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# WEIGHTED APPROXIMATION, VECTOR FIBRATIONS AND ALGEBRAS OF OPERATORS *

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## §1. INTRODUCTION

A vector fibration is a pair $(E,(F_x)_{x\varepsilon E})$ where E is a Hausdorff space and for each $x \varepsilon E$, $F_x$ is a real (resp. complex) vector space. A cross-section is any element of the Cartesian product $\prod_{x\varepsilon E} F_x$. A weight is a function v on E such that v(x) is a seminorm over $F_x$, for each $x \varepsilon E$. A weighted locally convex space $LV_\infty$ is a vector space L of cross-sections equipped with a locally convex topology determined by a family of seminorms

$$\| f\|_v = \sup \{v(x) \,[f(x)]; x \varepsilon E\}, \quad f \varepsilon L ,$$

where v ranges over a directed set V of weights, such that the real-valued function $x \mapsto v(x)[f(x)]$ is upper semicontinuous and null at infinity on E, for each $f \varepsilon L$. Given an algebra A of scalar-valued continuous functions on E, containing the constants, the vector space of all cross-sections is, in a natural way, an A-module.

The weighted approximation problem consists, then, in asking for a description of the closure of an A-module of $LV_\infty$; and, in particular, in finding necessary and sufficient conditions for an A-module to be dense.

In the particular case in which A is the algebra of all constant functions over E, an A-module W is, in general, only a vector subspace of $LV_\infty$. In such a case, the only thing we can do in general is to apply the Hahn-Banach theorem to describe the closure of W. To reduce the general case to this special case one introduces an equivalence relation on E, denoted by E/A, defined as follows: if x, y $\varepsilon$ E, then x = y modulo E/A if, and only if, u(x) = u(y) for all u in A. An A-submodule W is localizable under A in $LV_\infty$, by definition, if its closure consists of those $f \varepsilon L$ for which, given any $\varepsilon > 0$, any $v \varepsilon V$ and any equivalence class X modulo E/A, there is a $w \varepsilon W$ such that $v(x)[f(x) - w(x)] < \varepsilon$ for any $x \varepsilon X$. The strict weighted approximation consists, then, in asking for necessary and sufficient conditions for localizability.

The algebra A of scalar-valued continuous functions over E may be a general subalgebra of C(E) or a separating subalgebra. There arises naturally the question of deciding whether the special separating case implies the general one. In §5 we show that this is the case, answering in the affirmative the conjecture stated in ref. 6 and in ref. 7.

In §6 we extend to the context of vector fibrations the known sufficient conditions for localizability.

In §7 we consider the case of vector-valued functions, i.e. the case in which all the vector spaces $F_x$, $x \in E$, are equal to some fixed topological vector space F. In this paragraph we extend results about the strict topology $\beta$ due to Buck, Wells and Todd. We also prove a Weierstrass Theorem for locally convex Hausdorff spaces: if E and F are two such spaces, then P(E;F) the vector space of all continuous polynomials defined in E with values in F is dense in C(E;F) equipped with the compact-open topology.

In §8 we study the question of spectral synthesis for proper closed A - submodules W of $LV_\infty$, in the case which $LV_\infty$ is itself an A- module. We show that spectral synthesis holds for A- submodules that are localizable under A in $LV_\infty$. The results of this paragraph generalize the ones obtained by Todd in the case of $C_b(E;F)$ equipped with the strict topology.

Finally in §9 and §10 we turn to the case of operator algebras, which we proceed to described briefly. Let $\mathcal{L}$ be a real (resp. complex) locally convex Hausdorff space, and let $\mathcal{A}$ be a commutative algebra of linear operators over $\mathcal{L}$, not necessarily continuous; further assume that $\mathcal{A}$ contains the identity. The point co-spectrum of $\mathcal{A}$ is the set E of all homomorphism h of $\mathcal{A}$ onto $\underset{\sim}{R}$ (resp. $\underset{\sim}{C}$), such that the vector subspace $S_h$ of $\mathcal{L}$ spanned by u(x), with u in $h^{-1}(0)$ and x in $\mathcal{L}$, is not dense in $\mathcal{L}$. Consider over E the weakest topology which makes all the functions $h \mapsto h(u)$ are continuous. Clearly E is then a Hausdorff space, and we consider the vector fibration $(E,(F_h)_{h\varepsilon E})$, where $F_h = L/S_h$. We then establish the necessary condition for representing $\mathcal{L}$ as a weighted locally convex $LV_\infty$. This representation theorem uses a special concept of local convexity, which was first introduced by one of the Authors in ref. 5. We then extend to this context, via the representation theorem, the known criteria of localizability.

Historically, the concept of semicontinuous sums of locally convex spaces goes back to Silov and von Neumann. (See ref. 4, 11 and 12).

Related questions about weighted locally convex spaces of continuous scalar valued functions are now being studied by H. W. Summers (See ref. 13). We should also mention that several results on weighted polynomial approximation have been generalized recently by J. P. Ferrier (See ref. 3).

## §2. NOTATIONS AND PRELIMINARIES

In this section we give the notation and terminology which will be used throughout the paper.

$\underset{\sim}{R}$ and $\underset{\sim}{C}$ will denote the set of all real numbers and of all complex numbers, respectively. $\underset{\sim}{R}_+$ will denote the subset of $\underset{\sim}{R}$ consisting of all nonnegative real numbers. When referring to either $\underset{\sim}{R}$ or $\underset{\sim}{C}$ without being specific we shall use the symbol $\underset{\sim}{K}$. All vector spaces considered will be over $\underset{\sim}{K}$.

