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Excitations of the Gravitational Field — I *††

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ABSTRACT

We treat the geometry of spacetime as a stochastic variable. Fluctuations induce a deviation from Einstein's system of equations for the average geometry. A model is presented to deal with the fluctuations by expanding the perturbations on a series in the average geometry. As a consequence, some qualitatively new features appear. The influences on galaxy formation and on the propagation of gravitational waves are analyzed.

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I. INTRODUCTION

Recently there has been an increase of interest in modifications of Einstein's equations of General Relativity (GR) due to vacuum fluctuations of the gravitational field. There are many reasons for this. One is related to the moderate success of the covariant regularization procedure to treat the divergences that appear in quantum theory. Indeed, it has been shown[1,2] that by modifying Einstein's equations through an effective Lagrangian, which contains nonlinear terms in the curvature tensor, one obtains a covariant way to eliminate the divergences.

These vacuum corrections to classical GR have been considered by many authors.[3-6] It is our purpose here to discuss the effects of these modifications on some specific configurations of the gravitational field. In Sec. II we introduce some definitions and our notation, and we recall the quasi-Maxwellian approach to gravity which will be used throughout the paper. Section III presents the main idea of treating vacuum corrections in the quasi-Maxwellian scheme. The extra terms in Einstein's modified equations are decomposed as a series in the electric and magnetic parts of the Weyl conformal tensor. Section IV discusses the effects of these corrections on a gravitational wave propagating in a Minkowskii background. We use the analogy with Maxwell's electrodynamics in order to interpret the new terms induced by vacuum corrections of the gravitational field. In Sec. V we analyze the propagation of perturbations in a Friedmann-type Universe in our modified version of GR. We end with Sec. VI in which an outline of how to treat inhomogeneous corrections due to vacuum fluctuations is presented.

II. DEFINITIONS AND NOTATIONS

Our metric has signature (+ - - -).

Greek indices run 0, 1, 2, 3.

Latin indices run 1, 2, 3.

A single vertical bar means partial derivative.

A double vertical bar means covariant derivative.

A square bracket [ ] means antisymmetrization and a round bracket ( ), symmetrization.

We denote

$$\eta^{\alpha\beta\mu\nu} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\alpha\beta\mu\nu} ,$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric Levi-Civita symbol.

A dot means derivative in the direction of the 4-velocity v^{α}.

The Weyl tensor $C_{\alpha\beta\mu\nu}$ will be decomposed into its electric and magnetic parts as seen by an observer with velocity $v^{\alpha} = \delta^{\alpha}_{o}$:

$$E_{\alpha\beta} = -\,C_{\alpha\mu\beta\nu}\,v^{\mu}\,v^{\nu} \tag{1}$$

$$H_{\alpha\beta} = \overset{*}{C}_{\alpha\mu\beta\nu}\,v^{\mu}\,v^{\nu} \equiv \tfrac{1}{2}\,\eta_{\alpha\mu}{}^{\rho\sigma}C_{\rho\sigma\beta\lambda}\,v^{\mu} \tag{2}$$

Thus, we can write

$$\begin{aligned} C_{\alpha\beta}{}^{\mu\nu} = {} & 2\,v_{[\alpha}\,E_{\beta]}{}^{[\mu}\,v^{\nu]} + \delta^{[\mu}_{[\alpha}\,E^{\nu]}_{\beta]} - \\ & -\,\eta_{\alpha\beta\lambda\sigma}\,v^{\lambda}\,H^{\sigma[\mu}\,v^{\nu]} - \eta^{\mu\nu\rho\sigma}\,v_{\rho}\,H_{\sigma[\alpha}\,v_{\beta]} \end{aligned} \tag{3}$$

Einstein's equations of gravity, with suitable boundary conditions are equivalent to the following set,[7-9]

$$C^{\alpha\beta\mu\nu}{}_{\|\nu} = -\frac{k}{2} T^{\mu[\alpha\|\beta]} + \frac{k}{12} g^{\mu[\alpha} T^{|\beta]} \qquad (4)$$

we will call the right-hand side the current $J^{\alpha\beta\mu}$. From now on we will set $k = 8\pi G = 1$ and $C \equiv$ (velocity of light) $= 1$. Using the projector operator on the 3-dimensional rest-space of the observer v^α defined by $h_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu$, we can separate Eq. (4) into a set which has a striking resemblance to Maxwell's equations of electrodynamics.[7] This is achieved by multiplying the current $J^{\alpha\beta\mu}$ respectively by

i) $v_\beta\, v_\mu\, h_\alpha{}^\rho$,

ii) $v^\lambda\, v_\mu\, \eta_{\rho\lambda\alpha\beta}$,

iii) $h_\mu{}^{(\sigma}\, \eta^{\rho)\lambda}{}_{\alpha\beta}\, v_\lambda$,

iv) $v_\beta\, h_\mu{}^{(\rho}\, h^{\sigma)}{}_\alpha$.

