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ON ALGEBRAICALLY SPECIAL METRICS

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ON ALGEBRAICALLY SPECIAL METRICS

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ABSTRACT

Classes of algebraically special Einstein's geometries are examined, associated to a congruence of null geodesics, hypersurface orthogonal. Writing field equations in a null frame[2] adapted to the congruence, we make explicit the role of null rotations of the frame in generating new solutions (or in eliminating them). A simple case is treated as an example.

We have considered solutions of Einstein's vacuum field equations in which the metric has the form[1]

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - 2m\, k_\mu\, k_\nu \qquad (1)$$

where $g_{\mu\nu}$ is itself a solution of the vacuum field equations $R_{\mu\nu} = 0$, m is an arbitrary constant and k_μ is a null vector field with respect to $g_{\mu\nu}$, hypersurface orthogonal:

$$k_{[\mu}\, k_{\nu;\, \rho]} = 0 \qquad (2)$$

Equation (2) is a necessary and sufficient condition for

$$k_\mu = \lambda(x)\ \phi,_\mu \qquad (3)$$

where $\phi(x) =$ cte is a null hypersurface in the space-time equipped with the metric $g_{\mu\nu}$ and λ is a scalar function. The inverse of $\tilde{g}_{\mu\nu}$ is

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} + 2m\, k^\mu\, k^\nu \qquad (4)$$

(the index of k^μ is raised with $g^{\mu\nu}$).

The Ricci tensor $\tilde{R}_{\mu\nu}$ constructed with (1) is a polynomial of the fourth order in m, and since m is an arbitrary constant

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + 2m\, A_{\mu\nu} + 4m^2\, B_{\mu\nu} + m^3\, C_{\mu\nu} + m^4\, D_{\mu\nu} = 0 \qquad (5)$$

implies that each coefficient of the polynomial must vanish separately. The condition $D_{\mu\nu} = 0$ is automatically satisfied because k_μ is a null vector;

the condition $C_{\mu\nu} = 0$ implies that k_μ defines a congruence of null geodesics

$$k_{\mu;\alpha} k^\alpha = H k_\mu \tag{6}$$

where H is a scalar function. From (3), (6) and from the fact that $k_\mu k^\mu = 0$ we get

$$H = \frac{\lambda_{,\alpha} k^\alpha}{\lambda} \tag{7}$$

By using (6), $B_{\mu\nu} = 0$ yields

$$(H_{,\alpha} k^\alpha + k^\alpha{}_{;\alpha} H) k_\mu k_\nu = 0 \tag{8}$$

and form $A_{\mu\nu} = 0$ we get

$$R_{\alpha\beta\rho\nu} k^\alpha k^\rho + \overset{(1)}{A}_{\beta\nu} = 0 \tag{9}$$

where

$$\overset{(1)}{A}_{\beta\nu} = - k_{\alpha;\beta} k^\alpha{}_{;\nu} + (k_{\nu;\beta} + k_{\beta;\nu}) k^\alpha{}_{;\alpha} + H k_{\beta;\nu} - H \frac{\lambda_{,\nu} k_\beta}{\lambda} +$$

$$+ k^\alpha (\phi_{,\nu} \lambda_{,\beta} + \phi_{,\beta} \lambda_{,\nu})_{;\alpha} - \lambda k^\alpha{}_{;\alpha} (\phi_{,\nu})_{;\beta} +$$

$$- (\phi_{,\nu} \lambda^{,\alpha} k_\beta)_{;\alpha} \tag{10}$$

Let us consider now a local null frame - a quasi orthonormal complex basis[2] for vectors in space-time adapted to the congruence k^α, namely

$$(k^\alpha, \ell^\alpha, m^\alpha, \overline{m}^\alpha) \tag{11}$$

where the only non-zero scalar products of basis vectors are

$$k_\mu \ell^\mu = 1 \quad , \quad m_\mu \overline{m}^\mu = -1 \tag{12}$$

We can expand $k_{\mu;\nu}$ in this basis

$$k_{\mu;\nu} = \theta k_\mu k_\nu - \gamma k_\mu m_\nu - \overline{\gamma} k_\mu \overline{m}_\nu - \overline{\delta} m_\mu k_\nu +$$

$$+ \overline{\sigma} m_\mu m_\nu + \rho m_\mu \overline{m}_\nu - \delta \overline{m}_\mu k_\nu + \rho \overline{m}_\mu m_\nu + \sigma \overline{m}_\mu \overline{m}_\nu + H k_\mu \ell_\nu \tag{13}$$

