_1(t - r) + F_2(t + r) ; & \phi_3(t, r) &= \frac{f_1(t - r) + f_2(t + r)}{r} \end{aligned} \quad (4)$$

where  $F_1, F_2, f_1, f_2$  are arbitrary functions or symbolic functions. If we choose  $\delta$  functions for them, the above solutions represent waves which are: for  $d = 1$  moving to the right/left, and for  $d = 3$  outgoing/incoming.

Here  $d = 1$  and  $d = 3$  play a special role, since for no other dimension there is a solution of the type:

$$\phi_d(t, r) = f(t \pm r)r^\alpha \quad (\alpha \in \mathbb{R}) \quad (5)$$

This can be seen by inserting this ansatz in (1), which yields the conditions:  $2\alpha = 1 - d$  and  $\alpha^2 = \alpha(2 - d)$ , with the only solutions:  $\alpha = 0, d = 1$  or  $\alpha = -1, d = 3$ .

In [2] it was shown that if  $\phi_d$  is a solution of (1), then the same holds for

$$\phi_{d+2}(t, r) = -\frac{1}{r} \partial_r \phi_d(t, r) \quad (6)$$

With this respect, we can identify in (4):  $F'_1 = f_1, F'_2 = f_2$  and we obtain a set of solutions for all odd  $d$ , generated by the choice for  $F_1, F_2$ , e.g.

$$\phi_5(t, r) = \frac{f_1(t - r) + f_2(t + r)}{r^3} + \frac{f'_1(t - r) - f'_2(t + r)}{r^2} \quad \text{etc.} \quad (7)$$

Let's consider the two signs individually and first let  $F_2 \equiv 0$ . Then the solutions for all odd  $d$  - generated from  $F_1$  by means of (6) - have the form:

$$\phi_{2k+1}(t, r) = \sum_{n=0}^k a_n^{(k)} \frac{f^{(n)}(t - r)}{r^{2k-n-1}} \quad (8)$$

The coefficients  $a_n^{(k)}$  are the elements of a "modified Pascalian triangle":

$k = 1$	1						
$k = 2$	1	1					
$k = 3$	3	3	1				
$k = 4$	15	15	6	1			
$k = 5$	105	105	45	10	1		
$k = 6$	945	945	420	105	15	1	
...	...	...	...	...	...	...	...
	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	...

( $d = 1$  is not represented here.)

The elements at the margins are:  $a_0^{(k)} = (2k - 1)!!$ ,  $a_k^{(k)} = 1$ , and the rest is determined by the recursion formula:

$$a_n^{(k)} = a_{n-1}^{(k-1)} + (2k - n - 1)a_n^{(k-1)} \quad (10)$$

As in the Pascalian triangle, the elements off the margin are built from the two elements vertically and on the left above. The only modifications are the left margin and the factor  $(2k - n - 1)$  in (10).

We add two observations for the elements next to the margins, which can easily be proved by induction. For all  $k \in \mathbb{N}$  these elements take the form:

$$a_1^{(k)} = a_0^{(k)}, \quad a_{k-1}^{(k)} = \binom{k+1}{2} \quad (11)$$

Now we consider the other case:  $F_1 \equiv 0$ . The solutions generated by (6) from  $F_2$  can be represented in the same way as (8):

$$\phi_{2k+1}(t, r) = \sum_{n=0}^k \bar{a}_n^{(k)} \frac{f^{(n)}(t+r)}{r^{2k-n-1}} \quad (12)$$

The triangle for  $\bar{a}_n^{(k)}$ , however, is different:

$k = 1$	1						
$k = 2$	1	-1					
$k = 3$	3	-1	1				
$k = 4$	15	-1	2	-1			
$k = 5$	105	9	9	-2	1		
$k = 6$	945	177	72	-3	3	-1	
...	...						
	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	...

The left margin is the same as in (9) and also the recursion relation (10) still holds, but the right margin is oscillating:  $\bar{a}_k^{(k)} = (-1)^k$ , thus altering all the off margin elements. For the second column from the right we have the general expression:

$$\bar{a}_{k-1}^{(k)} = \begin{cases} (k+1)/2 & k \text{ odd} \\ -k/2 & k \text{ even} \end{cases} \quad (14)$$

### 3 The spherical wave equation in $p + q$ dimensions

We consider a flat space with coordinates  $(t_1, \dots, t_q, x_1, \dots, x_p)$  and search for solutions of the wave eq., which depend only on  $r \doteq \sqrt{\sum_{i=1}^p x_i^2}$  and  $\tau \doteq \sqrt{\sum_{i=1}^q t_i^2}$ . They have to fulfill:

$$\left[ \partial_r^2 + \frac{p-1}{r} \partial_r - \partial_\tau^2 - \frac{q-1}{\tau} \partial_\tau \right] \phi_{p,q}(\tau, r) = 0 \quad (15)$$

i.e. we generalize the case  $q = 1$  considered in section 1. Accordingly, we also generalize the first ansatz to:

$$\phi_{p,q}(\tau, r) = f(\tau \pm r) r^\alpha r^\beta \quad (\alpha, \beta \in \mathbb{R}) \quad (16)$$

Again the wave eq. imposes constraints, which allow  $\alpha$  and  $\beta$  only to take the values 0 or  $-1$ , and therefore:  $p, q \in \{1, 3\}$ . The only new solution that we obtain is:

$$\phi_{3,3}(\tau, r) = \frac{f(\tau \pm r)}{\tau r} \quad (17)$$

Now we search again for a rule how to generate solutions in higher dimensions starting from these particular ones.