We then obtain the following set of equations for a perfect fluid with density ρ and pressure p:

$$E_{\alpha\mu\|\nu}\, h^{\mu\nu}\, h^\alpha{}_\varepsilon + 3\, H_{\alpha\varepsilon}\, \omega^\alpha + \eta_{\varepsilon\beta\lambda\tau}\, H^{\tau\nu}\, v^\beta\, \sigma^\lambda{}_\nu = \frac{1}{3}\, \rho_{|\alpha}\, h^\alpha{}_\varepsilon\ , \qquad (5a)$$

$$H_{\alpha\mu\|\nu}\, h^{\mu\nu}\, h^\alpha{}_\varepsilon - 3\, E_{\alpha\varepsilon}\, \omega^\alpha - \eta_{\varepsilon\beta\lambda\tau}\, E^{\tau\mu}\, \sigma_\mu{}^\beta\, v^\lambda = (\rho + p)\, \omega_\varepsilon\ , \qquad (5b)$$

$$\dot{H}^{\lambda\alpha}\, h_\alpha{}^\varepsilon\, h_\lambda{}^\rho + \frac{1}{2}\, E_\beta{}^\mu{}_{\|\alpha}\, h_\mu{}^{(\varepsilon}\, \eta^{\rho)\lambda\alpha\beta}\, v_\lambda + \theta\, H^{\varepsilon\rho} - \frac{3}{2}\, H_\nu{}^{(\varepsilon}\, \sigma^{\rho)\nu} +$$
$$+ H_{\mu\nu}\, \sigma^{\mu\nu}\, h^{\varepsilon\rho} - \frac{1}{2}\, H_\nu{}^{(\rho}\, \omega^{\varepsilon)\nu} - a_\alpha\, E_\beta{}^{(\rho}\, \eta^{\varepsilon)\lambda\alpha\beta}\, v_\lambda = 0\ , \qquad (6a)$$

$$\dot{E}^{\lambda\alpha}\, h_\alpha{}^\varepsilon\, h_\lambda{}^\rho - \frac{1}{2}\, H^{\alpha\mu\|\nu}\, h_\mu{}^{(\rho}\, \eta^{\varepsilon)}{}_{\lambda\nu\alpha}\, v^\lambda + \theta\, E^{\varepsilon\rho} - \frac{3}{2}\, E_\nu{}^{(\varepsilon}\, \sigma^{\rho)\nu} +$$
$$+ E_{\mu\nu}\, \sigma^{\mu\nu}\, h^{\varepsilon\rho} - \frac{1}{2}\, \varepsilon_\nu{}^{(\rho}\, \omega^{\varepsilon)\nu} + a_\alpha\, H_{\beta}{}^{(\rho}\, \eta^{\varepsilon)\lambda\alpha\beta}\, v_\lambda = -\frac{1}{2}\, (\rho + p)\, \sigma^{\varepsilon\rho}, \qquad (6b)$$

in which $\omega^{\alpha\beta}$ is the vorticity, $\sigma^{\mu\nu}$ the shear, Θ the expansion, a^{α} the acceleration of the congruence generated by v^{α}.

III. VACUUM QUANTUM FLUCTUATIONS

We will follow here a procedure outlined before[3,4,10] by means of which Einstein's equations of GR are to be considered a model for microscopic fields represented by the metric tensor $g_{\mu\nu}$; macroscopic fields contain fluctuations, represented by $\delta g_{\mu\nu}$, around some mean metric $\langle g_{\mu\nu}\rangle$. We write

$$g_{\mu\nu} = \langle g_{\mu\nu}\rangle + \delta g_{\mu\nu}. \qquad (7)$$

Due to fluctuations the mean metric satisfies a modified set of equations which we can write[3]

$$R^{\mu}{}_{\nu} - \tfrac{1}{2} R\delta^{\mu}{}_{\nu} = - T^{\mu}{}_{\nu} + \varphi^{\mu}{}_{\nu} \; . \qquad (8)$$

The left-hand side is constructed from $\langle g_{\mu\nu}\rangle$; the $\varphi^{\mu}{}_{\nu}$ term on the right-hand side depends on the fluctuations. Ginzburg et al.[3] argue that a good model for the perturbing term $\varphi^{\mu}{}_{\nu}$ can be obtained by developing it as a series in the unperturbed mean geometry. This is certainly the simplest useful hypothesis one can assume about this term. As Ginzburg et al.[3] show, this hypothesis leads to nonlinear terms in the Lagrangian for $\langle g_{\mu\nu}\rangle$. This is a known fact in electrodynamics where vacuum fluctuations of the quantum theory can be described roughly by nonlinearities in the classical system.

Rewriting Eq. (8) in terms of the Weyl tensor we obtain:

$$C^{\alpha\beta\mu\nu}{}_{\Vert\nu} = J^{\alpha\beta\mu} + Q^{\alpha\beta\mu} \qquad (9)$$

where $Q^{\alpha\beta\mu}$ depends on the perturbing terms $\delta E^{\alpha\mu}$, $\delta H^{\alpha\mu}$ and the left-hand side depends only on the mean tensors $\langle E_{\alpha\beta}\rangle \equiv \mathcal{E}_{\alpha\beta}$ and $\langle H_{\alpha\beta}\rangle \equiv \mathcal{H}_{\alpha\beta}$. Then, the hypothesis is made that we can write

$$Q^{\alpha\beta\mu} = \sum_k c_k\, \theta^{\alpha\beta\mu}_{(k)} \tag{10}$$

in which $\theta^{\alpha\beta\mu}_{(k)}$ are polynomials in the unperturbed Weyl tensor. The actual values of the constants c_k depend on quantum physics. Their calculation is outside the purpose of the present paper. For a preliminary rough calculation see Ref. 3.

