WILSON LOOP AND RELATED STRINGS FOR THE INSTANTON AND THEIR VARIATIONAL DERIVATIVES

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ABSTRACT - We compute ordered integrals along arcs (or circles) and segments for the instanton. We use them to obtain variational and partial derivatives for open and closed strings. We also compute the D'Alembertian for Wilson loops.

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Lately, the Wilson loop [1] [2] has arisen considerable attention in gauge theories, due mainly to the fact that its behaviör serves as a confinement criterium. It could also act as a dynamical variable which may eventually replace the fields in the lagrangean, with its own evolution equation which should involve their variational derivatives [3] [4] [5] [6] [7].

On the other hand, instantons [8] [9], being solutions of Yang-Mills equations may have important contributions in effective calculations.

Instantons and instantons-like configurations (like merons) present a symmetry property which allows the actual calculation of the ordered exponential integral along particular paths, closed or opened. These paths are arcs of circles which we choose in the plane X_1X_2 with the center at the origin, and also straight line segments. The center of the instanton is at any point of the plane X_3X_4.

In this way, we can consider loops which are complete circles, or closed paths formed by arcs and straight lines.

We consider first some simple examples of loop (or string) integral computations for the instanton, whose potential is:

$$A_\mu = 2i \frac{\sigma_{\mu\nu}(X_\nu - X^0_\nu)}{(X - X^0)^2 + \lambda^2} \qquad (1)$$

although it will become clear that the same procedure can be applied to a more general set of potentials, not necessarily solutions of Yang-Mills equations.

In the specific case of the instantons having the center at $X_1^0 = X_2^0 = 0$, we have explicitly, with X on the plane $X_1\ X_2$

$$A_\mu = 2i\ \frac{\sigma_{\mu 1}\ X_1 + \sigma_{\mu 2}\ X_2 - \sigma_{\mu 3}\ X_3^0 - \sigma_{\mu 4}\ X_4^0}{X^2 + \lambda^2} \qquad (2)$$

with $X^2 = X_1^2 + X_2^2 + X_3^{0\,2} + X_4^{0\,2}$

If we now consider polar variables in the $X_1 X_2$ plane, and remember that

$$\sigma_{ij} = \frac{1}{2}\ \varepsilon_{ijk}\ \sigma_k \qquad \sigma_{i4} = \frac{1}{2}\ \sigma_i \qquad (4)$$

we find:

$$A_\mu\ dx^\mu = i\ B\ d\Theta \qquad (5)$$

$$-i\ A_\Theta(\Theta,r) = B(\Theta,r) = -\frac{r^2}{X^2+\lambda^2}\left[\sigma_1\left(\cos\Theta\frac{X_3^0}{r} - \operatorname{sen}\Theta\frac{X_4^0}{r}\right) + \right.$$

$$\left. + \sigma_2\left(\operatorname{sen}\Theta\frac{X_3^0}{r} + \cos\Theta\frac{X_4^0}{r}\right) + \sigma_3\right] \qquad (6)$$

We note that

$$e^{i\,B(\Theta)d\Theta} = e^{-i\frac{\Theta}{2}\sigma_3}\; e^{i\,B(0)d\Theta}\; e^{i\frac{\Theta}{2}\sigma_3} \qquad (7)$$

Let us consider now an angle interval $\Theta_i \leq \Theta \leq \Theta_f$. We divide it in N equal parts $\frac{\Theta_f - \Theta_i}{N}$ and express the value of $\exp i\,B(\Theta)d\Theta$ at the corresponding points $\Theta_n = \frac{n}{N}(\Theta_f - \Theta_i)$ $(n=1,2,\ldots,N)$ by means of (7). Then we take the limit $N \to \infty$. We get:

$$W_{\Theta_f\,\Theta_i}(r) = P\; e^{\int_{\Theta_i}^{\Theta_f} B(\Theta)d\Theta} \qquad \text{and} \qquad (8)$$