In particular we note that

$$k^\alpha{}_{;\alpha} = -2\rho + H \tag{14}$$

and from (3) it follows

$$\overline{\gamma} - \delta = \frac{\lambda_{,\alpha} m^\alpha}{\lambda} \tag{15}$$

Contracting equations (7) and (9) with the various pairs of (11) and using (14) and (15) we obtain (apart from some trivial equations) the set of independent equations

$$H_{;\alpha} k^\alpha + k^\alpha{}_{;\alpha} H = 0 \tag{16a}$$

$$|\sigma|^2 - \rho^2 + 2\rho H = 0 \tag{16b}$$

$$R_{\alpha\beta\rho\nu} k^\alpha k^\rho m^\beta m^\nu + 2H\sigma = 0 \tag{16c}$$

$$R_{\rho\beta\alpha\nu} k^\rho k^\alpha \ell^\beta \ell^\nu - 2\rho(\theta - \frac{\lambda_{,\beta} \ell^\beta}{\lambda}) - 2H\theta - \frac{\lambda^{,\alpha}{}_{;\alpha}}{\lambda} +$$

$$+ \frac{\lambda_{,\alpha} \ell^\alpha}{\lambda} H + 2H_{,\beta} \ell^\beta + \frac{\lambda_{,\alpha} \lambda^{,\alpha}}{\lambda^2} + 4|\gamma|^2 - 2|\delta|^2 = 0 \tag{16d}$$

$$R_{\rho\beta\alpha\nu} k^\alpha k^\rho \ell^\beta m^\nu + \bar{\gamma}(H + \rho) + H_{,\nu} m^\nu - 2\rho\,\delta + \sigma\gamma = 0 \tag{16e}$$

which are equivalent to

$$\tilde{R}_{\mu\nu} = 0 \tag{17}$$

Since we can parametrize the congruence such that in (5) $H = 0$, we have by (16c) that, in this case, k^α must be a Debever-Penrose vector concerning $g_{\mu\nu}$ * .

If $\lambda = 1$ then $H = 0$,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - 2m \; \phi_{,\mu} \, \phi_{,\nu} \tag{18}$$

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* This can be more easily seen by considering the spinorial expression of basis (11) and writing $R_{\rho\beta\alpha\nu}$ in Penrose's spinorial formalism[2].

and equations (16) reduce to

$$|\sigma|^2 = \rho^2$$

$$R_{\rho\beta\alpha\nu}\, k^{\rho}\, k^{\alpha}\, \ell^{\beta}\, \ell^{\nu} + 2|\delta|^2 - 2\rho\,\theta = 0 \tag{19}$$

$$R_{\rho\beta\alpha\nu}\, k^{\rho}\, k^{\alpha}\, \ell^{\beta}\, m^{\nu} + \rho\,\overline{\gamma} - 2\rho\,\delta + \sigma\,\gamma = 0$$

In this case the curvature tensor becomes

$$\tilde{R}^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu} + 2m(k_{\beta;\nu}\, k^{\alpha}{}_{;\mu} - k_{\mu;\beta}\, k^{\alpha}{}_{;\nu} + k_{\rho}\, k^{\alpha}\, R^{\rho}{}_{\beta\mu\nu})$$

This expression shows that $\tilde{g}_{\mu\nu}$ is algebraically special if and only if $g_{\mu\nu}$ is*. Besides, let us calculate the curvature scalar $\tilde{R}^2 = \tilde{R}^{\alpha}{}_{\beta\mu\nu}\, \tilde{R}_{\alpha}{}^{\beta\mu\nu}$ where indices are raised and lowered with $\tilde{g}^{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ respectively. We obtain

$$\tilde{R}^2 = R^2 \tag{20}$$

if and only if $\sigma = 0$. So if $\lambda = 1$ and $g_{\mu\nu}$ is algebraically special $(\sigma = 0)$[3], (18) can be absored by a coordinate transformation on $g_{\mu\nu}$.