We will be interested here in two main types of expansion: i) Local series:

$$Q^{\alpha\beta\mu}_{I} = m\, v^{[\alpha}\, \mathcal{E}^{\beta]\mu} + n\, v^{[\alpha}\, \mathcal{H}^{\beta]\mu} + p\, \eta^{\alpha\beta\lambda\sigma}\, \mathcal{E}_{\lambda}{}^{\mu}\, v_{\sigma} + q\, \eta^{\alpha\beta\lambda\sigma}\, \mathcal{H}_{\lambda}{}^{\mu}\, v_{\sigma}\ . \tag{11}$$

ii) Nonlocal series:

$$Q^{\alpha\beta\mu}_{II} = a\, \eta^{\varepsilon\sigma\alpha\beta}\, \mathcal{E}^{\mu}{}_{\varepsilon|\sigma} + b\, \eta^{\varepsilon\sigma\alpha\beta}\, \mathcal{H}^{\mu}{}_{\varepsilon|\sigma} + c\, v^{[\alpha}\, \dot{\mathcal{E}}^{\beta]\mu} + d\, v^{[\alpha}\, \dot{\mathcal{H}}^{\beta]\mu} + r\, v^{[\alpha}\, \eta^{\beta]\lambda\rho\nu}\, v_{\lambda}\, \mathcal{E}^{\mu}{}_{\rho|\nu} + s\, v^{[\alpha}\, \eta^{\beta]\lambda\rho\nu}\, \mathcal{H}^{\mu}{}_{\rho|\nu}\, v_{\lambda}. \tag{12}$$

These are the most general linear series in the unperturbed Weyl tensor. Both terms $Q^{\alpha\beta\mu}_{I}$ and $Q^{\alpha\beta\mu}_{II}$, may be regarded as consequences of nonlinear Lagrangians due to vacuum perturbation.

IV. QUANTUM CORRECTIONS TO THE PROPAGATION OF GRAVITATIONAL WAVES IN MINKOWSKII SPACETIME

In order to obtain insight into the modifications induced by vacuum corrections let us analyze some specific configurations for the gravitational field. In this section we discuss the case of a gravitational wave propagating in a flat spacetime; and in the next section we consider the evolution of perturbations in a Friedmann-like background.

Four possible corrections for the current $Q^{\alpha\beta\mu}$ will be analyzed:

<u>Case i</u>:

$$Q_I^{\alpha\beta\mu} = m\, v^{[\alpha}\, \mathcal{E}^{\beta]\mu} + n\, v^{[\alpha}\, \mathcal{H}^{\beta]\mu}. \tag{13}$$

Equations (9) give:

$$\mathcal{E}^{\ell}{}_{k|\ell} = 0, \tag{14a}$$

$$\mathcal{H}^{\ell}{}_{k|\ell} = 0, \tag{14b}$$

$$\dot{\mathcal{H}}^{k\ell} + \tfrac{1}{2}\, \mathcal{E}^{(k}{}_{m|n}\, \eta^{\ell)mn} = 0, \tag{15a}$$

$$\dot{\mathcal{E}}^{k\ell} - \tfrac{1}{2}\, \mathcal{H}^{(k}{}_{m|n}\, \eta^{\ell)mn} = -\, m\, \mathcal{E}^{\ell k} - n\, \mathcal{H}^{\ell k}, \tag{15b}$$

in which $\eta^{\ell mn} \equiv \epsilon^{o\ell mn}$.

Let us consider the case in which n is zero. Then by multiplying equations (15a,b) by the factor $\frac{1}{2}\, \delta^{m}{}_{(i}\, \eta_{j)}{}^{k\ell}\, (\partial/\partial x^{k})$ and using equations (15a,b) we obtain

$$\Box\mathcal{H} \equiv \ddot{\mathcal{H}}_{\ell m} - \nabla^2\, \mathcal{H}_{\ell m} = -\, m\, \dot{\mathcal{H}}_{\ell m}\,, \tag{16}$$

$$\Box\mathcal{E} \equiv \ddot{\mathcal{E}}_{\ell m} - \nabla^2\, \mathcal{E}_{\ell m} = -\, m\, \dot{\mathcal{E}}_{\ell m}\,, \tag{17}$$

in which ∇^2 represents the 3-dimensional Laplacian operator. A typical solution of the wave equation (16) is given by

$$\mathcal{H}_{k\ell} = A_{k\ell}\, \ell^{i(kx - \omega t)}\, e^{-mt/2}\,. \tag{18}$$

The case in which m is positive will give rise to an attenuation of the wave. The energy of the wave will decrease by a factor e^{-mt}. This case has been presented in Ref. 10. The factor m can be interpreted in terms of a conductivity of the vacuum, by analogy with electrodynamics.