$$W_{\Theta_f\,\Theta_i}(r) = e^{-i\frac{\Theta_f}{2}\sigma_3}\; e^{i(B(0)+\frac{1}{2}\sigma_3)(\Theta_f - \Theta_i)}\; e^{i\frac{\Theta_i}{2}\sigma_3} \qquad (9)$$

or

$$W_{\Theta_f\,\Theta_i}(r) = e^{-i\frac{\Theta_f}{2}\sigma_3}\left[\cos(\Theta_f-\Theta_i)L + i\,\Sigma\,\mathrm{sen}(\Theta_f-\Theta_i)L\right] e^{i\frac{\Theta_i}{2}\sigma_3} \qquad (10)$$

where $L^2 = B_1^2 + B_2^2 + (B_3 + \frac{1}{2})^2$, $\vec{B}\cdot\vec{\sigma} = B(0)$. (11)

Explicitly $$L^2 = \frac{1}{4} - \frac{r^2 \lambda^2}{(X^2 + \lambda^2)^2} \qquad (12)$$

and:

$$\Sigma = \vec{\Sigma} \cdot \vec{\sigma} ; \; \Sigma_1 = \frac{B_1}{L} \; , \; \Sigma_2 = \frac{B_2}{L} \; , \; \Sigma_3 = \frac{B_3 + \frac{1}{2}}{L} \; . \qquad (13)$$

(10) represents the value of the "integral" (8) for any arc of radius r with $\Theta_f \leq \Theta \leq \Theta_i$.

In particular, for the complete circle:

$$W_{2\pi,0}(r) = -e^{2\pi i (B(0) + \frac{1}{2} \sigma_3)} \qquad (14)$$

$$W_{2\pi,0}(r) = -\cos 2\pi L - i\Sigma \text{ sen } 2\pi L \qquad (15)$$

If we take the trace of (15) we get the corresponding Wilson loop:

$$W(r) = \text{Tr } W_{2\pi,0}(r) = -2 \cos 2\pi L. \qquad (16)$$

If we now consider - for simplicity - a radial segment in the (x,y) plane, the integral is abelian and can be computed straight forwardly with the result,

$$W_{r_2 r_1}(\Theta) = P\, e^{\int_{r_1}^{r_2} A_r \, dr} = e^{-i \frac{\Theta}{2} \sigma_3} \, e^{-\frac{i}{\Lambda} \text{arctg}\left[\frac{(r_2 - r_1)\Lambda}{\Lambda^2 + r_1 r_2}\right] (X_4^0 \sigma_1 - X_3^0 \sigma_2)} \, e^{i \frac{\Theta}{2} \sigma_3} \qquad (17)$$

with $\Lambda^2 = X_3^{o\,2} + X_4^{o\,2} + \lambda^2$

A completely similar result can be obtained for a segment

$$X_\mu = \alpha_\mu + \beta_\mu \tau \qquad \tau_1 \leq \tau \leq \tau_2 \quad .$$

With formulae (9) and (17) one can compute more general loops. In particular, for the "plaquette" $r_1 \leq r \leq r_2$, $\Theta_1 \leq \Theta \leq \Theta_2$:

$$W(r_1 r_2 \Theta_1 \Theta_2) = \mathrm{Tr}\left\{ W_{\Theta_1\Theta_2}(r_1) W_{r_1 r_2}(\Theta_2) W_{\Theta_2\Theta_1}(r_2) W_{r_2 r_1}(\Theta_1) \right\} \tag{18}$$

Let us now consider the variational derivatives and related partial derivatives of strings.

It can be shown that for an open string

$$\frac{\delta W_{\sigma_2 \sigma_1}}{\delta X^\nu(\sigma)} = W_{\sigma_2\sigma} f_{\nu\lambda}(\sigma) \frac{dX^\lambda}{d\sigma} W_{\sigma\sigma_1} + \delta(\sigma-\sigma_2) A_\nu(\sigma_2) W_{\sigma_2 \sigma_1} - \delta(\sigma-\sigma_1) W_{\sigma_2\sigma_1} A_\nu(\sigma_1) \tag{19}$$

Where σ is a parameter along the curve, and $f_{\mu\nu}$ the field intensity. See also refs. [2] to [7] and [10] and [11].