If E and F are topological spaces, $C(E;F)$ is the set of all continuous mappings from E into F. If $F = \underset{\sim}{K}$, we often denote $C(E;\underset{\sim}{K})$ simply by $C(E)$. In what follows, E is always a Hausdorff space and $C(E)$ is endowed with the compact-open topology, unless the contrary is explicitly stated. $C(E)$ is then a locally convex topological algebra with unity. Assume now that F is a topological vector space. $C_b(E;F)$ is then the vector subspace of $C(E;F)$ consisting of all f in $C(E;F)$ such that $f(E)$ is a bounded set of in F. $C_\infty(E;F)$ is the subspace of $C_b(E;F)$ consisting of all f in $C(E;F)$ that vanish at infinity, i.e. those f in $C(E;F)$ for which, given any neighbourhood N of the origin of F, there exists a compact subset K of E such that $f(x) \in N$, for every $x \in E$ outside of K. This is surely the case when f has compact support, i.e. when f vanishes identically outside some compact subset of E. We shall denote by $K(E;F)$ the subspace of $C_\infty(E;F)$ consisting of all f in $C(E;F)$ with compact support.

A <u>weight on</u> $\underset{\sim}{R}^n$ is any upper semicontinuous positive real-valued function defined on $\underset{\sim}{R}^n$. Let $w \geq 0$ be a weight on $\underset{\sim}{R}^n$. $Cw_\infty(\underset{\sim}{R}^n;\underset{\sim}{K})$ denotes the vector subspace of $C(\underset{\sim}{R}^n;\underset{\sim}{K})$ consisting of all $f \in C(\underset{\sim}{R}^n;\underset{\sim}{K})$ such that fw vanishes at infinity. $Cw_\infty(\underset{\sim}{R}^n;\underset{\sim}{K})$ is endowed with the topology determined by the seminorm $\|f\|_w = \sup \{w(x)\ |f(x)|;\ x \in \underset{\sim}{R}^n\}$.

A locally bounded $\underset{\sim}{K}$- valued function f on $\underset{\sim}{R}^n$ is said to be <u>rapidly decreasing at infinity</u>, if the following equivalent conditions hold true:

(1) pf is bounded on $\underset{\sim}{R}^n$, for all p in $P(\underset{\sim}{R}^n)$.

(2) $pf \to 0$ at infinity for all p in $P(\underset{\sim}{R}^n)$.

Let $w \geq 0$ be a <u>weight on</u> $\underset{\sim}{R}^n$ which is rapidly decreasing at infinity. Then $P(\underset{\sim}{R}^n) \subset Cw_\infty(\underset{\sim}{R}^n;\underset{\sim}{K})$, and in this case w is said to be a <u>fundamental weight</u> if $P(\underset{\sim}{R}^n)$ is dense in $Cw_\infty(\underset{\sim}{R}^n;\underset{\sim}{K})$. We shall denote by $\Omega_n$ the set of all fundamental weights on $\underset{\sim}{R}^n$, and by $\Gamma_n$ the subset of $\Omega_n$ consisting of all $\gamma \in \Omega_n$ such that

$\gamma^k \in \Omega_n$, for all $k > 0$.

We shall consider $\underset{\sim}{R}^n$ as a vector lattice in the usual way: if $u = (u_1,\ldots,u_n)$ and $t = (t_1,\ldots,t_n)$ belong to $\underset{\sim}{R}^n$ we write $u \leq t$ provided $u_i \leq t_i$ for all $i = 1,\ldots,n$; and define $|u| = (|u_1|,\ldots,|u_n|)$. A real-valued function f defined on $\underset{\sim}{R}^n$ is then said to be <u>decreasing</u> if $u, t \in R^n$ and $|u| \leq |t|$ imply $f(u) \geq f(t)$. Denote by $\Omega_n^d$ the subset of $\Omega_n$ consisting of those fundamental weights which are decreasing. Denote by $\Gamma_n^d$ the intersection $\Gamma_n \cap \Omega_n^d$.

If E is a topological space and $A \subset C(E)$ is a sub-algebra containing the constants, E/A denotes the equivalence relation defined on E as follows: if $x,y \in E$, then $x \equiv y$ modulo E/A if and only if $f(x) = f(y)$ for all f in A. The subalgebra A is said to be <u>separating</u> on E, if the equivalence classes of E modulo E/A are reduced to points, i.e. if for any pair of distinct points $x, y \in E$, there exists some f in A such that $f(x) \neq f(y)$. Let F denote the quotient space of E modulo E/A, and $\pi: E \to F$ the quotient map. Then F is a Hausdorff space, since it admits a separating subalgebra. To each $f \in A$ there corresponds a unique $h \in C(F)$ such that $f = h \circ \pi$. The set $B = \{h; f = h \circ \pi, f \in A\}$ is such a separating subalgebra on F, containing the constants. If A is self-adjoint in the complex case, so is B. Finally, if E is compact, then F is compact; $\pi$ is a closed map and the saturated neighbourhood of each equivalence class $\pi^{-1}(y)$, $y \in F$, form a basis of neighbourhoods of that class. (A subset $X \subset E$ is saturated with respect to E/A if it contains the union of all subsets $\pi^{-1}(\pi(x))$ when x ranges over X).

## §3. THE SPACES $LV_b$ and $LV_\infty$

Let E be a Hausdorff space and for each $x \in E$, let $F_x$ be a real (resp. complex) vector space. By a cross-section f over E we mean a function f on E such that f(x) belongs to $F_x$ for each x in E, i.e. a point in the Cartesian product $\prod_{x \in E} F_x$. The Cartesian product $\prod_{x \in E} F_x$ is made a real (resp. complex) vector space in the usual way, and a vector space of cross-sections is a vector subspace of $\prod_{x \in E} F_x$. A weight on E, say v, is a function on E such that v(x) is a seminorm over $F_x$, for each x in E. A set V of weights on E is directed, by definition, if for every pair $v_1, v_2 \in V$, there exist $v \in V$ and $t > 0$ such that $v_i(x) \leq tv(x)$ for all $x \in E$, $i = 1, 2$. From now on, V always denotes a directed set of weights. If f is a cross-section over E and v is a weight on E, the positive real-valued function $x \mapsto v(x)[f(x)]$, defined on E, will be denoted by $v[f]$.

Definition 1. Let L be a vector space of cross-section over E. A weight v on E is said to be

(1) L- bounded

(2) L- upper semicontinuous,

(3) L- null at infinity,

in the case the function $v[f]$ is, respectively,

(1) bounded on E,

(2) upper semicontinuous on E,

(3) vanishes at infinity on E,

for every cross-section f in L.