When n is not null, but m = 0 equations (14a,b) do not change; but equations (16), (17) go into the set:

$$\Box \mathcal{H}^{k\ell} = n \; (\mathrm{curl}\; \mathcal{H})^{k\ell}, \tag{16'}$$

$$\Box \mathcal{E}^{k\ell} = n \; (\mathrm{curl}\; \mathcal{E})^{k\ell}, \tag{17'}$$

in which we have used the definition

$$(\mathrm{curl}\; N)^{k\ell} \equiv \tfrac{1}{2} N^{(k}{}_{m|n} \; \eta^{\ell)mn} . \tag{19}$$

<u>Case ii:</u>

$$Q^{\alpha\beta\mu}_{II} = p \; \eta^{\alpha\beta\lambda\sigma} \; \mathcal{E}_{\lambda}{}^{\mu} \; v_{\sigma} + q \; \eta^{\alpha\beta\lambda\sigma} \; \mathcal{H}_{\lambda}{}^{\mu} \; v_{\sigma} . \tag{20}$$

Equations (9) projected will give rise to

$$\mathcal{E}^{k}{}_{\ell|k} = 0 , \tag{21a}$$

$$\mathcal{H}^{k}{}_{\ell|k} = 0 , \tag{21b}$$

$$\dot{\mathcal{H}}^{k\ell} + (\mathrm{curl}\; \mathcal{E})^{k\ell} = p \; \mathcal{E}^{k\ell} + q \; \mathcal{H}^{k\ell} , \tag{22a}$$

$$\dot{\mathcal{E}}^{k\ell} - (\mathrm{curl}\; \mathcal{H})^{k\ell} = 0. \tag{22b}$$

Using the same procedure as above we obtain

$$\Box \mathcal{E}^{k\ell} = p \; \mathrm{curl}\; \mathcal{E}^{k\ell} + q \; \mathrm{curl}\; \mathcal{H}^{k\ell}, \tag{23}$$

$$\Box \mathcal{H}^{k\ell} = p \; \mathrm{curl}\; \mathcal{H}^{k\ell} - q \; \mathrm{curl}\; \mathcal{E}^{k\ell}, \tag{24}$$

in which we have neglected terms of second order on the constans p, q. There is a simple analogy between these cases and electrodynamics (see Table I) which certainly can be used as a guide for future studies of the properties of these equations.

Table I

Case iii:

$$Q^{\alpha\beta\mu}_{III} = a\ \eta^{\varepsilon\sigma\alpha\beta}\ \mathcal{E}^{\mu}{}_{\varepsilon|\sigma} + b\ \eta^{\varepsilon\sigma\alpha\beta}\ \mathcal{H}^{\mu}{}_{\varepsilon|\sigma}\ . \tag{25}$$

This will give rise to the set

$$\mathcal{E}^{k}{}_{\ell|k} = 0, \tag{26a}$$

$$\mathcal{H}^{k}{}_{\ell|k} = 0, \tag{26b}$$

$$\dot{\mathcal{H}}^{k\ell} + (\mathrm{curl}\ \mathcal{E})^{k\ell} = a\ \dot{\mathcal{E}}^{k\ell} + b\ \dot{\mathcal{H}}^{k\ell}\ , \tag{27a}$$

$$\dot{\mathcal{E}}^{k\ell} - (\mathrm{curl}\ \mathcal{H})^{k\ell} = a\ (\mathrm{curl}\ \mathcal{E})^{k\ell} + b\ (\mathrm{curl}\ \mathcal{H})^{k\ell}. \tag{27b}$$

Let us investigate the case in which a = 0. A direct calculation gives

$$\ddot{\mathcal{H}}^{k\ell} - \frac{1+b}{1-b}\nabla^2\ \mathcal{H}^{k\ell} = 0, \tag{28}$$

$$\ddot{\mathcal{E}}^{k\ell} - \frac{1+b}{1-b}\nabla^2\ \mathcal{E}^{k\ell} = 0, \tag{29}$$

The weight b of the expansion gives rise to a modification of the velocity of propagation of the waves. We can define an index-of-refraction for the vacuum by

$$n_b = \left(\frac{1-b}{1+b}\right)^{\frac{1}{2}}\ . \tag{30}$$

For the case in which b is null and $a \neq 0$ Eqs. (27) change to

$$\dot{\mathcal{H}}^{k\ell} + \mathrm{curl}\ \mathcal{E}^{k\ell} = a\ \dot{\mathcal{E}}^{k\ell}\ , \tag{31a}$$

$$\dot{\mathcal{E}}^{k\ell} - (\mathrm{curl}\ \mathcal{H})^{k\ell} = a\ \mathrm{curl}\ \mathcal{E}^{k\ell}\ . \tag{31b}$$

The analogy with electrodynamics is straightforward, but not illuminating.

Case iv:

$$Q^{\alpha\beta\mu}_{IV} = r\ v^{[\alpha}\ \eta^{\beta]\lambda\rho\sigma}\ v_\lambda\ \varepsilon^{\mu}{}_{\rho|\sigma} + s\ v^{[\alpha}\ \eta^{\beta]\lambda\rho\sigma}\ v_\lambda\ \mathcal{H}^{\mu}{}_{\rho|\sigma}\ . \tag{32}$$

The case in which $r = 0$ gives rise to the equations:

$$\varepsilon^{\ell}{}_{k|\ell} = 0, \tag{33a}$$

$$\mathcal{H}^{\ell}{}_{k|\ell} = 0, \tag{33b}$$

$$\dot{\mathcal{H}}^{1\ell} + \mathrm{curl}\ \varepsilon^{k\ell} = 0, \tag{34a}$$

$$\dot{\varepsilon}^{k\ell} - \mathrm{curl}\ \mathcal{H}^{k\ell} = -\ s\ \mathrm{curl}\ \mathcal{H}^{k\ell}. \tag{34b}$$