Form.(19) can be written in a more compact way.

$$\frac{\delta W_{\sigma_2 \sigma_1}}{\delta X^\mu(\sigma)} = W_{\sigma_2\sigma} f_{\mu\nu}(\bar{A}(\sigma)) W_{\sigma\sigma_1} \frac{dX^\nu}{d\sigma} \tag{20}$$

or also:

$$\frac{\delta W_{\sigma_2\sigma_1}}{\delta \sigma^{\mu\nu}} = W_{\sigma_2\sigma} f_{\mu\nu}(\bar{A}(\sigma)) W_{\sigma\sigma_1} \tag{21}$$

where the field $f_{\mu\nu}(\bar{A}(r))$ is to be computed from:

$$\bar{A}(\sigma,u,v,w)=\varepsilon(\sigma,u,v,w)\ A_\mu(\sigma,u,v,w)$$

$$\varepsilon(\sigma,u,v,w)=\varepsilon(\sigma)\begin{cases} 1 & \sigma_1 \leq \sigma \leq \sigma_2 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

ie: $\varepsilon(\sigma)$ is the characteristic function of the string coordinate; the string itself being given by

$$u = u_0 \ , \quad v = v_0 \ , \quad w = w_0 \ , \quad \sigma_1 \leq \sigma \leq \sigma_2$$

It is not difficult to understand the following relation for an open string: (See also form (9) of ref. [11])

$$\int_{\sigma_1}^{\sigma_2} d\sigma \frac{\delta W_{\sigma_2\sigma_1}}{\delta X^\nu(\sigma)} = \frac{\partial W_{\sigma_2\sigma_1}}{\partial X^\nu} \tag{23}$$

(valid also for general coordinate systems).

Let us see how (23) works for the instanton.

We take $W_{\Theta_2\Theta_1}(r)$ given by (10), X_ν being the radial coordinate r. In this case, we have

$$\frac{\partial W_{\Theta_f\Theta_i}(r)}{\partial r} = \int_{\Theta_i}^{\Theta_f} W_{\Theta_f\Theta}\ f_{r\Theta}\ W_{\Theta\Theta_i}\ r d\Theta + A_r(\Theta_r) W_{\Theta_f\Theta_i} - W_{\Theta_f\Theta}\ A_r(\Theta_i)$$

(24)

where

$$f_{r\Theta} = - \frac{2\ i\ \lambda^2\ \sigma_3}{(X^2 + \lambda^2)^2} \tag{25}$$

$$A_r(\Theta) = e^{-i\sigma_3 \frac{\Theta}{2}} \qquad A_r(0)\ e^{i\ \sigma_3 \frac{\Theta}{2}} \tag{26}$$

$$A_r(0) = - \frac{i}{(X^2 + \lambda^2)} (X_4^0\ \sigma_1 - X_3^0\ \sigma_2) \tag{27}$$

The validity of (24) can then be explicitly checked by independent computation of both members.

Besides radial dilatation we can consider angular desplacements of the string sector as a whole.

In this case, from form (23) and (19) we get:

$$\frac{\partial W_{\Theta_f\Theta_i}}{\partial \Theta} = \int_{\Theta_i}^{\Theta_f} \frac{dX^\nu}{d\Theta} \frac{\delta W_{\Theta_f\Theta_i}}{\delta X^\nu(\Theta)}\ d\Theta = A_\Theta(\Theta_f)\ W_{\Theta_f\Theta_i} - W_{\Theta_f\Theta_i}\ A_\Theta(\Theta_i) \tag{28}$$

In particular, for $\Theta_i=\Theta, \Theta_f=\Theta + 2\pi$ (28) gives:

$$\frac{\partial W_{\Theta+2\pi,\Theta}}{\partial\Theta} = \left[A_\Theta(\Theta),\, W_{\Theta+2\pi,\Theta}\right] \tag{29}$$

which is a Heisenberg eq. for the loop $\Theta \to \Theta + 2\pi$.