Let us investigate now the behaviour of equations (16) under a local null rotation[2]

$$k^{2} \rightarrow Ak^{\alpha} \tag{21}$$

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* This can be more easily seen by considering the spinorial expression of basos (11) and writing $R_{\rho\beta\alpha\nu}$ in Penrose's spinorial formalism .

$$m^{\alpha} \to e^{ic}(m^{\alpha} - A\,\bar{B}\,k^{\alpha})$$

$$\ell^{\alpha} \to A^{-1}\ell^{\alpha} - Bm^{\alpha} - \bar{B}\,\bar{m}^{\alpha} - A\,B\,\bar{B}\,k^{\alpha} \tag{21}$$

where A, C are real, A > 0 and B complex. We restrict curselves to the case A = cte. Under (21) the parameters of expansion (13) transform as

$$\begin{aligned}
&\theta \to A^{-1}\theta - (B\,\bar{\gamma} + \bar{B}\,\gamma) - A\,B\,\bar{B}\,H + 2\,A\,B\,\bar{B}\,\rho - (B\delta + \bar{B}\,\bar{\delta}) + (A\bar{B}^2\bar{\sigma} + AB^2\sigma) \\
&\bar{\gamma} \to e^{ic}\,(\bar{\gamma} - A\bar{B}H - AB\sigma - A\bar{B}\rho) \\
&\delta \to e^{ic}\,(\delta - AB\sigma - A\bar{B}\rho) \\
&\sigma \to e^{2ic}\,A\sigma \\
&\rho \to A\rho \\
&H \to AH
\end{aligned} \tag{22}$$

and accordingly equations (16): equations (16a), (16b) and (16c) are invariant under (21) and (22), but the last two equations turn into a linear combination of (16) plus some terms added. For instance (16e) goes into

$$e^{ic}\left[A\,\{16e\} - A^2B\,\{16c\} - A^2\gamma\bar{B}\{16b\} - A^2\,\bar{B}\,\{16a\} - 2A^2\,\bar{B}\,\rho H\right] = 0 \tag{23}$$

which is not invariant because of the last term. If we consider equation (16d) more carefully we can see that it contains two terms which are invariant under (21), (22), namely

$$- \frac{\Box^2 \lambda^2}{\lambda} + \frac{\lambda_{,\alpha} \lambda^{,\alpha}}{\lambda^2} \qquad \text{where } \Box \lambda \equiv \lambda^{,\alpha}{}_{;\alpha} \tag{24}$$

and (16d) can be written

$$\frac{\Box^2 \lambda}{\lambda} - \frac{\lambda_{,\alpha} \lambda^{,\alpha}}{\lambda^2} - T = 0 \tag{25}$$

where T is a scalar function which transforms under (21), (22) as

$$\begin{aligned} T \to T &- 2AB\,\{\overline{16e}\} - AB\left[2\sigma(\gamma - \bar{\delta}) + 2\rho(\bar{\gamma} - \delta) + H(\bar{\gamma} - 3\delta)\right] + \\ &- 2A\bar{B}\,\{\overline{16e}\} - A\bar{B}\left[2\bar{\sigma}(\bar{\gamma} - \delta) + 2\rho(\gamma - \bar{\delta}) + H(\gamma - 3\bar{\delta})\right] + \\ &+ A^2 B^2\{16c\} + A^2 \bar{B}^2\{\overline{16c}\} - 2A^2 B\bar{B}\,\{16a\} + 2A^2 B\bar{B}\,\{16b\} + \\ &+ 2A^2 B\bar{B}\,(\rho^2 - 2\rho H + 6H^2) \end{aligned} \tag{26}$$