**Rule 1** If  $\phi_{p,q}$  is a solution of the wave eq. (15), then also

$$\phi_{p+2,q} \doteq -\frac{1}{r}\partial_r\phi_{p,q} \quad \text{and} \quad \phi_{p,q+2} \doteq -\frac{1}{r}\partial_r\phi_{p,q}$$

are solutions (always for the corresponding dimensions).

This generalizes the rule (6) and yields for example the solution (17). The proof by induction is straightforward, if we just consider

$$\left[\partial_r^2 + \frac{p+1}{r}\partial_r\right]\frac{1}{r}\partial_r = \frac{1}{r}\partial_r\left[\partial_r^2 + \frac{p-1}{r}\partial_r\right]$$

and the same for  $\tau$ .

Hence starting from any  $\phi_{1,1} = F_1(\tau - r) + F_2(\tau + r)$  we can generate immediately solutions for all odd  $p$  and  $q$ . If we choose  $F_1, F_2$  to be step functions, then the degree of the pole in a solution  $\phi_{p,q}$ , which is generated by application of rule 1, is  $(p-1)/2 + (q-1)/2$ . In particular this set of solutions includes the physical monopole solution in 3+1 dimensions.

Now we want to postulate a second rule for generating solutions in higher dimensions from a given one:

**Rule 2** If  $\phi_{p,q}$  is a solution of (15), then also

$$\bar{\phi}_{p+2,q+2} \doteq -\left[\frac{1}{r}\partial_r + \frac{1}{\tau}\partial_\tau\right]\phi_{p,q}$$

is a solution.

The proof consists just of inserting rule 1. But we emphasize that  $\bar{\phi}_{p+2,q+2}$  does not coincide with  $\phi_{p+2,q+2} = \frac{1}{r\tau}\partial_r\partial_\tau\phi_{p,q}$ . To clarify the situation, we introduce the concept of equivalence classes of solutions in different dimensions:

*Definition:* The solutions  $\phi_{p+2,q}$  ( $\phi_{p,q+2}$ ) and  $\phi_{p,q}$  belong to the same equivalence class, if they are related as  $\phi_{p+2,q} = -\frac{1}{r}\partial_r\phi_{p,q}$  ( $\phi_{p,q+2} = -\frac{1}{\tau}\partial_\tau\phi_{p,q}$ ), where we really mean "equal to", excluding different additive or multiplicative constants.

Hence every solution  $\phi_{1,1}$  defines an equivalence class with a unique  $\phi_{p,q}$  for all odd  $p, q$ . Now rule 2 can be formulated like this:

If  $\phi_{p+2,q}$  and  $\phi_{p,q+2}$  are solutions belonging to the same equivalence class, then their superposition  $\bar{\phi}_{p+2,q+2} = \phi_{p+2,q} + \phi_{p,q+2}$  is a solution too, which does, however, not belong to the same equivalence class.

Of course we may also define a second type of equivalence classes of solutions related by rule 2. In such classes,  $p - q$  must be fix. Starting from a given  $\phi_{p,q}$ , all

$$\bar{\phi}_{p+2n,q+2n} = \sum_{k=0}^n \binom{n}{k} \phi_{p+2k,q+2(n-k)} \quad (18)$$

belong to the same second type equivalence class as  $\phi_{p,q}$ , if all the  $\phi_{p+2k,q+2(n-k)}$  belong to the same first type equivalence class as  $\phi_{p,q}$ . This generalizes the above formulation.

Let's reconsider  $p = q = 3$  and start from  $\phi_{1,1}$  given in (4). We obtain:

$$\begin{aligned}\phi_{3,3}(\tau, r) &= \frac{1}{\tau r}[-F_1''(\tau - r) + F_2''(\tau + r)] \\ \bar{\phi}_{3,3}(\tau, r) &= \frac{1}{\tau r}[(\tau - r)F_1'(\tau - r) + (\tau + r)F_2'(\tau + r)]\end{aligned}\quad (19)$$

These functions are different, but not basically different in the sense that both of them fit with solution (17).

If we choose  $F_1, F_2$  to be step functions, then  $F_1'', F_2'' = \delta'$ , i.e. we get from the first rule a dipole solution in 3+3 dimensions. (We noted before that in general the pole has the degree  $(p/2 + q/2 - 1)$ ). The second rule seems to generate a monopole solution. But in fact it vanishes since  $x\delta(x) \equiv 0$ .

To see that the two prescriptions do provide basically different solutions in general, we consider as an example  $p = q = 5$  and start from  $\phi_{3,3} = f(\tau - r)/\tau r$ :

$$\phi_{5,5}(\tau, r) = \frac{1}{(\tau r)^3}[f(\tau - r) - f'(\tau - r)(\tau - r) - f''(\tau - r)\tau r] \quad (20)$$

$$\bar{\phi}_{5,5}(\tau r) = \frac{1}{(\tau r)^3}[f(\tau - r)(\tau^2 + r^2) + f'(\tau - r)\tau r(\tau - r)] \quad (21)$$

Clearly, these solutions are not related any more by a redefinition of  $f$  in one of them; this we see already from the different pole structure for  $f = \delta'$ .

As a last example we start from an outgoing wave  $\phi_{3,1} = f(\tau - r)/r$  and proceed to 5 + 3 dimensions:

$$\phi_{5,3}(\tau, r) = -\frac{1}{\tau r^3}[r f''(\tau - r) + f'(\tau - r)] \quad (22)$$

$$\bar{\phi}_{5,3}(\tau, r) = -\frac{1}{\tau r^3}[f'(\tau - r)r(\tau - r) - \tau f(\tau - r)] \quad (23)$$

If we choose  $f = \delta$ , then the above solution  $\bar{\phi}_{5,3}$  vanishes.