The wave equation now has the form:

$$\ddot{\varepsilon}^{k\ell} - (1 + 2s)\ \nabla^2\ \varepsilon^{k\ell} = 0\ , \tag{35}$$

$$\ddot{\mathcal{H}}^{k\ell} - (1 + 2s)\ \nabla^2\ \mathcal{H}^{k\ell} = 0\ . \tag{36}$$

We obtain a result similar to the previous one. We can define an index of refraction n_s given by

$$n_s = (1 + 2s)^{-\frac{1}{2}}. \tag{37}$$

Case v:

$$Q^{\alpha\beta\mu}_{V} = c\ v^{[\alpha}\ \dot{\varepsilon}^{\beta]\mu} + d\ v^{[\alpha}\ \dot{\mathcal{H}}^{\beta]\mu}\ . \tag{38}$$

Equations (9) give:

$$\varepsilon^{\ell}{}_{k|\ell} = 0\ , \tag{39a}$$

$$\mathcal{H}^{\ell}{}_{k|\ell} = 0\ , \tag{39b}$$

$$\dot{\mathcal{H}}^{k\ell} + \mathrm{curl}\ \varepsilon^{k\ell} = 0\ , \tag{40a}$$

$$\dot{\varepsilon}^{k\ell} - (\mathrm{curl}\ \mathcal{H})^{k\ell} = c\ \dot{\varepsilon}^{k\ell} + d\ \dot{\mathcal{H}}^{k\ell}. \tag{40b}$$

In the case where d is zero the wave equations obtained by using Eqs. (40) are:

$$\ddot{\mathcal{X}}^{k\ell} - \frac{1}{1-c}\nabla^2 \mathcal{X}^{k\ell} = 0, \tag{41}$$

$$\ddot{\varepsilon}^{k\ell} - \frac{1}{1-c}\nabla^2 \varepsilon^{k\ell} = 0. \tag{42}$$

Once again, we can interpret the weight c as a sort of index of refraction given by

$$n_c = (1-c)^{\frac{1}{2}} \tag{43}$$

(see Table II).

V. PERTURBATIONS OF FRIEDMANN UNIVERSES: THE ROLE OF VACUUM FLUCTUATIONS

Let us turn now to a discussion of the effects of the vacuum corrections in a strong gravitational background. We will concentrate our analysis on the study of stability properties of cosmological models. For the sake of completeness we will review briefly the pertinent perturbation equations.

The geometry of the background is assumed to be of Friedmann type, i.e.,

$$ds^2 = dt^2 - a^2(t)\, d\sigma^2 \tag{44}$$

in which $d\sigma^2$ is the line element of a 3-dimensional homogeneous space. We use a comoving system of coordinates in which the velocity field of the matter fluid is given by $v^\alpha = \delta^\alpha_0$. The fluid has no shear and no rotation, but has a nonzero expansion $\Theta = 3\,\dot{a}/a$. For a pressure-free fluid $a(t) = a_{(0)} t^{2/3}$ where a_0 is a constant. In this background we consider an arbitrary change of the configuration described by small perturbations of the metric and of the fluid parameters:

$$\begin{aligned} g_{\mu\nu} &\to g_{\mu\nu} + \delta g_{\mu\nu}\,, \\ v^\mu &\to v^\mu + \delta v^\mu\,, \\ \rho &\to \rho + \delta\rho\,. \end{aligned} \tag{45}$$

In terms of these perturbations the parameters of the congruence generated by the fluid motion are:

i) Rotation:

$$\delta\omega_{k\ell} = \frac{1}{2}\,\delta\, v_{[k\|\ell]} = \frac{1}{2}\,\delta\, v_{[k|\ell]}\,, \tag{46}$$

$$\delta\omega_{0\alpha} = 0.$$

ii) Shear:

$$\delta\sigma_{ij} = \frac{1}{2}\,\delta\, v_{(i\|j)} - \frac{1}{3}\,\delta\Theta\, g_{ij} - \delta\Gamma^{o}_{ij} \tag{47}$$

where $\delta\Gamma^{o}_{ij}$ is the perturbed Christoffel symbol.

iii) Expansion:

$$\delta\Theta = \delta v^{i}{}_{\|i} + \delta\Gamma^{\alpha}_{0\alpha}\,. \tag{48}$$

iv) Acceleration:

$$\delta a^{k} = (\delta v^{k})^{\cdot} + \frac{\Theta}{3}\,\delta v^{k}. \tag{49}$$

In these formulas we have specialized the gauge of our perturbations by using a coordinate transformation to set

$$\delta g_{0\alpha} = 0 \quad \text{and so} \quad \delta v^{o} = 0.$$

The equation for the perturbation reads

$$C^{\alpha\beta\mu\nu}{}_{\|\nu} = \delta I^{\alpha\beta\mu} + Q^{\alpha\beta\mu} \tag{50}$$

in which $\delta I^{\alpha\beta\mu}$ represents the contribution of the perturbed fluid and $Q^{\alpha\beta\mu}$ depends on vacuum corrections. Since the background geometry is conformally flat, we will denote the perturbations by $C^{\alpha\beta\mu\nu}$, $E^{\alpha\beta}$, $H^{\alpha\beta}$ instead of $\delta C^{\alpha\beta\mu\nu}$, $\delta E^{\alpha\beta}$, $\delta H^{\alpha\beta}$. The complete system of equations for the perturbation is given by (compare with Hawking[11])