From the above definition, it follows that any weight v which is L- bounded determines a seminorm over L, namely

$$f \mapsto |f|_v = \sup \{v(x)[f(x)] ; x \in E\}.$$

Notice also that if the weight v is L- upper semicontinuous and L- null at infinity, the set $\{x \in E; v(x)[f(x)] \geq \varepsilon\}$ is compact, for all $\varepsilon > 0$ and f in L. Hence v is L- bounded.

Definition 2. Let L be a vector space of cross-sections over E and V a directed set of weights which are L- upper semicontinuous and L- bounded. $LV_b$ will denote the vector space L endowed with the locally convex topology determined by the family of seminorms $f \mapsto |f|_v$, when v ranges over V. In the particular case in which the weights $v \in V$ are L- upper semicontinuous and L- null at infinity, we shall write $LV_\infty$ for this space. The spaces $LV_b$ and $LV_\infty$ are called weighted locally convex spaces of cross-sections.

Since we assumed V to be directed, the sets of the form $\{f \in L; v(x)[f(x)] < \varepsilon$ for all $x \in E\}$, where $v \in V$ and $\varepsilon > 0$, form a basis of neighbourhoods of the origin in $LV_b$ or $LV_\infty$. Moreover, $LV_b$ or $LV_\infty$ is a Hausdorff space if, given any $f \in L$, $f \neq 0$, there exist $v \in V$ and $x \in E$ such that $v(x)[f(x)] > 0$.

Remark 1. Given a weighted locally convex space $LV_b$ of cross-sections over E, the vector subspace $L_0$ of L of all cross-sections f such that $v[f]$ vanishes at infinity, for every $v \in V$, is closed in $LV_b$. Indeed, let $f \in L$ be a cross-section which belongs to the closure of $L_0$ in $LV_b$. Given any $v \in V$ and $\varepsilon > 0$, there is a $g \in L_0$ such that $v(x)[f(x) - g(x)] < \varepsilon/2$ for any $x \in E$. Hence $\{x \in E; v(x)[f(x)] \geq \varepsilon\} \subset \{x \in E; v(x)[g(x)] \geq \varepsilon/2$ . Since the right-hand side of this inclusion is compact, the same is true of its left-hand side, it being closed. Thus $v[f]$ vanishes at infinity, for any v in V, and therefore $f \in L_0$. Notice that $L_0$ endowed with the relative topology of subspace of $LV_b$ is precisely $L_0V_\infty$.

When X is a closed subset of E and $L \subset \prod_{x\in E} F_x$ is a vector space of cross-sections over E, we shall denote by $L|X$ the vector space $\pi_X(L)$, where $\pi_X$ is the canonical projection of $\prod_{x\in E} F_x$ onto $\prod_{x\in X} F_x$. If $f \in \prod_{x\in E} F_x$, we shall denote by $f|X$ the cross-section $\pi_X(f)$. Let now V be a set of weights on E. We shall denote by $V|X$ the set of all weights on X of the form $(v(x))_{x\in X}$, where v ranges over V. Notice that $v|X[f|X] = v[f]|X$ for all $v \in V$ and $f \in L$. Hence, if v is L- upper semicontinuous, then $v|X$ is $L|X$- upper semicontinuous. Similarly, if v is L- bounded or L- null at infinity, then $v|X$ is $L|X$- bounded or $L|X$- null at infinity, respectively. Therefore, it follows that, if $LV_b$ or $LV_\infty$ are defined, then the weighted locally convex spaces $(L|X)(V|X)_b$ or $(L|X)(V|X)_\infty$ are also defined. In such a case, we shall write $LV_b|X$ or $LV_\infty|X$ instead of the longer expressions $(L|X)(V|X)_b$ or $(L|X)(V|X)_\infty$ respectively.

## §4. THE WEIGHTED APPROXIMATION PROBLEM. LOCALIZABILITY

The vector space $\prod_{x\in E} F_x$ of all cross-sections is an A- <u>module</u>, for any <u>sub</u>algebra $A \subset C(E;\underset{\sim}{K})$ containing the constants, under the following multiplication operation: if $u \in A$ and $f = (F(x))_{x\in E}$ is a cross-section then uf is the cross--section $(u(x)f(x))_{x\in E}$. Thus, if W is a vector space of cross-section, we say that W is an A- module, where A is as above, if $AW = \{uf;\ u \in A,\ f \in W\} \subset W$, i.e. if W is an A- submodule of $\prod_{x\in E} F_x$.

Given an A- module $W \subset LV_\infty$, the <u>weighted</u> <u>approximation</u> <u>problem</u> consists, then, in asking for a description of the closure of W in $LV_\infty$; and , in particular, in finding necessary and sufficient conditions for W to be dense in $LV_\infty$.

In the particular case in which A is the algebra of all constant functions over E, an A- module W is, in general, only a vector subspace of $LV_\infty$. In such a case, the only thing we can do in general is to apply the Hahn-Banach theorem: once known the dual space of $LV_\infty$, the closure of W consists of those $f \in L$ such that any continuous linear functional $\phi$ over $LV_\infty$ which vanishes on W, vanishes at f too.

We shall try to reduce the general case to this special case. We first notice that for any equivalence class $X \subset E$ module E/A, the subalgebra $A|X \subset C(X;\underset{\sim}{K})$ is precisely the algebra of all constant functions over X. This remark suggests the following definition.

Definition 3. An A- module $W \subset LV_\infty$ is localizable under A in $LV_\infty$, when its closure consists of those $f \in L$ such that, for any equivalence class $X \subset E$ modulo E/A, given $v \in V$ and $\varepsilon > 0$, there exists $w \in W$ such that $v(x)[f(x) - - w(x)] < \varepsilon$ for any $x \in X$.

The strict weighted approximation problem consists, then, in asking for necessary and sufficient conditions for localizability.

## §5. THE SEPARATING CASE

Let us consider the following conditions:

(a) The subalgebra $A \subset C(E)$ is separating on E.