The first prescription is more powerful, since the second one is restricted to a simultaneous increase of spatial and temporal dimensions (hence it does, e.g. not yield solutions in  $1 + q$  or  $p + 1$  dimensions). But the latter is useful to complete the first one, because it provides basically different solutions. We also note that the application of the two prescriptions commutes. Hence if we proceed e.g. from a  $\phi_{3,5}$  to  $\bar{\phi}_{7,7}$ , it does not matter if we first go to  $\phi_{5,5}$  and then to  $\bar{\phi}_{7,7}$ , or if we start with  $\bar{\phi}_{5,7}$  and then apply the first prescription to obtain the same  $\bar{\phi}_{7,7}$ . On the other hand, if we go e.g. from  $\phi_{3,3}$  to a solution in  $7 + 7$  dimensions, we have to distinguish if we use the second rule not at all, once or twice (regardless of the order of the operators). If we start from the form (17), then the solution in  $7 + 7$  dimensions contains maximally 4, 3, 2 derivatives of  $f$ , respectively.

Generally: if we replace two applications of the first rule by one application of the second rule, then the maximal pole strength is reduced by one.

## 4 Solutions, which depend only on $\xi \doteq (\tau - r)(\tau + r)$

Here we treat the two signs between  $\tau$  and  $r$  democratically. Let  $\psi_n(\xi)$  be a solution in  $n = p + q$  dimensions. In [2] it was noted that such solutions are:

$$\psi_2(\xi) = \ln \xi, \quad \psi_n(\xi) = \xi^{1-n/2} \quad (n \geq 3) \quad (24)$$

where for  $n > 2$  we can not identify  $p$  and  $q$  separately. Note that  $\psi_2(\xi)$  is a special case of the form given in (4).

This string of solutions corresponds to rule 1, since

$$-\frac{1}{r}\partial_r\psi_n(\xi) = 2f'_n(\xi) = \frac{1}{\tau}\partial_\tau\psi_n(\xi) \quad (25)$$

is indeed the solution given in (24) for  $n + 2$ .

The second rule is supposed to generate a different solution for  $n + 4$ , which would be strange since the set (24) is complete (up to constants). But to look at (25) we recognize that the second rule only yields trivial solutions here:  $\psi_{n+4}(\xi) \equiv 0$ . So for this special type of solutions only the first rule is useful. For further details, see [2].

## 5 Generalization to the Klein Gordon equation

Now we generalize (15) to the spherically symmetric Klein Gordon equation, i.e. the relativistic equation of motion for a *massive* scalar particle:

$$\left[\partial_r^2 + \frac{p-1}{r}\partial_r - \partial_\tau^2 - \frac{q-1}{\tau}\partial_\tau + m^2\right]\phi_{p,q}^{(m)}(\tau, r) = 0 \quad (26)$$

It is easy to check that both rules still hold for this generalized case. The reason is simply that the factor  $m^2$  commutes with the differential operators. (However, additive constants are not arbitrary any more in  $\phi_{p,q}^{(m)}$ .)

Again we search for solutions of the form  $\psi_n^{(m)}(\xi)$ . In terms of  $\xi$ , eq. (26) reads:

$$\left[\xi\partial_\xi^2 + \frac{n}{2}\partial_\xi - \left(\frac{m}{2}\right)^2\right]\psi_n^{(m)}(\xi) = 0 \quad (27)$$

In [5], p. 972 we find the solution of the differential eq.

$$u''(z) + \frac{1-\nu}{z}u'(z) - \frac{1}{4z}u(z) = 0 \quad (28)$$

namely:

$$u(z) = z^{\nu/2}Z_\nu(i\sqrt{z}) \quad (29)$$

where  $Z_\nu$  is any Bessel function. For  $m \neq 0$  we can transform (27) to this form by substituting  $z \doteq m^2\xi$ . Thus the solution is:

$$\psi_n^{(m)}(\xi) = (m\sqrt{\xi})^{1-n/2}Z_{1-n/2}(im\sqrt{\xi}) \quad (30)$$

(where we assume  $m > 0$ ). In

$$\partial_{m^2}\psi_n^{(m)}(\xi) = \frac{2-n}{4m^2}\psi_n^{(m)}(\xi) + \frac{1}{2}i\xi(m\sqrt{\xi})^{-n/2}Z'_{1-n/2}(im\sqrt{\xi}) \quad (31)$$

the derivated Bessel function can be expressed in various ways in terms of (non derivated)  $Z_{1-n/2\pm 1}$ , which describes the dynamics in different dimensions.

Of course, in the limit  $m \rightarrow 0$  we recover the solutions of the preceding section.



## 6 Transition to even and fractional dimensions

The first rule in the form of section 3 only permits steps of two dimensions. To arrive at non odd dimensions from the explicit solutions given above by means of that rule, we need a fractional application of the operator, which provides the dimensional shift. This concept becomes much simpler if we use for this operator the identity:

$$-\frac{1}{r}\partial_r \equiv -2\partial_{r^2} \quad (32)$$

We ignore the factor -2 and postulate the natural *generalization of rule 1*:

If  $\phi_{p,q}$  is a solution of the spherical wave eq. (15) (or Klein Gordon eq. (26)), then also

$$\phi_{p+2\alpha,q} \doteq \partial_{r^2}^\alpha \phi_{p,q} \quad \text{and} \quad \phi_{p,q+2\beta} \doteq \partial_{\tau^2}^\beta \phi_{p,q} \quad (33)$$

are solutions in  $(p+2\alpha)+q$  and  $p+(q+2\beta)$  dimensions, respectively, for all  $\alpha, \beta \in \mathbb{R}$ , if only  $p+2\alpha \geq 1$  resp.  $q+2\beta \geq 1$ .

We saw this before for integer  $\alpha, \beta$ , hence we just consider non integers now. Clearly the two statements are equivalent, so let's only consider the first one.