Up to now, as we had $W_{\Theta_f\Theta_i}$ as an explicit function of (r,Θ) we could check (23) for radial and angular variables.

For cartesian coordinates, we compute the r.h.s. Nevertheless even if we only know $W_{\sigma\sigma'}$ at a certain position, formulae (19) and (23) allow the calculation of any translational derivative. This remain also true for higher derivatives with repeated use of those formulae.

In particular, for a closed loop:

$$\frac{\partial W_{\Theta+2\pi,\Theta}}{\partial X^\mu} = \int_\Theta^{\Theta+2\pi} W_{\Theta+2\pi,\Theta'}\, f_{\mu\Theta'}\, W_{\theta'\theta}\, r d\Theta' + \left[A_\mu(\Theta), W_{\Theta+2\pi,\Theta}\right] \tag{30}$$

where

$$f_{\mu\Theta} = f_{\mu\nu} \frac{dX^\nu}{d\Theta}$$

Taking the trace of (30), we have for the Wilson loop:

$$\frac{\partial W}{\partial X^{\mu}} = \mathrm{Tr} \, . \int_0^{2\pi} f_{\mu\Theta} \, W_{\Theta + 2\pi, \Theta} \; r d\Theta \tag{31}$$

Now we can compute the D'Alembertian of the Wilson loop by the simple expedient of taking derivatives and using (30). Thus (in cartesian coordinates):

$$\Box W = \partial_{\mu} \partial_{\mu} W = \mathrm{Tr} \int_0^{2\pi} \left\{ \partial_{\mu} f_{\mu\Theta} \; W_{\Theta + 2\pi, \Theta} + f_{\mu\Theta} \, \partial_{\mu} W_{\Theta + 2\pi, \Theta} \right\} r d\Theta \tag{32}$$

$$\Box W = \mathrm{Tr} \int_0^{2\pi} \left\{ D_{\mu} f_{\mu\Theta} \, W_{\Theta + 2\pi, \Theta} + f_{\mu\Theta} \int_{\Theta}^{\Theta + 2\pi} W_{\Theta + 2\pi, \Theta'} \, f_{\mu\Theta'} \, W_{\Theta' \Theta} \, r d\Theta' \right\} r d\Theta \tag{33}$$

which, for a solution of Yang-Mills equation reduces to

$$\Box W = \mathrm{Tr} \int_0^{2\pi} \left\{ f_{\mu\Theta} \int_{\Theta}^{\Theta + 2\pi} W_{\Theta + 2\pi, \Theta'} \; f_{\mu\Theta'} \, W_{\Theta' \Theta} \, r d\Theta' \right\} \; r d\Theta \tag{34}$$

A similar result can be obtained for open strings. In the case of the instanton, using (10) to (13), we find:

$$\Box W = \frac{16 \pi \lambda^4}{(X^2 + \lambda^2)^4} \left\{ 2\pi (1 - \Sigma_3^2) \cos 2\pi L + \frac{1 + \Sigma_3^2}{L} \operatorname{sen} 2\pi L + \right.$$

$$\left. + \frac{4L(1 - \Sigma_3^2)}{4L^2 - 1} \operatorname{sen} 2\pi L \right\} \tag{35}$$

Analogously, by taking (21) for a closed, traced loop, and then using (30), we obtain:

$$\partial_\mu \frac{\delta W}{\delta \sigma^{\mu\nu}(X)} = \mathrm{Tr}\left\{ D_\mu f_{\mu\nu} W_{\Theta+2\pi,\Theta} + f_{\mu\nu} \int_{\Theta}^{\Theta+2\pi} W_{\Theta+2\pi,\Theta'} f_{\mu\Theta'} W_{\Theta'\Theta} \, r d\Theta' \right\} . \qquad (36)$$

(For the quantum case see form (14) of reference [6])

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