(b) For all $f \in L$ and $x \in E$, given $v \in V$ and $\varepsilon > 0$, there exist $w \in W$ and neighbourhood $U$ of x in E, such that $v(t)[f(t) - w(t)] < \varepsilon$ for all $t \in U$.

(c) For any $x \in E$, $W(x) = \{w(x); w \in W\}$ is dense in $L(x) = \{f(x); f \in L\} \subset F_x$, when $F_x$ is equipped with the locally convex topology determined by the family of seminorms $V(x) = \{v(x); v \in V\}$.

Proposition 1. Conditions (b) and (c) are equivalent. Proof. It is obvious that (b) implies (c). Conversely, suppose that (c) is valid. Let $f \in L$ and $x \in E$. Given any $v \in V$ and $\varepsilon > 0$, there exists $w \in W$ such that $v(x)[f(x) - w(x)] < \varepsilon$. Since $f - w$ belongs to L, $v[f - w]$ is upper semicontinuous. Hence, there exists a neighbourhood $\mathcal{U}$ of x in E such that $v(t)[f(t) - w(t)] < \varepsilon$, for any $t \in \mathcal{U}$. Therefore (b) is satisfied.

Definition 4. The Separating Case of the strict weighted approximation problem occurs when conditions (a) and (b), or equivalently, conditions (a) and (c), are satisfied.

Proposition 2. In the Separating Case, W is dense in $LV_\infty$ if, and only if, W is localizable under A in $LV_\infty$.

Proof. By condition (a), any equivalence class of E modulo E/A is a set reduced to a point. Hence, W is localizable under A in $LV_\infty$ if, and only if, its closure in $LV_\infty$ consists of those cross-sections $f \in L$ such that $f(x)$ belongs to the closure of $W(x) \subset F_x$, for any $x \in E$. By condition (c), these are precisely all the elements of L.

Remark 2. In ref. 6 and 7 it was conjectured the equivalence between the separating and the general cases of the results of weighted approximation theory. This is established below (see Theorem 1) and its proof rests on the following crucial result.

Lemma 1. Let E and F be two Hausdorff spaces and $\pi : E \to F$ a continuous mappings from E onto F. For any upper semicontinuous function $g : E \to \mathbb{R}_+$ that vanishes at infinity, let $h: F \to \mathbb{R}_+$ be defined as $h(y) = \sup \{g(x); x \in \pi^{-1}(y)\}$, for all $y \in F$. Then h is upper semicontinuous and vanishes at infinity.

Proof. We first notice that h is well defined, since g is bounded and attains its maximum at each closed set $\pi^{-1}(y)$ for y in F. Let then $\varepsilon > 0$ be given. The set $X = \{x \in E; g(x) \geq \varepsilon\}$ is compact. Since $\pi$ is continuous, $\pi(X)$ is also compact. We claim that $\{y \in F; h(y) \geq \varepsilon\} = \pi(X)$. For, if $y \in \pi(X)$, then $y = \pi(x)$ for some $x \in X$, and then $h(y) \geq g(x) \geq \varepsilon$. Conversely, if $y \notin \pi(X)$ and t is any point in $\pi^{-1}(y)$, then $g(t) < \varepsilon$. By the remark made at the beginning of the proof, it follows that $h(y) < \varepsilon$. This establishes our claim, and ends the proof.

Remark 3. If in Lemma 1 we omit the hypothesis that g vanishes at infinity, then h may fail to be upper semicontinuous or to vanish at infinity, as the following simple examples show.

Example 1. Let $E = \mathbb{R}^2$, $F = \mathbb{R}$ and $\pi: \mathbb{R}^2 \to \mathbb{R}$ be defined as $\pi(x,y) = y$. Let g be the characteristic function of the closed set $\{(x,y); y = e^x\}$. Then h defined in Lemma 1 is the characteristic function of the open set $\{y \in \mathbb{R}; y>0\}$ and therefore is not upper semicontinuous.

Example 2. Let E, F and $\pi$ be as in Example 1. Let now g be the characteristic function of the closed set $\{(x,y); y = x^2\}$. Then h is the characteristic function of the closed set $\{y \in \mathbb{R}; y \geq 0\}$. Thus h is upper semicontinuous, but does not vanish at infinity.

Let then $LV_\infty$ be a weighted locally convex space of cross-sections and $W \subset LV_\infty$ an A- module. Let F be the quotient space of E by the equivalence relation E/A, and $\pi: E \to F$ be the quotient map. Let $\pi_*: C(F) \to C(E)$ be the induced homomorphism: $\pi_*(b) = b \circ \pi$ for any $b \in C(F)$. Then $B = \pi_*^{-1}(A)$ is a subalgebra of C(F) containing the unity and separating on F. If A is self-adjoint, so is B. For every $y \in F$, $\pi^{-1}(y)$ is a closed subset of E and we shall denote by

$f|\pi^{-1}(y)$ the cross-section over $\pi^{-1}(y)$ given by $(f(x))_{x\in\pi^{-1}(y)}$, when f is a cross-section over E. Let $L|\pi^{-1}(y)$ be the vector space $\{f|\pi^{-1}(y);\ f \in L\}$. For every weight v on E consider the weight v' over F defined by

$$v'(y)[f|\pi^{-1}(y)] = \sup\{v(x)[f(x)];\ x \in \pi^{-1}(y)\} .$$

Let M be the vector space of cross-sections $g \in \prod_{y\in F} L|\pi^{-1}(y)$ such that,

1) there exists $f \in L$, such that $g(y) = f|\pi^{-1}(y)$ for all $y \in F$;

2) given $y \in F$, $v \in V$ and $\varepsilon > 0$, there exists $w \in W$ such that $v'(y)[w|\pi^{-1}(y) - g(y)] < \varepsilon$.

By Lemma 1, each weight v' on F is M- upper semicontinuous and M- null at infinity. Hence we may consider the space $MV'_\infty$, where $V' = \{v';\ v \in V\}$. Let $W' = \{(w|\pi^{-1}(y))_{y\in F};\ w \in W\}$. Then W' is a B- module.