For the fractional derivation - or integration for  $\alpha < 0$  - we refer to the definition given by the Weyl transformation (see e.g. [3, 4])

$$\begin{aligned} \phi_{p+2\alpha,q}(\tau, r) &= \frac{1}{\Gamma(-\alpha)} \int_{r^2}^{\infty} (u-r^2)^{-1-\alpha} \phi_{p,q}(\tau, \sqrt{u}) du \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} x^{-1-\alpha} \phi_{p,q}(\tau, \sqrt{r^2+x}) dx \end{aligned} \quad (34)$$

We substitute  $y \doteq \sqrt{r^2+x}$ , and we have to check now if the following expression vanishes:

$$\begin{aligned} & \left[ \partial_r^2 + \frac{p-1+2\alpha}{r} \partial_r - \partial_\tau^2 - \frac{q-1}{\tau} \partial_\tau \right] \int_0^{\infty} x^{-1-\alpha} \phi_{p,q}(\tau, y) \\ \propto & \int_0^{\infty} x^{-1-\alpha} \left[ \left( 1 - \frac{x}{y^2} \partial_y^2 + \frac{p+2\alpha-r^2/y^2}{y} \partial_y - \partial_\tau^2 - \frac{q-1}{\tau} \partial_\tau \right) \phi_{p,q}(\tau, y) \right. \\ = & \int_0^{\infty} x^{-1-\alpha} \left[ -\frac{2x}{y} \partial_x + \frac{2\alpha}{y} + \frac{x}{y^3} \right] \partial_y \phi_{p,q}(\tau, y) dx \end{aligned}$$

In the sense of analytic continuation we can perform partial integration in the first term without boundary terms. This shows immediately that this expression vanishes, as we postulated.

In principle also the second rule could be generalized accordingly, but the resulting fractional operator is very uneasy to handle.

Now we have a key for the construction of solutions in non odd dimensions. General solution are for instance:

$$\phi_{2,1}(t, r) = \int_0^{\infty} x^{-3/2} [F_1(t - \sqrt{r^2+x}) + F_2(t + \sqrt{r^2+x})] dx \quad (35)$$

$$\begin{aligned} \phi_{2,2}(\tau, r) &= \int_0^{\infty} dx \int_0^{\infty} dy (xy)^{-3/2} \cdot \\ & \quad [F_1(\sqrt{\tau^2+y} - \sqrt{r^2+x}) + F_2(\sqrt{\tau^2+y} + \sqrt{r^2+x})] \end{aligned} \quad (36)$$

etc.

Of course we would like to have really explicit solutions. But before evaluating them for particular functions  $F_1, F_2$ , we construct some solutions for even dimensions in an independent way.

## 7 Explicit solutions for even dimensions

In particular the case of  $2 + 1$  dimensions is motivated from solid state physics: some crystals consist of layers, where the interaction between layers is strongly suppressed with respect to the interactions inside the layers.

In view of our recursion rules for the generation of solutions in higher dimensions, we have to concentrate on finding particular solutions for  $2 + 1$ ,  $1 + 2$  and  $2 + 2$  dimensions.

So far, we only have the - manifestly Lorentz invariant - solutions of section 4 for even dimensions. Now we want to construct different ones, and then discuss them in the context of section 6.

If we look for polynomials in  $r, \tau$  fulfilling (15), the only non trivial - i.e. non constant - solution is (except for the linear solutions for  $\phi_{1,1}$ )

$$\phi_{p,p}(\tau, r) = \tau^2 + r^2 \quad (37)$$

So we have a new solution for any  $p = q$ , e.g.  $\phi_{2,2}$ , but it does not yield a non trivial string of solutions from our theorems. However, it is new, even in  $p = q = 1$ , and it will help us to find less obvious solutions.

We make an ansatz that generalizes somehow the solutions depending only on  $\xi$ :

$$\phi(\tau, r) = f(\tau, r)(\tau^2 \pm r^2)^\alpha \quad (\alpha \in \mathbb{R}) \quad (38)$$

For the upper sign we can only reproduce (37), so we concentrate on the lower sign (Lorentz invariant bracket), assume  $\tau^2 \neq r^2$  (off light cone) and denote  $f_r \doteq \partial_r f$  etc. We arrive at the condition:

$$(\tau^2 - r^2) \left\{ f_{\tau\tau} + \frac{p-1}{r} f_r - f_{\tau r} - \frac{q-1}{\tau} f_\tau \right\} = 4\alpha \left\{ r f_r + \tau f_\tau + \left( \alpha - 1 + \frac{n}{2} \right) f \right\} \quad (39)$$

(still with  $n = p + q$ ). We want the curly brackets to vanish, i.e.  $f$  should also obey the wave eq. If we take the trivial solution  $f \equiv \text{const.}$ , we obtain again the solutions (24), so we look for something more original.

Let's consider  $q = 1$  or  $p = 1$ : then a simple solution of (39) is  $f = \tau$  resp.  $f = r$  and  $\alpha = n/2$  and yields:

$$\phi_{p,1} = \frac{\tau}{(\tau^2 - r^2)^{(p+1)/2}} \quad ; \quad \phi_{1,q} = \frac{r}{(\tau^2 - r^2)^{(q+1)/2}} \quad (40)$$

In  $p = q = 1$  we get special cases of the form (4), but more interesting are  $\phi_{2,1}$  and  $\phi_{1,2}$  given in (40). If we proceed by means of rule 1 to  $\phi_{4,1}$ ,  $\phi_{1,4}$  etc. we stay inside the set described by (40). But in addition we obtain solutions for *all mixed*  $p$  and  $q$  (with respect to even/odd), e.g.

$$\phi_{2,3} = \frac{1}{\tau} \partial_r \frac{\tau}{\tau^2 - r^2} = \frac{1}{\tau(\tau^2 - r^2)} - \frac{2\tau}{(\tau^2 - r^2)^2} \quad (41)$$

etc.