$$(\delta\Theta)^{\bullet} - (\delta a^{\alpha})_{\|\alpha} + \frac{2}{3}\,\Theta\,\delta\Theta = -\frac{1}{2}\,\delta\rho\ , \qquad (51)$$

$$(\delta\omega_{k\ell})^{\bullet} - \frac{1}{2}\,\delta a_{[k|\ell]} + \frac{2}{3}\,\Theta\,\delta\omega_{k\ell} = 0, \qquad (52)$$

$$(\delta\sigma_{k\ell})^{\bullet} - \frac{1}{2}\,\delta a_{(k\|\ell)} + \frac{2}{3}\,\Theta\,\delta\sigma_{k\ell} + \frac{1}{3}\,(\delta a^{\lambda})_{\|\lambda}\,h_{k\ell} = -\,\mathcal{E}_{k\ell}\ , \qquad (53)$$

$$\mathcal{E}_{k\ \|\ell}^{\ \ell} = \frac{1}{3}\,(\delta\rho)_{|k} - \frac{1}{3}\,\dot{\rho}\,\delta v_k\ , \qquad (54)$$

$$\mathcal{H}_{k\ \|\ell}^{\ \ell} = \rho\,\delta\omega_k\ , \qquad (55)$$

$$\dot{\mathcal{H}}^{k\ell} + \frac{1}{2}\,\mathcal{E}_{m}{}^{(k}{}_{|n}\,\eta^{\ell)mn} + \Theta\,\mathcal{H}^{k\ell} = p\,\mathcal{E}^{k\ell} + q\,\mathcal{H}^{k\ell}\ , \qquad (56)$$

$$\dot{\mathcal{E}}^{k\ell} - \frac{1}{2}\,\mathcal{H}^{(k}{}_{m|n}\,\eta^{\ell)mn} + \Theta\,\mathcal{E}^{k\ell} = -\frac{\rho}{2}\,\delta\sigma^{k\ell} - m\,\mathcal{E}^{k\ell} - n\,\mathcal{H}^{k\ell}. \qquad (57)$$

The right-hand side of Eqs. (56)-(57) come from the contribution of local linear terms to the expansion of the vacuum fluctuation terms.

We will analyze here the situation in which the perturbation of the matter density is not accompanied by a perturbation of the fluid velocity; thus, we set

$$\boxed{\delta v^k = 0.} \qquad (58)$$

This implies immediately:

$$\delta\Theta = \delta\Gamma^{\alpha}_{0\alpha}\ ,$$

$$\delta\omega_k = 0\ ,$$

$$\delta\sigma_{k\ell} = -\,\delta\Gamma^{o}_{k\ell} - \frac{1}{3}\,(\delta\Theta)\,g_{k\ell}\ ,$$

$$\delta a^k = 0\ .$$

In the absence of perturbations of the fluid velocity the perturbation of the density $\delta\rho$ depends crucially on the pressure in the background. Indeed, if there is an equation of state relating pressure to density like $p = \epsilon\rho$, then conservation of matter will give rise to the relation

$$(1+\epsilon)\,\rho\,\delta a^{k} + \epsilon\,\rho_{|\ell}\,\delta(v^{k}v^{\ell}) - \epsilon(\delta\rho)_{|\ell}\,h^{\ell k} = 0 \tag{59}$$

If both $\epsilon \neq 0$ and $\delta v^{k} = 0$, Eq. (59) tells us that the perturbation $\delta\rho$ must be spatially homogeneous. However, since we choose the pressure to be zero we can have both inhomogeneity in the density and zero perturbation of the velocity. Let us investigate this situation here.

The most interesting case arises when $p = q = u = 0$. For this case it is easy to see that the only effect of m in Eq. (57) is to introduce an additional exponential dependence on time for the Weyl tensor. Indeed, we have from Eqs. (57) and (54):

$$\mathcal{E}^{k}{}_{\ell} = X^{k}{}_{\ell}(x^{i})\;t^{-4/3}\;e^{-mt}\;, \tag{60}$$

$$\delta\rho = (\delta\rho)_{0}\;t^{-4/3}\;e^{-mt}\quad. \tag{61}$$

The case in which m is negative is of great interest for the problem of galaxy formation.[4] Indeed, for negative values of m the contrast factor $\delta\rho/\rho$ <u>increases</u> exponentially. As has been remarked previously many times[12-14] this effect could be of crucial importance to permit the creation of inhomogeneous regions in our Universe.

VI. INHOMOGENEOUS CORRECTIONS

In the preceding sections we have discussed the effects on the average geometry due to perturbations of Einstein's equations induced by vacuum quantum fluctuations of the gravitational field. Our analysis was limited to a model given by the linear expansion, Eqs. (11) and (12), of the fluctuating geometry in terms of the average curvature. The factors c_k were assumed to be constant. For more general situations, c_k fails to be constant and can depend on spacetime position in a complicated manner. For instance, the conductivity coefficient m of Eq. (13) could be inhomogeneous throughout spacetime — a situation which has an analogy in electrodynamics. In order to deal with such cases we have to enlarge our model. In this section we present two examples of the inhomogeneous case. We will limit once again our expansion to terms linear in the Weyl tensor.