THEOREM 1. *W is localizable under A in $LV_\infty$ if, and only if, W' is dense in $MV'_\infty$.*

Proof. Suppose that W is localizable under A in $LV_\infty$. Let $g \in M$, $v' \in V'$ and $\varepsilon > 0$ be given. Let $f \in L$ be such that $g(y) = f|\pi^{-1}(y)$ for all $y \in F$. Let $X \subset E$ be any equivalence class modulo E/A, $u \in V$ and $\delta > 0$ be given. If $y_0 \in F$ is such that $X = \pi^{-1}(y_0)$, by definition of M there exists $w_0 \in W$ such that

$$u'(y_0)[w_0|\pi^{-1}(y_0) - f|\pi^{-1}(y_0)] < \delta .$$

Hence $u(x)[w_0(x) - f(x)] < \delta$ for all $x \in X$. Therefore f belongs to the closure of W in $LV_\infty$. Hence there exists $w \in W$ such that $v(x)[w(x) - f(x)] < \varepsilon$ for all $x \in E$. But then $v'(y)[w|\pi^{-1}(y) - f|\pi^{-1}(y)] < \varepsilon$ for all $y \in F$, i.e. g belongs to the closure of W' in $WV'_\infty$.

Conversely, suppose that W' is dense in $MV'_\infty$. Let $f \in L$ be such that, for any equivalence class $X \subset E$ modulo E/A, given $v \in V$ and $\varepsilon > 0$, there exists

$w \in W$ such that $v(x)[f(x) - w(x)] < \varepsilon$, for any $x \in X$. Consider $g = (f|\pi^{-1}(y))_{y\in F}$. By the above property, g belongs to M. Therefore, given $u \in V$ and $\delta > 0$, there exists $w \in W$ such that $u'(y)[f|\pi^{-1}(y) - w|\pi^{-1}(y)] < \delta$ for all $y \in F$. Hence $u(x)[f(x) - w(x)] < \delta$ for all $x \in E$, and f belongs to the closure of W in $LV_\infty$.

## §6. SUFFICIENT CONDITIONS FOR LOCALIZABILITY

If A is a subalgebra of C(E) containing the constants, G(A) will denote a subset of A which topologically generates A as an algebra over $\underset{\sim}{K}$ with unity, i.e. the subalgebra over $\underset{\sim}{K}$ of A generated by G(A) and the function identically one is dense in A for the compact-open topology of C(E). If $W \subset LV_\infty$ is an A-module, G(W) will denote a subset of W which topologically generates W as a module over A, i.e. the submodule over A of W generated by G(W) is dense in W for the topology of $LV_\infty$.

The proof of the theorems in this paragraph are entirely similar to the analogous results proved in ref. 10, and are therefore omitted. The first theorem below reduces the search of sufficient conditions of localizability to the search of fundamental weights in the sense of Bernstein on $\underset{\sim}{R}^n$, i.e. to the Finite Dimensional Bernstein Approximation Problem.

THEOREM 2. *Suppose that there exist* G(A) *and* G(W) *such that:*

(1) G(A) *consists only of real-valued functions;*

(2) *given any* $v \in V$, $a_1,\ldots,a_n \in G(A)$ and $w \in G(W)$, *there exist* $a_{n+1},\ldots, a_N \in G(A)$, *where* $N \geq n$, *and* $\omega \in \Omega_N$ *such that for all* $x \in E$:

$$v(x)[w(x)] \leq \omega(a_1(x),\ldots, a_n(x),\ldots, a_N(x)) .$$

*Then* W *is localizable under* A *in* $LV_\infty$.

Corollary 1. Suppose that there exist G(A) and G(W) such that:

(1) G(A) consists only of real-valued functions;

(2) G(A) and G(W) are finite; say $G(A) = \{a_1, \ldots, a_n\}$;

(3) given any $v \in V$ and any $w \in G(W)$, there exists $\omega \in \Omega_n$ such that for all $x \in E$:

$$v(x)[w(x)] \leq \omega(a_1(x), \ldots, a_n(x)) .$$

Then W is localizable under A in $LV_\infty$.

THEOREM 3. Suppose that A is self-adjoint and that there exist G(A) and G(W) such that, given any $v \in V$, $a_1, \ldots, a_n \in G(A)$ and $w \in G(W)$, there exist $a_{n+1}, \ldots, a_N \in G(A)$, where $N > n$, and $\omega \in \Omega_N^d$ such that

$$v(x)[w(x)] \leq \omega(|a_1(x)|, \ldots, |a_n(x)|, \ldots, |a_N(x)|)$$

for all $x \in E$. Then W is localizable under A in $LV_\infty$.

Corollary 2. Assume that A is self-adjoint and that there exist G(A) and G(W) such that:

(1) G(A) and G(W) are finite; say $G(A) = \{a_1, \ldots, a_n\}$ ;

(2) given any $v \in V$ and any $w \in G(W)$, there exists $\omega \in \Omega_n^d$ such that for all $x \in E$:

$$v(x)[w(x)] \leq \omega(|a_1(x)|, \ldots, |a_n(x)|) .$$

Then W is localizable under A in $LV_\infty$.

THEOREM 4. Assume that the hypothesis of Theorem 1 holds; and that we are in the Separating Case. Then W is dense in $LV_\infty$.

Proof. Apply Proposition 2 and Theorem 1.

Remark 4. Theorem 2 and Theorem 4 are equivalent. Indeed, given A, $LV_\infty$

and W satisfying the hypothesis of Theorem 2, let B, $MV_\infty'$ and W' be as in §5. Taking $G(B) = \pi_*^{-1}(G(A))$ and $G(W') = \{(w|\pi^{-1}(y))_{y\varepsilon F}; w \varepsilon G(W)\}$, they satisfy the hypothesis of Theorem 4. Hence W' is dense in $MV_\infty'$. By Theorem 1, W is localizable under A in $LV_\infty$.

Our next theorem reduces the search of sufficient conditions of localizability of modules to the search of fundamental weights on $\underset{\sim}{R}$, i.e. to the One Dimensional Bernstein Approximation Problem.