We should still find new solutions for  $p$  and  $q$  both even. The above solutions for  $f$  fails (it leads to a trivial  $\phi_{p,q}$ ), therefore we try the polynomial solution:  $f = \tau^2 + r^2$  for  $p = q$ . We obtain:

$$\phi_{p,p} = \frac{\tau^2 + r^2}{(\tau^2 - r^2)^{1+p}} \quad (42)$$

Now we have in particular the desired solution for  $\phi_{2,2}$  and thus a string of explicit solutions for all even  $p$  and  $q$ . If we use the first rule to produce  $\phi_{p+2,p+2}$  etc., we keep the form of (42), and the second rule leads back to the form (24). But we can generate new solutions for  $p \neq q$ , such as

$$\phi_{4,2} = \frac{2\tau^2 + r^2}{(\tau^2 - r^2)^4} \quad ; \quad \phi_{2,4} = \frac{\tau^2 + 2r^2}{(\tau^2 - r^2)^4} \quad (43)$$

Last we want to relate these solutions to the concept of fractional derivatives described in section 6. If we start from  $\phi_{1,1} = \delta(\tau - r)$ , then

$$\phi_{2,1} = \int_0^\infty x^{-3/2} \delta(\tau - \sqrt{r^2 + x}) dx = |\tau| \int_0^\infty x^{-3/2} \delta(x - \tau^2 + r^2) \quad (44)$$

If we drop the proportionality constant (including  $\text{sign}(\tau)$ ), we recover  $\phi_{2,1}$  given in (40).

If we begin with  $\phi_{1,1} = \Theta(\tau - r)$  or  $\phi_{3,1} = \delta(\tau - r)/r$  and move to  $\phi_{2,1}$  by half a derivation resp. integration with respect to  $r^2$ , then we end up with the form (24). If we proceed from this solution to  $\phi_{2,2}$  by applying  $\partial_{r^2}^{1/2}$ , so we are still in the set of solutions (24).

## 8 Construction of new solutions for fixed $p$ and $q$

Up to now we considered procedures to build up solutions in different dimensions from one given solution. Now that we already know solutions for all integer  $p + q$ , we look for different solutions in the same dimension as the given one.

For this we need an operator  $A$ , which commutes with the spherical D'Alembert operator:

$$[(\partial_r^2 + \frac{p-1}{r} \partial_r - \partial_\tau^2 - \frac{q-1}{\tau} \partial_\tau), A] = 0 \quad (45)$$

(Of course,  $A = \text{const.}$  is not interesting.)

**Rule 3** For  $p = 1$  resp.  $q = 1$  the operator  $A = \partial_r^\alpha$  resp.  $\partial_\tau^\alpha$  works for all  $\alpha \in \mathbb{R}^+$ .

Consider as an example  $\phi_{2,1} = (\tau^2 - r^2)^{-1/2}$  (included in (24)). One derivation by  $\tau$  yields immediately the solution given in (40), and in addition we obtain:

$$\tilde{\phi}_{2,1}(\tau, r) = \partial_\tau^2 (\tau^2 - r^2)^{-1/2} \propto \frac{2\tau^2 + r^2}{(\tau^2 - r^2)^{5/2}} \quad \text{etc.} \quad (46)$$

But if the dimension, which is  $> 1$ , is also odd, we don't win anything because there we already have solutions involving arbitrary functions of  $(\tau \pm r)$ . Take e.g.  $\phi_{3,1}$  from (4): application of rule 3 just reproduces the same structure.

What can we do if  $p$  and  $q$  are both  $> 1$ ? It can be seen easily that any ansatz  $A = P_1(\tau, r)P_2(\partial_\tau, \partial_r)$ , where  $P_1, P_2$  are finite polynomials, fails.

But we can combine the generalized rule 1 with rule 3 to obtain:

**Rule 4** If  $\phi_{p,q}$  is a solution, then also

$$\partial_{r^2}^{(p-1)/2} \partial_r^\alpha \partial_{r^2}^{-(p-1)/2} \phi_{p,q} \quad \text{and} \quad \partial_{r^2}^{(q-1)/2} \partial_r^\beta \partial_{r^2}^{-(q-1)/2} \phi_{p,q}$$

are solutions in  $p + q$  dimensions, for any  $\alpha, \beta \in \mathbb{R}^+$ .

As an example we consider  $\phi_{3,2}$  from (24). For  $\alpha = 1, 2, \dots$  we get further solutions in  $3 + 2$  dimensions, namely:

$$\frac{\tau^2/r + 2r}{(\tau^2 - r^2)^{5/2}}, \quad \frac{3\tau^2 + 2r^2}{(\tau^2 - r^2)^{7/2}} \quad \text{etc.} \quad (47)$$

Note, however, that the practical application of this theorem is often complicated. In particular if  $p$  and  $q$  are both even, we are forced to use fractional integration and derivation.

We may also build new solutions by a combination of the first and second rule, which is different from unity, e.g. with the operator:

$$[\partial_{r^2} + \partial_{r^2}][\partial_{r^2} \partial_{r^2}]^{-1} = \partial_{r^2}^{-1} + \partial_{r^2}^{-1} \quad (48)$$

## 9 Definition of $\square^\lambda$ for $q > 1$

In [1, 2] the analytic continuation of  $(\tau^2 - r^2)^\lambda$  with respect to real  $\lambda$  was discussed and applied extensively in  $d + 1$  dimensions. The discussion revealed that its singularities and residues are strongly related to physical properties. In particular in [2] it was shown that a displacement in dimensions, not necessarily integers, can be interpreted as solutions for different powers of the D'Alembert operator.