We allow the conductivity m to depend on the cosmical time through the expansion factor Θ. This dependence could be a very general one but we take the simplest case in which m is directly proportional to the expansion. Finally, $Q^{\alpha\beta\mu}$ can depend also on the vorticity of the congruence. These conditions are fulfilled by the current

$$Q^{\alpha\beta\mu} = \Theta \, v^{[\alpha} \, \mathcal{E}^{\beta]\mu} + q \, \eta^{\alpha\beta\rho\nu} \, \mathcal{E}_{\rho\lambda} \, \omega^{\lambda} \, v_{\nu} v^{\mu} \tag{62}$$

in which ω^{λ} is the vorticity vector defined in terms of the vorticity tensor $\omega_{\mu\nu}$ by $\omega^{\tau} = \frac{1}{2} \eta^{\alpha\beta\rho\tau} \omega_{\alpha\beta} v_{\rho}$. Equation (9) projected on the v^{α}-basis gives, for the above current, the following set of equations:

$$\mathcal{E}_{\alpha\mu\|\nu} \, h^{\mu\nu} \, h^{\alpha}{}_{\varepsilon} + 3 \, \mathcal{H}_{\alpha\varepsilon} \, \omega^{\alpha} + \eta_{\varepsilon\beta\lambda\tau} \, \mathcal{H}^{\tau\nu} \, v^{\beta} \, \sigma^{\lambda}{}_{\nu} = \frac{1}{3} \, \rho_{|\alpha} \, h^{\alpha}{}_{\varepsilon} \tag{63a}$$

$$\mathcal{H}_{\alpha\mu\|\nu} \, h^{\mu\nu} \, h^{\alpha}{}_{\varepsilon} - 3 \, \mathcal{E}_{\alpha\varepsilon} \, \omega^{\alpha} - \eta_{\varepsilon\beta\lambda\tau} \, \mathcal{E}^{\tau\nu} \, \sigma_{\nu}{}^{\lambda} \, v^{\beta} = (\rho + p) \, \omega_{\varepsilon} - 2q\omega^{2} \omega_{\rho} , \tag{63b}$$

$$\dot{\mathcal{H}}^{\lambda\alpha} h_{\alpha}{}^{\varepsilon} h_{\lambda}{}^{\rho} + \frac{1}{2} \, \mathcal{E}_{\beta}{}^{\mu}{}_{\|\alpha} \, h_{\mu}{}^{(\varepsilon} \, \eta^{\rho)\lambda\alpha\beta} \, v_{\lambda} + \Theta \, \mathcal{H}^{\varepsilon\rho} - \frac{3}{2} \mathcal{H}_{\nu}{}^{(\varepsilon} \, \sigma^{\rho)\nu} +$$
$$+ \mathcal{H}_{\mu\nu} \, \sigma^{\mu\nu} \, h^{\varepsilon\rho} - \frac{1}{2} \mathcal{H}_{\nu}{}^{(\rho} \, \omega^{\varepsilon)\nu} - a_{\alpha} \, \mathcal{E}_{\beta}{}^{(\rho} \, \eta^{\varepsilon)\lambda\alpha\beta} \, v_{\lambda} = 0, \tag{64a}$$

$$\dot{\mathcal{E}}^{\lambda\alpha} h_{\alpha}{}^{\varepsilon} h_{\lambda}{}^{\rho} - \frac{1}{2} \mathcal{H}^{\alpha\mu\|\nu} \, h_{\mu}{}^{(\varepsilon} \, \eta^{\rho)}{}_{\lambda\nu\alpha} \, v^{\lambda} + \Theta \, \mathcal{E}^{\varepsilon\rho} - \frac{3}{2} \, \mathcal{E}_{\nu}{}^{(\varepsilon} \, \sigma^{\rho)\nu} +$$
$$+ \mathcal{E}_{\mu\nu} \, \sigma^{\mu\nu} \, h^{\varepsilon\rho} - \frac{1}{2} \, \mathcal{E}_{\nu}{}^{(\rho} \, \omega^{\varepsilon)\nu} + a_{\alpha} \, \mathcal{H}_{\beta}{}^{(\rho} \, \eta^{\varepsilon)\lambda\alpha\beta} \, v_{\lambda} = - \frac{1}{2} \, (\rho + p) \sigma^{\varepsilon\rho} - \Theta\mathcal{E}^{\varepsilon\rho}, \tag{64b}$$

in which we used $\omega^2 \equiv \omega_\alpha \omega^\alpha$. The presence of the vorticity term in the current introduces qualitatively new properties to the system of equations for the average geometry. For instance, as we show next, the set of equations (63), (64) admits an expanding shear-free rotating cosmological solution without matter. As is well-known, such properties are incompatible with Einstein's equations of GR.

The absence of shear implies that the vorticity ω_λ is an eigenvector of the electric tensor $\mathcal{E}_{\alpha\beta}$. Indeed, we have

$$\mathcal{E}_{\alpha\beta} = \omega_\alpha \omega_\beta - \frac{1}{3}\omega^2 h_{\alpha\beta}. \tag{65}$$

When the magnetic part $H_{\alpha\beta}$ is set equal to zero Eqs. (63) and (64) reduce to

$$\mathcal{E}^{\alpha\mu}{}_{\|\alpha} = 0, \tag{66}$$

$$\dot{\mathcal{E}}^{\alpha\beta} + 2\theta\, \mathcal{E}^{\alpha\beta} = 0, \tag{67}$$

$$\mathcal{E}_{\mu\beta\|\alpha}\, h^{\mu(\varepsilon}\, \eta^{\rho)\lambda\alpha\beta}\, v_\lambda = 0, \tag{68}$$

in which we have made $q = 1$ and used the fact that $\omega_{\alpha\beta}\,\omega^\alpha = 0$.