THEOREM 5. *Suppose that there exist* G(A) *and* G(W) *such that:*

(1) G(A) *consists only of real-valued functions;*

(2) *given any* $v \varepsilon V$, $a \varepsilon G(A)$ *and* $w \varepsilon G(W)$ *there exists* $\gamma \varepsilon \Gamma_1$ *such that for all* $x \varepsilon E$:

$$v(x)[w(x)] \leq \gamma(a(x)) .$$

*Then* W *is localizable under* A *in* $LV_\infty$.

THEOREM 6. *Assume that* A *is self-adjoint and that there exist* G(A) *and* G(W) *such that, given any* $v \varepsilon V$, $a \varepsilon G(A)$ *and* $w \varepsilon G(W)$ *there exists* $\gamma \varepsilon \Gamma_1^d$ *such that for all* $x \varepsilon E$:

$$v(x)[w(x)] \leq \gamma(|a(x)|).$$

*Then* W *is localizable under* A *in* $LV_\infty$.

THEOREM 7. *Assume that the hypothesis of Theorem 6 holds, and that we are in the Separating Case. Then* W *is dense in* $LV_\infty$.

*Remark 5.* Theorem 6 and Theorem 7 are equivalent. The proof of this fact in entirely similar to that of Remark 4.

THEOREM 8. (Analytic Criterion of Localizability). Assume that $A$ is self-adjoint in the complex case and that there exist $G(A)$ and $G(W)$ such that, given any $v \in V$, $a \in G(A)$ and $w \in G(W)$, there exist constants $C > 0$ and $c > 0$ such that for all $x \in E$:

$$v(x)\left[w(x)\right] \leq C\, e^{-c|a(x)|} .$$

Then $W$ is localizable under $A$ in $LV_\infty$.

Proof. The function $\gamma(t) = C\, e^{-c|t|}$ defined in $\underset{\sim}{R}$ belongs to $\Gamma_1^d$ (see Lemma 2 §28, Nachbin ref. 10). It remains to apply Theorem 6.

THEOREM 9. (Quasi-analytic Criterion of Localizability). Assume that $A$ is self-adjoint in the complex case and that there exist $G(A)$ and $G(W)$ such that, given any $v \in V$, $a \in G(A)$ and $w \in G(W)$ we have

$$\sum_{m=1}^{\infty} (M_m)^{-1/m} = +\infty$$

where $M_m = \sup\{v(x)\left[a^m(x)\, w(x)\right];\ x \in E\}$ for $m = 0, 1, 2, \ldots$ Then $W$ is localizable under $A$ in $LV_\infty$.

Proof. Let $\gamma$ be defined on $\underset{\sim}{R}$ by setting

$$\gamma(t) = \inf\{M_m / t^m\ ;\ m = 0, 1, 2, \ldots\},$$

if $t \neq 0$ and $\gamma(0) = 0$, if some $M_m = 0$, or $\gamma(0) = M_0$ otherwise. Then $\gamma \geq 0$ is upper semicontinuous and

$$\sup\{\gamma(t)|t^m|\ ;\ t \in \underset{\sim}{R}\} \leq M_m \text{ for } m = 0, 1, 2, \ldots$$

By Lemma 2, §29, Nachbin ref. 10, $\gamma$ belongs to $\Gamma_1^d$. From the definition of $M_m$ we have $v(x)\left[a^m(x)\, w(x)\right] \leq M_m$ for every $x \in E$ and all $m = 0, 1, 2, \ldots$ Hence $v(x)\left[w(x)\right] \leq \gamma(|a(x)|)$ for all $x \in E$. By Theorem 6, $W$ is localizable under $A$ in $LV_\infty$.

Remark 6. Theorem 8 is based on the uniqueness of analytic continuation, whereas Theorem 9 rests on the Denjoy-Carleman Theorem.

Definition 5. The Bounded Case of the strict weighted approximation problem is the one in which there exist $G(A)$ and $G(W)$ such that every $a \in G(A)$ is bounded on the support of the function $v[w]$, for any $v \in V$ and any $w \in G(W)$.

Remark 7. Each of the following hypothesis leads to an instance of the Bounded Case:

(1) $A \subset C_b(E)$.

(2) Every $a \in G(A)$ is bounded on the support of any $v \in V$.

(3) Every $a \in G(A)$ is bounded on the support of any $w \in G(W)$.

(4) Every $v[w]$ has compact support, when $v \in V$ and $w \in G(W)$.

(5) Every $v \in V$ has compact support.

(6) Every $w \in G(W)$ has compact support.

THEOREM 10. *Assume that A is self-adjoint in the complex case and that we are in the Bounded Case. Then W is localizable under A in $LV_\infty$.*

*Proof*. Let $v \in V$, $a \in G(A)$ and $w \in G(W)$ be given. Let $m = \sup\{|a(x)|;\ x$ in the support of $v[w]\}$ and $M = \sup\{v(x)[w(x)];\ x \in E\}$. Let $c > m$ and $C > M$. If $\gamma$ is the characteristic function of the interval $[-c, c] \subset \mathbb{R}$ times the constant $C$, then $\gamma \in \Gamma_1^d$ and

$$v(x)[w(x)] \leq \gamma(|a(x)|) \quad \text{for every } x \in E.$$

By Theorem 6, W is localizable under A in $LV_\infty$.

THEOREM 11. *Assume that A is self-adjoint in the complex case and that we are in the Bounded Case. Then, if A is separating on E, W is dense in $LV_\infty$ if,* *and only if*, W(x) *is dense in* L(x) *for each* x *in* E.

*Proof*. The condition is obviously necessary. On the other hand, if A is separating on E and W(x) is dense in L(x) for each x in E, then we are in the Separating Case, and therefore W is dense in $LV_\infty$ if W is localizable under A in $LV_\infty$, by Proposition 2. Since we are in the Bounded Case, this always occurs.