Now also this consideration shall be generalized to  $p + q$  dimensions. For that purpose, we need a definition of  $\square^\lambda$  for multiple times. The extension of the analytic properties is not straightforward, but displays interesting and qualitatively new properties, as we will see.

Let us summarize the analytic properties of the distributions

$$P_{+-}^\lambda \doteq (\tau^2 - r^2)_{+-}^\lambda \quad \text{and} \quad (P + i0)^\lambda$$

(see [3] p. 350 ff.).  $+-$  means zero outside, inside the light cone, respectively and  $(P + i0)^\lambda$  means the limit  $(\tau^2 - r^2 + i\varepsilon)^\lambda$  when  $\varepsilon \rightarrow 0$ . The main results are the following.

a)  $p$  odd,  $q$  even or v.v.

$P_+^\lambda$  has simple poles at  $\lambda = -1, -2, \dots$  and  $\lambda = -n/2, -n/2 - 1, \dots$ , where  $n \doteq p + q$ . For  $k \in \mathbb{N}$  the residues are:

$$\text{res}_{\lambda \rightarrow -k} P_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(P) \quad (49)$$

$$\text{res}_{\lambda \rightarrow -n/2-k} P_+^\lambda = \frac{(-1)^{p/2} \pi^{n/2}}{4^k k! \Gamma(n/2 + k)} \square^k \delta(x_1, \dots, x_n) \quad (50)$$

(here the components  $x_i$  run over both types of coordinates).

b)  $p, q$  even

$P_+^\lambda$  has simple poles for  $\lambda = -1, -2, \dots, -k, \dots$

$$\operatorname{res}_{\lambda \rightarrow -k} P_+^\lambda = (-1)^{k-1} \delta^{(k-1)}(P) \quad (k \leq n/2) \quad (51)$$

$$\operatorname{res}_{\lambda \rightarrow -n/2-k} P_+^\lambda = \frac{(-1)^{n/2+k-1}}{(n/2+k-1)!} \delta^{(n/2+k-1)}(P) + \frac{(-1)^{p/2} \pi^{n/2}}{4^k k! \Gamma(n/2+k)} \square^k \delta(x_1, \dots, x_n) \quad (52)$$

c)  $p, q$  odd

$P_+^\lambda$  has simple poles for  $k = -1, -2, \dots, -n/2 + 1$  with

$$\operatorname{res}_{\lambda \rightarrow -k} P_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(P) \quad (53)$$

and single and double poles for  $\lambda = -n/2, -n/2 - 1, \dots$  with (observe the difference to  $p, q$  even)

$$\operatorname{res}_{\lambda \rightarrow -n/2-k} P_+^\lambda = c_1 \frac{\square^k \delta(P)}{(\lambda + n/2 + k)^2} + \frac{c_2 \square^k \delta(x_1, \dots, x_n) + c_2' \delta^{(n/2+k-1)}(P)}{\lambda + n/2 + k} \quad (54)$$

For  $P_-$  exchange  $p$  and  $q$  and replace  $\delta^{(k-1)}(P)$  by  $\delta^{(k-1)}(-P)$  and  $\square$  by  $-\square$ .

d) The poles of  $(P + i0)^\lambda$  are simple for  $\lambda = -n/2 - k, k = 0, 1, 2, \dots$  and

$$\operatorname{res}_{\lambda \rightarrow -n/2-k} (P + i0)^\lambda = \frac{e^{-i\pi p/2} \pi^{n/2}}{4^k k! \Gamma(n/2+k)} \square^k \delta(x_1, \dots, x_n) \quad (55)$$

and its complex conjugate for  $(P - i0)^\lambda$ .

In [1] a definition of  $\square^\lambda$  for real  $\lambda$  was given, which reduces to  $\square^k$  if  $\lambda \in \mathbb{N}$ . This was achieved as follows

$$\square_{R,A}^\lambda * \phi(x) = \frac{2^{2\alpha+1} (t^2 - r^2)^{-\alpha-n/2} \Theta(-t)}{\pi^{n/2-1} \Gamma(1-\alpha-n/2) \Gamma(-\alpha)} * \phi(x) \quad (56)$$

$R, A$  stand for retarded, advanced and refer to the negative, positive sign in the argument of the step function  $\Theta$ . It is easy to verify that (56) reduces to  $\square^k \phi(x)$  when  $\lambda = k$  and  $q = 1$ , due to the properties a), b) and c).

This is generally the case when  $p, q$  are both odd. E.g. in dimensions 3+3 or 3+1 the double pole plays an essential role. (In 3+3  $t$  is to be understood as  ${}^\pm \tau$ .)

So for a classical theory with  $p, q$  both odd,  $\square^\lambda$  given by (56) is well defined, retarded as well as advanced.

The same happens if we consider a quantum theory, where (see [1])

$$\square_{\pm}^\lambda * \phi(x) = {}^\pm i e^{\pm i\pi(\lambda+n/2)} 4^\lambda (4\pi)^{n/2} \frac{\Gamma(\lambda+n/2)}{\Gamma(-\lambda)} (t^2 - r^2 \pm i0)^{\lambda-n/2} * \phi(x) \quad (57)$$

Thus if  $\lambda = k \in \mathbb{N}$  the numerator as well as the denominator pick up a pole, and we are left with the residue  $\square^k \delta$  (see property d)).

If we try to repeat the same procedure for  $p, q$  even, e.g. 4+2 dimensions, the classical theory as described by eq. (56), runs into trouble. The above reasoning doesn't work any more, since  $(t^2 - r^2)_+^\lambda$  has only single poles, and there is no way to compensate the double poles in the denominator occurring in (56). Everything would be swept away.