For the case

$$\sigma = H = 0 \tag{27}$$

field equations imply $\rho = 0$ and we have by (23) that (16e) is invariant and hence, by (26), (16d) or (25) is invariant. And T assumes the form

$$T = R_{\alpha\beta\rho\nu}\, k^{\rho} k^{\alpha} \ell^{\beta} \ell^{\nu} + 4|\gamma|^2 - 2|\delta|^2$$

for $\sigma = H = 0$.

But if (27) does not hold we are faced with a problem: what role does a null rotation play in the space of solutions of $\tilde{R}_{\mu\nu} = 0$ and what does the non-invariance of equations(16d),(16e) mean in terms of solutions?We argue that null rotations generate new solutions of $\tilde{R}_{\mu\nu} = 0$, i.e., for each different basis (11) we must have a new class of solutions (1) of (17). To see this we shall treat a simple case.

We consider the case $g_{\mu\nu}$ flat and the 3-dimensional null hypersurfa hypersurface in Minkowski Space as the future light-cone

$$\phi = x^{o} - r = cte \tag{28}$$

where $r^2 = x^2 + y^2 + z^2$. This implies

$$\phi_{,\alpha} = (1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}) \tag{29}$$

and

$$k_{\mu;\nu} = \lambda_{,\nu}\, \phi_{,\mu} + \lambda\, \phi_{,\mu\nu}$$

Adapted to the congruence $k_\alpha = \lambda \phi_{,\alpha}$ we construct the null frame

$$\begin{aligned} k_\alpha &= \lambda\, (1, - x/r, - y/r, - z/r) \\ \ell_\alpha &= \frac{1}{\lambda} (\frac{1}{1 - z^2/r^2}, \frac{\cos\theta}{\sqrt{1 - z^2/r^2}}, \frac{\sin\theta}{\sqrt{1 - z^2/r^2}}, \frac{-z/r}{1 - z^2/r^2} \end{aligned} \tag{30}$$

$$m_\alpha = \frac{e^{2i\theta}}{\sqrt{2}} \left(\frac{z/r}{\sqrt{1 - z^2/r^2}} , \; i \sin\theta, \; - i \cos\theta, \; - \frac{1}{\sqrt{1 - z^2/r^2}} \right), \qquad (30)$$

in Cartesian coordinates, where $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2 + y^2}}$. Clearly (30) satisfies (12) and this basis is uniquely defined up to a local null rotation. The parameters of expansion (13) are calculated to be

$$\theta = \frac{\lambda_{,\alpha} \ell^\alpha}{\lambda} + \theta_o$$

$$\overline{\gamma} = \frac{\lambda_{,\alpha} m^\alpha}{\lambda} + \delta$$

$$\theta_o = \frac{1}{2\lambda r} \frac{1}{(1 - z^2/r^2)^2} \qquad (31)$$

$$\delta = - \frac{e^{2i\theta}}{\sqrt{2}} \frac{z^2/r^2}{\sqrt{1 - z^2/r^2}}$$

$$\rho = - \frac{\lambda}{r}$$

$$\sigma = 0$$

where we make use of (29) and (30). Equations (16) become

$$H_{,\alpha} k^\alpha - 2\rho H + H^2 = 0 \qquad (32a)$$

$$\rho = 2H \qquad (32b)$$

$$\frac{\Box^2 \lambda}{\lambda} - \frac{\lambda_{,\alpha} \lambda^{,\alpha}}{\lambda^2} - T_o = 0 \qquad (32c)$$

$$H_{,\nu} m^\nu + H(3\overline{\gamma} - 2\delta) = 0 \qquad (32d)$$

where

$$T_o = -2\rho(\theta - \frac{\lambda_{,\alpha}\ \ell^{\alpha}}{\lambda}) - 2H\theta + \frac{\lambda_{,\alpha}\ \ell^{\alpha}}{\lambda} H + 2H_{,\beta}\ \ell^{\beta} + 4|\gamma|^2 - 2|\delta|^2$$

At this point we must note that the parametrization of the congruence cannot be arbitrarily chosen. For instance, if we take an affine parameter ($H = 0$), using (7) and (30) we have

$$\frac{\partial\lambda}{\partial x^o} + \frac{\partial\lambda}{\partial r} = 0$$

and hence $\lambda = \lambda(x^o - r)$, which does not satisfy (32) since $\rho \neq 0$.