## §7. VECTOR-VALUED FUNCTIONS

In this paragraph we shall consider the case in which all the vector spaces $F_x$, $x \in E$, are equal to some fixed vector space F. A cross-section f is then any mapping from E into F, and a weight v on E is a mapping whose values v(x) are seminorms over F. We shall restrict our attention to the case in which F is a locally convex space and the weights are continuous seminorm-valued. As a first example of this situation, let E be a locally compact Hausdorff space and F a locally convex space whose topology is determined by a family Γ of seminorms over F. Let V be the family of weights of the form $x \mapsto |\phi(x)|p$, when $\phi$ ranges over $C_\infty(E)$, the space of all continuous scalar-valued functions that vanish at infinity, and p ranges over the set Γ. If $L = C_b(E;F)$, then all weights $v \in V$ are L- upper semicontinuous and L- null at infinity, and $LV_\infty$ is precisely the space $C_b(E;F)$ equipped with the strict topology β, first introduced by Buck in ref. 1. Our next theorem extends results of Buck (see Theorem 5, ref. 2), Wells (see Theorem 2, ref. 15) and Todd (see Theorem 3, ref. 14).

THEOREM 12. *Let* E *be a locally compact Hausdorff space, and* F *be a locally convex space. Let* $A \subset C_b(E)$ *be a separating subalgebra containing the unity* *and self-adjoint in the complex case. Let* $W \subset C_b(E;F)$ *be an* $A$*- module. If* $W(x)$ *is dense in* $F$ *for each* $x$ *in* $E$, *then* $W$ *is* $\beta$*-dense in* $C_b(E;F)$.

*Proof.* Since $A \subset C_b(E)$ we are in the Bounded Case. Therefore we have just to apply Theorem 11.

Our next theorem is a Weierstrass Theorem for locally convex Hausdorff spaces. If E and F are two such spaces, $P(E;F)$ denotes the vector space of all continuous polynomials defined in E with values in F. On $C(E;F)$ we will consider the compact-open topology; this is defined by weights of the form $x \in E \mapsto \chi_K(x)p$, where $\chi_K$ is the characteristic function of K, and K ranges over the compact subsets of E and p over the continuous seminorms of F.

*THEOREM 13. Let E and F be two locally convex Hausdorff spaces, both real. Then* $P(E;F)$ *is dense in* $C(E;F)$.

*Proof.* Let A be the algebra of all continuous real-valued polynomials defined in E. The algebra A contains the constants, is separating over E, and is self-adjoint. The vector space $P(E;F)$ is an A- module. For every x in E, the set $P(x) = \{p(x); p \in P(E;F)\}$ is equal to F. Since the weights have compact support, we are in the Bounded Case, and therefore by Theorem 11, $P(E;F)$ is dense in $C(E;F)$.

*Remark 8.* If E and F are complex, theorem 13 holds with the following modifications. If $m \geq 1$ is an integer, let $\phi: E^m \to F$ be a continuous function which in each variable is either linear or anti-linear and let $p(x) = \phi(x_1, \ldots, x_m)$ where $x_1 = \ldots = x_m = x$. Let $P_m^*(E;F)$ be the set of all such p's. For $m = 0$, let $P_0^*(E;F)$ be the set of all constant mappings from E to F. Then $P^*(E;F)$ is the vector space of all finite sums $\Sigma\, p_i$, where $p_i \in P_i^*(E;F)$.

When $F = C$, $A = P^*(E;C)$ is a separating and self-adjoint subalgebra of $C(E)$, containing the constants. Moreover $P^*(E;F)$ is an A- module under pointwise operations. Since $P^*(x) = F$ for all $x \in E$, it follows that $P^*(E;F)$ is dense in $C(E;F)$ in the compact-open topology. Theorem 13 remains true if we restrict attention to elements of $P(E;F)$ with range generating a finite dimensional subspace. Likewise for its complex analogue.

When F is a normed space the arguments in Nachbin ref. 10, §19, Theorem 1 can be applied to prove Theorem 13 above, directly.

## §8. SPECTRAL SYNTHESIS

In this paragraph $A \subset C(E;\underset{\sim}{K})$ is a subalgebra containing the constants and $LV_\infty$ is an A- module.

THEOREM 14. *Every proper closed A- submodule $W \subset LV_\infty$, which is localizable under A in $LV_\infty$, is contained in some closed A- submodule of co-dimension one in L, and is the intersection of all proper closed A- submodules of codimension one in L which contain it.*

*Proof.* Let $f \in L$ be a cross-section which does not belong to $\overline{W} = W$. Since W is localizable under A in $LV_\infty$, there exists an equivalence class $X \subset E$ modulo E/A such that $f|X$ does not belong to the closure of $W|X$ in $LV_\infty|X$. By the Hahn-Banach Theorem, there exists a continuous linear functional $\phi \in (LV_\infty|X)'$ such that $\phi(f|X) \neq 0$ and $\phi(w|X) = 0$ for all $w \in W$. Let them M be the closed A-submodule of codimension 1 in L defined by $M = \{g \in L; \phi(g|X) = 0\}$. It is clear that $W \subset M$ and $f \notin M$.

Let $\mathcal{R}$ denote the set of all equivalence classes $X \subset E$ modulo E/A. If

$X \in \mathcal{R}$ and $\phi \in (LV_\infty|X)'$, let $W(X,\phi) = \{g \in L;\ \phi(g|X) = 0\}$. If $W \subset LV_\infty$ is a proper closed A- submodule, which is localizable under A in $LV_\infty$, let $\mathfrak{X}$ be the set of all $X \in \mathcal{R}$ such that there exists $0 \neq \phi \in (LV_\infty|X)'$ such that $W \subset W(X,\phi)$. By Theorem 14, $\mathfrak{X} \neq \emptyset$ and $W = \bigcap_{X\in\mathfrak{X}} W(X,\phi)$.