And even if we write a phenomenological  $\square^\lambda$  with a single pole in the denominator, we arrive for integer  $\lambda$  at a linear combination

$$a_1 \square^k + a_2 \delta^{(k-1)}(P)$$

and *not* the desired result  $\square^k$ .

However, this is not the case for a quantum theory of the form (57). The latter leads in fact to  $\square^k$  for  $\lambda \rightarrow k$ . Hence everything works for the quantum operator  $\square_\pm^\lambda$  with multiple times when  $p$  and  $q$  are both even or both odd, whereas the classical definition for  $\square^\lambda$  fails for even dimensions.

So if we consider the quantum definition of  $\square^\lambda$ , we can extend the result of [1], section V, according to which a derivation with respect to  $r^2$  ( $\tau^2$ ) increases the spatial (temporal) dimension by 2 or diminishes the power  $\alpha$  by 1. If we apply the operator  $\partial_{r^2}^\gamma$  ( $\partial_{\tau^2}^\gamma$ ) on a radial solution, we increase  $p$  ( $q$ ) by  $2\gamma$  or diminish  $\alpha$  by  $\gamma$ .

The results for the "mixed case" (with respect to even/odd) are obvious and not plagued by any problems due to double poles.

## 10 The wave equation in curved space

In particular, the previous formalism can be used to transform a time coordinate into a space coordinate. The operator, which does this job, is

$$\partial_{r^2}^{-1/2} \partial_{\tau^2}^{1/2} \tag{58}$$

Of course, this reminds us of the process taking place when we enter a black hole. When crossing the Schwarzschild radius, a time coordinate becomes spatial and vice versa, as we see from the Schwarzschild metrics in polar coordinates:

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + r^2 d\Omega_q^2 - \frac{1}{1 - 2M/r} dr^2 - r^2 d\Omega_p^2 \tag{59}$$

We refer to a spherically symmetric, static black hole solution.  $d\Omega_q$ ,  $d\Omega_p$  are the surface elements on the temporal, spatial unit sphere, respectively. <sup>1</sup> (Generally, a sensible dimensional continuation of general relativity is given by the Lovelock eqs, [7]. For its application on static black holes, see [8].)

Formula (59) displays immediately that in a transition from the outside to the inside of a black hole (of radius  $2M$ ) the radial temporal component  $\tau$  becomes spatial while the

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<sup>1</sup>The appearing singularity of the metrics on the boundary is an artifact of the choice of the coordinates and can be cured by a different choice, as Kruskal showed for 3+1 dimensions [6].

radial spatial component  $r$  becomes temporal. The character of the further components remains unchanged, so we have a transition  $p + q \rightarrow p + q$ . (However, inside the black hole, the angular terms in (59) suffer from a mismatch between the radial and the angular factor.)

Let's consider this in view of spherically symmetric waves. The time-to-space transition we might describe with the operator (58), but to include also the simultaneous space-to-time transition we have no reasonable alternative to its inverse operator, so we don't end up with anything instructive.

We could describe a transition of  $3 + 1 \rightarrow 3 + 1$  - or generally:  $p + q \rightarrow p + q$  - dimensions different from unity, e.g. by using the procedures of section 8, but a motivation for this remains to be found.

A highly interesting transition, which is *not*  $p + q \rightarrow p + q$  is the *Wick rotation*. It transforms e.g. Schrödinger's eq. to the diffusion eq., hence deterministic and reversible quantum mechanics to an irreversible stochastic process (Brownian motion).

But unfortunately our transformation rules 1 and 2 fail as soon as  $p$  or  $q$  vanishes, since we have always assumed  $\partial_r^2 - \partial_r^2$  to be part of the D'Alembert operator. In Euclidean space we would be left with the search for harmonic functions, which is well established in the literature.

Still, the most attractive feature in this context seems to us the exchange of a time against a space coordinate in gravitation theory. But in order to approach such questions as the transition across the boundary of black holes seriously, we have to expand our discussion to a curved space.

We add some simple remarks on this generalization. However, much of this extension remains to be worked out. Here we want to illustrate one interesting property, which is related to the previous discussion of dimensional shifts. As we will see, certain types of curvatures can be described in a flat metrics by altering the dimensions.

Assume the temporal and spatial sector of the metrics to be decoupled:

$$g = \begin{pmatrix} g^{(\tau)} & 0 \\ 0 & g^{(r)} \end{pmatrix}, \quad (60)$$

where  $g^{(\tau)}$ ,  $g^{(r)}$  is a  $q \times q$ ,  $p \times p$  matrix, respectively.

First we consider the spatial part of the generalized D'Alembert operator in this metrics, which is the Laplace-Beltrami operator:

$$\Delta = \frac{1}{\sqrt{|\det g^{(r)}|}} \partial_\mu \left[ \sqrt{|\det g^{(r)}|} g^{(r)\mu\nu} \right] \partial_\nu \quad (61)$$

In flat space and polar coordinates  $r, \theta_1 \dots \theta_{p-1}$  we have  $g^{(r)rr} = 1$  and  $\sqrt{|\det g^{(r)}|} = r^{p-1} \sin^{p-2} \theta_{p-1} \sin^{p-3} \theta_{p-2} \dots \sin \theta_2$ . Even if we generalize this to

$$\sqrt{|\det g^{(r)}|} = r^{p-1} F(\theta_1, \dots, \theta_{p-1}), \quad (62)$$

where  $F$  is an arbitrary function, we always end up with the radial part

$$\Delta_{(r)} = \partial_r^2 + \frac{p-1}{r} \partial_r \quad (63)$$

that we have used so far.