Under a local null rotation (by local we mean that B can be a function of coordinates)[4] (32a) and (32b) are invariant, (32d) becomes

$$\{32d\} - 8\,A\bar{B}H^2 = 0 \tag{33}$$

or

$$\{32d\} - A\bar{B}\{32b\} - A^2\bar{B}\{32a\} - 4A\bar{B}H^2 = 0 \tag{33'}$$

since we can make use of (32a), (32b) at ease because they are invariant; and

$$T_o \to T_o - 20\,A^2\,B\bar{B}H^2 - AB(5H\bar{\gamma} - 7H\delta)$$

$$- A\bar{B}(5H\gamma - 7H\bar{\delta}) \tag{34}$$

(34)

(in obtaining (34) we have made use of (33).

Thus it is always possible to construct a local null rotation which makes (34) zero and so reduces the set (32) to the equations

$$H_{,\alpha} k^{\alpha} - 2\rho H + H^2 = 0$$

$$\rho = 2H$$

$$\Box^2 \lambda - \frac{\lambda_{,\alpha} \lambda^{,\alpha}}{\lambda} = 0 \qquad (35)$$

If we consider now $\lambda = \lambda(r)$ the last equation (35) gives a general solution

$$\lambda = r^{-n} \qquad (36)$$

for any real n, up to a multiplicative constant. Using (7), (30) and (31) the first equation (35) gives

$$\frac{d^2 \lambda}{dr^2} + \frac{2}{r} \frac{d\lambda}{dr} + \frac{1}{\lambda} \left(\frac{d\lambda}{dr} \right)^2 = 0$$

and first integral is the second equation (35)

$$\frac{d\lambda}{dr} = - \frac{\lambda}{2r}$$

which restricts (36) to $n = \frac{1}{2}$. The solution

$$\lambda = r^{-1/2} \qquad (37)$$

corresponds to a Schwarzschild metric

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} - 2m\,\lambda^2\, \phi_{,\mu}\, \phi_{,\nu} \tag{38}$$

where $\phi_{,\mu}$ is given by (29) and m is the mass of the source, and which can be put back into its well known form by an Eddington mapping[5].

In general, for $\lambda = \lambda(x)$ the set (35) can be considered as a principal equation

$$\Box^2 \lambda - \frac{\lambda_{,\alpha}\, \lambda^{,\alpha}}{\lambda} = 0$$

whose solutions are restricted by two equations.

$$H_{,\alpha}\, k^{\alpha} - 2\rho H + H^2 = 0$$

$$\rho = 2H$$

closely associated to the null congruence.

Solution (37) is not, by any way, a solution of equations (32) with $T_o \neq 0$; conversely, equations (32) must have compatible solutions which are not solutions of (35).

Thus by a null rotation (21) we are able to generate and/or eliminate solutions of equations (16). We proved this in a very simple case making a rotation and obtaining the well-known Schwarzschild solution. We believe that not only does it work well in the simple example considered but it also constitutes a method for treating more general cases and obtaining new solutions.

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(1) Notations and conventions: signature of $g_{\mu\nu}$ is -2; Greek indices range 0 - 3; a comma (,) denotes simple partial derivative and (;) denotes covariant derivative with respect to $g_{\mu\nu}$; (x^0,x,y,z) are Cartesian coordinates and $\eta_{\mu\nu} = (+, -, -, -)$ is the flat space metric; a bar denotes complex conjugation.

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