Besides Eqs. (66), (67), and (68) there are two more equations that give the time evolution of the expansion and of the rotation:

$$\dot{\omega}_\lambda + \frac{2}{3}\theta\, \omega_\lambda = 0 \tag{69}$$

$$\dot{\theta} + \frac{\theta^2}{3} + 2\omega^2 = 0. \tag{70}$$

Equation (67) is a consequence of Eqs. (65) and (69). Thus, we are left with Eqs. (66), (68), (69), and (70) which are manifestly compatible.

VII. CONCLUSION

We have examined the modifications of Einstein's equations of GR due to fluctuations of the geometry. In Sec. III we presented a model by means of which the extra terms — due to fluctuations — can be expanded in a series in the average geometry. We have analyzed here only the linear terms of the series. This is sufficient to produce new features in the theory, some of which we have examined.

A weak gravitational field propagates in the form of a wave with a velocity that depends on the properties of the fluctuation. The analogy with electrodynamics permitted us to define an index of refraction for the perturbed medium.

Then, in Sec. V, we showed that an inhomogeneous perturbation of the matter density grows faster than in Einstein's theory. This behavior is a very new feature which could prove to be of importance in our understanding of galaxy formation.

Finally, in Sec. VI, we study the effects of an inhomogeneous rotation-dependent current $Q^{\alpha\beta\mu}$, on cosmological models. As a consequence of the presence of rotation, the perturbed system admits a shear-free rotating and expanding Universe, without matter.

The above properties seem appealing enough to suggest further investigations of our model.

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TABLE I. Correspondence between the gravitational current $Q^{\alpha\beta\mu}$ due to local vacuum corrections in Einstein's theory, and electrodynamics

$Q^{\alpha\beta\mu}$ (Gravitational Current)	$-\frac{1}{4} Q^{mn(\ell}\ \eta^{k)}{}_{mn}$ (3^{rd} Projection)	$-\frac{1}{2}\left[Q^{\ell ok} + Q^{ko\ell}\right]$ (4^{th} Projection)	J^k (Electric Current)	I^k (Magnetic Current)
$v^{[\alpha}\ \mathcal{E}^{\beta]\mu}$	0	$\mathcal{E}^{\ell k}$	E^k	0
$v^{[\alpha}\ \mathcal{H}^{\beta]\mu}$	0	$\mathcal{H}^{\ell k}$	H^k	0
$\eta^{\alpha\beta\lambda\sigma}\ \mathcal{E}_\lambda{}^\mu\ v_\sigma$	$\mathcal{E}^{k\ell}$	0	0	E^k
$\eta^{\alpha\beta\lambda}\ \mathcal{H}_\lambda{}^\mu\ v_\sigma$	$\mathcal{H}^{k\ell}$	0	0	H^k

$v^{[\alpha}\ \eta^{\beta]\lambda\rho\nu}\ \mathcal{E}^\mu{}_{\rho\|\nu}\ v_\lambda$	0	$(\text{curl}\ \mathcal{E})^{k\ell}$	$(\text{curl E})^k$	0

TABLE II. Correspondence between the gravitational current $Q^{\alpha\beta\mu}$ due to nonlocal vacuum corrections to Einstein's equations, and electrodynamics

$Q^{\alpha\beta\mu}$ (Gravitational Current)	$-\frac{1}{4} Q^{mn(\ell} \eta^{k)}{}_{mn}$ (3^{rd} Projection)	$-\frac{1}{2} (Q^{ko\ell} + Q^{\ell ok})$ (4^{th} Projection)	J^k (Electric Current)	I^k (Magnetic Current)
$\eta^{\varepsilon\sigma\alpha\beta} \mathcal{E}^{\mu}{}_{\varepsilon\|\sigma}$	$\dot{\mathcal{E}}^{k\ell}$	$(\text{curl } \mathcal{E})^{k\ell}$	$(\text{curl E})^k$	$\dot{E}^k$
$\eta^{\varepsilon\sigma\alpha\beta} \mathcal{H}^{\mu}{}_{\varepsilon\|\sigma}$	$\dot{\mathcal{H}}^{k\ell}$	$(\text{curl } \mathcal{H})^{k\ell}$	$(\text{curl H})^k$	$\dot{H}^k$
$v^{[\alpha} \eta^{\beta]\lambda\rho\nu} \mathcal{E}^{\mu}{}_{\rho\|\nu} v_\lambda$	0	$(\text{curl } \mathcal{E})^{k\ell}$	$(\text{curl E})^k$	0
$v^{[\alpha} \eta^{\beta]\lambda\rho\nu} \mathcal{H}^{\mu}{}_{\rho\|\nu} v_\lambda$	0	$(\text{curl } \mathcal{H})^{k\ell}$	$(\text{curl H})^k$	0
$v^{[\alpha} \dot{\mathcal{E}}^{\beta]\mu}$	0	$\dot{\mathcal{E}}^{k\ell}$	$\dot{E}^k$	0
$v^{[\alpha} \dot{\mathcal{H}}^{\beta]\mu}$	0	$\dot{\mathcal{H}}^{k\ell}$	$\dot{H}^k$	0

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