A generalization of  $\Delta_{(r)}$  can be achieved, however, if we let the matrix elements  $g^{(r)\mu\nu}$  depend on  $r$ . We still consider the spatial part separately and assume that it takes the form:

$$(g^{(r)\mu\nu}) = \begin{pmatrix} g^{rr} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{ij} & \\ 0 & & & \end{pmatrix} \quad (64)$$

with  $i, j = 1, \dots, p-1$ , i.e. in addition the radial and the angular part are decoupled. Hence:

$$\det g^{(r)} = g^{rr} \cdot \det(g_{ij}) \quad (65)$$

Let us consider the general case where both factors in (65) pick up a non trivial dependence on  $r$ .

$$g^{rr} = f_1(r) \quad (66)$$

$$\det(g_{ij}) = f_2(r)r^{p-1}F(\phi_i) \quad (67)$$

where  $f_1, f_2$  are functions  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ . Inserting this into (61) we obtain:

$$\Delta_{(r)} = f_1(r) \left[ \partial_r^2 + \left\{ \frac{p-1}{r} + \frac{3}{2}(\partial_r \ln f_1(r)) + \frac{1}{2}(\partial_r \ln f_2(r)) \right\} \partial_r \right] \quad (68)$$

In general, such functions  $f_1, f_2$  require new types of solutions, completely different from the case  $f_1 = f_2 = \text{const.}$  considered above. An exception is the case, where they take the simple monomial form:

$$f_1(r) = c_1 r^\alpha \quad (69)$$

$$f_2(r) = c_2 r^\beta \quad (c_1, c_2, \alpha, \beta : \text{constants} \in \mathbb{R}) \quad (70)$$

Here the modification of the dynamics due to the metrics – with respect to the flat space – corresponds to a dimensional shift in the flat space (as long as  $c_1, c_2 \neq 0$ ,  $3\alpha + \beta \geq 2(1-p)$ ):

$$p \rightarrow p + \frac{3}{2}\alpha + \frac{1}{2}\beta \quad (71)$$

Hence ironing the curved space – as it is very standard – corresponds for the special metrics given in (66,67,69,70) – to a shift in the flat space dimension.

Thus in a Euclidean space we have to look for harmonic functions in a modified dimension. E.g. for critical phenomena it would be most crucial if we could argue that the effective dimension of the flat space, where most models are defined, is slightly different from four, due to a weak curvature of this type.

Now we remember the temporal part, which we have ignored for the moment. We still assume that the metrics does not couple it to the spatial part, see (60). We are interested now in cases where the metrics causes a shift in the flat dimensions  $p$  and/or  $q$ , the same effect that was obtained before from other operations.

If we want to leave the temporal part in the flat form, then the factor  $f_1(r)$  in (68) disturbs in the sense that we do not arrive at the flat D'Alembert operator in a modified



dimension unless  $f_1(r) \equiv 1$ . For different functions  $f_1$ , the flat solutions are only valid if they are separable:  $\phi_{p,q}(\tau, r) = \psi_1(\tau)\psi_2(r)$ ;  $\psi_1, \psi_2$  being harmonic functions in  $p, q$  dimensions, respectively. Also an angle dependent factor in (66) would disturb in this sense.

In order to use flat solutions which do not have this separable form, we have to deal with  $f_1(r) \equiv 1$ , e.g. by using Riemann normal coordinates, and arrange the dimensional shift only by a non constant  $f_2(r)$  in (67) of the form (70).

Of course the analogous statements hold if we want to introduce a curved temporal metrics and keep a flat spatial metrics. If we only use curvatures of the type (67,70), then we can easily shift  $p$  and  $q$  simultaneously.

Finally, we can also cause a simultaneous modification of the flat  $p$  and  $q$  by the following choice:

$$g^{(r)rr} = g^{(\tau)\tau\tau} = c_1 r^\alpha \tau^{\bar{\alpha}} \quad (c_1, \alpha, \bar{\alpha} : \text{constants} \in \mathbb{R}) \quad (72)$$

If we combine this with

$$f_2(r) = c_2 r^\beta, \quad \bar{f}_2(\tau) = \bar{c}_2 \tau^{\bar{\beta}} \quad (73)$$

where  $\bar{f}_2(\tau)$  is the temporal analogue to  $f_2(r)$ , then we arrive at the dimensional transformation:

$$p \rightarrow p + \frac{3}{2}\alpha + \frac{1}{2}\beta, \quad q \rightarrow q + \frac{3}{2}\bar{\alpha} + \frac{1}{2}\bar{\beta} \quad (74)$$

## 11 Conclusions

We have provided a reservoir of solutions of the spherically symmetric wave equation in  $p + q$  dimensions and gave some insight into their structure. We gave a large number of explicit solutions and a set of prescriptions for constructing new solutions out of them. In particular we showed how to modify the dynamics such that it fulfills the wave eq. in different temporal or spatial dimensions. The same prescriptions also hold for the Klein Gordon equation.

The transition in steps of two dimensions is very simple. Its analytic continuation allows also for transitions by one or by fractional dimensions, but its application is somewhat more involved. Such transitions in the flat space also correspond to certain types of curvature, i.e. special curved metrics can be described in the flat space by altering its dimensions.

We found an operator transforming a space into a time coordinate or vice versa, a transition that actually takes place when we cross the boundary of a black hole. However, our description of this process is not complete yet.

The analytic continuation of  $\square^\lambda$  with respect to  $\lambda$  corresponds in some cases again to the dynamics of the standard D'Alembert operator in modified dimensions. This continuation is feasible for the classical as well as for the quantum definition in odd  $p$  and/or odd  $q$ . If  $p, q$  are *both* even, however, the classical definition fails and only the quantum definition yields a sensible result.

**Acknowledgement** One of us (J.J.G.) is indebted to O. Obregon for motivating discussions.

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