Conversely, suppose $\mathfrak{X} \neq \emptyset$, is a subset of $\mathcal{R}$ such that for all $X \in \mathfrak{X}$, there exists $0 \neq \phi \in (LV_\infty|X)'$. The $W = \bigcap_{X\in\mathfrak{X}} W(X,\phi)$ is a localizable under A in $LV_\infty$. To prove this consider $f \in L$ such that $f|X$ belongs to the closure of $W|X$ in $LV_\infty|X$, for every $X \in \mathcal{R}$. Suppose $f \notin \overline{W}$. Since W is closed, $f \notin W$ and therefore $f \notin W(X,\phi)$ for some $X \in \mathfrak{X}$, i.e. $\phi(f|X) \neq 0$. However $W \subset W(X,\phi)$, hence $\phi(f|X) = 0$, a contradiction. Hence $f \in \overline{W}$, and W is localizable under A in $LV_\infty$.

We have thus proved the following:

<u>THEOREM 15</u>. <u>A proper closed A- submodule</u> $W \subset LV_\infty$ <u>is localizable under A in</u> $LV_\infty$ <u>if, and only if, it is of the form</u> $W = \bigcap_{X\in\mathfrak{X}} W(X,\phi)$, <u>for some</u> $\mathfrak{X} \neq \emptyset$, <u>such that</u> $\mathfrak{X} \subset \mathcal{R}$ <u>and for all</u> $X \in \mathfrak{X}$ <u>there is some</u> $0 \neq \phi \in (LV_\infty|X)'$.

<u>Corollary</u>. <u>A closed A- submodule of codimension one in</u> $LV_\infty$ <u>is localizable under A in</u> $LV_\infty$ <u>if, and only if, it is of the form</u> $W(X,\phi)$ <u>for some</u> $X \in \mathcal{R}$ <u>such that there exists</u> $0 \neq \phi \in (LV_\infty|X)'$.

<u>Remark 9</u>. In the particular case in which $LV_\infty$ is $C_b(E;F)$ equipped with the strict topology $\beta$ and $A \subset C_b(E;\underset{\sim}{K})$ is a self-adjoint subalgebra containing the constants, we are in the Bounded Case of the strict weighted approximation problem, and by Theorem 10 any A- submodule of $C_b(E;F)$ is localizable under A in $C_b(E;F)$. Therefore, Theorem 14 above generalizes Theorem 5 of Todd ref. 14 and Theorem 15 and its Corollary generalize Theorem 4 of Todd ref. 14.

## §9. OPERATOR ALGEBRAS

In what follows $\mathcal{L}$ denotes a real (resp. complex) locally convex Hausdorff topological vector space and $\mathcal{A}$ denotes a commutative algebra of linear operators over $\mathcal{L}$, not necessarily continuous. We further assume that $\mathcal{A}$ contains the identity operator.

Definition 6. The point co-spectrum of $\mathcal{A}$ is the set of all homomorphisms $h$ of $\mathcal{A}$ onto $\mathbb{R}$ (resp. $\mathbb{C}$) such that there exists $f \in L'$, $f \neq 0$, such that $f(u(x)) = h(u)f(x)$ for all $u \in \mathcal{A}$ and $x \in \mathcal{L}$.

We shall denotes by E the point co-spectrum of $\mathcal{A}$.

Notice that E is also the set of all homomorphisms h of $\mathcal{A}$ onto $\mathbb{R}$ (resp. $\mathbb{C}$) such that there exists an $f \in L'$, $f \neq 0$, such that $f(u(x)) = 0$ for any u in the kernel $h^{-1}(0)$ and x in $\mathcal{L}$.

Or, equivalently, by the Hahn-Banach Theorem, E is the set of all homomorphisms h of $\mathcal{A}$ onto $\mathbb{R}$ (resp. $\mathbb{C}$) such that the vector subspace $S_h$ of $\mathcal{L}$ spanned by $\{u(x);\ u \in h^{-1}(0),\ x \in L\}$ is not dense in $\mathcal{L}$.

We shall consider over E the Gelfand topology, i.e. the weakest topology over E such that all the functions $h \mapsto h(u)$ from E into $\mathbb{R}$ (resp. $\mathbb{C}$) are continuous, when u ranges over $\mathcal{A}$. Clearly E is then a Hausdorff space.

For each $h \in E$, let $F/S_h$. The vector fibration $(E,(F_h)_{h\in E})$ is called the vector fibration associated with the point co-spectrum of $\mathcal{A}$. For each $h \in E$, let $x \mapsto x_h$ denote the quotient map of $\mathcal{L}$ onto $F_h$. Then, for each $x \in \mathcal{L}$, the family $(x_h)_{h\in E}$ is a cross-section over E, which we shall denote by $\hat{x}$. The mapping $x \mapsto \hat{x}$ from $\mathcal{L}$ into $\prod_{h\in E} F_h$ is linear; hence the image of $\mathcal{L}$ under this mapping is a vector space of cross-sections, which we shall denote by L.

For each $u \in \mathcal{A}$, let $\tilde{u}$ denote the continuous mapping $h \mapsto h(u)$. The mapping $u \mapsto \tilde{u}$ from $\mathcal{A}$ into $C(E;\underset{\sim}{K})$ is a homomorphism; hence the image of $\mathcal{A}$ under it is a subalgebra of $C(E;\underset{\sim}{K})$, which we shall denote by A. Notice that A is separating over E. We claim that

$$u(x)^{\wedge} = \tilde{u} \cdot \hat{x}$$

for all $u \in \mathcal{A}$ and $x \in \mathcal{L}$. Indeed, for all $h \in E$, $u(x) - h(u)x = (u-h(u)I)(x)$. Notice that $u - h(u)I$ belongs to $h^{-1}(0)$. Hence $u(x) - h(u)x$ belongs to $S_h$ for all $h \in E$, and $(u(x))_h = (h(u)x)_h = h(u)x_h$ for all $h \in E$, i.e. $u(x)^{\wedge} = \tilde{u} \cdot \hat{x}$, as we wanted to prove.

From this it follows that L is an A- module and that the operators $u \in \mathcal{A}$ appear as multiplication by $\tilde{u} \in A$.

For each continuous seminorm p over $\mathcal{L}$, we shall denote by $p_h$ the quotient seminorm $p_h(x_h) = \inf \{p(y); y \in x_h\}$ for all $x_h \in F_h$. The mapping $h \mapsto p_h$ is then a weight over E, which we shall denote by $\hat{p}$. Notice that every weight $\hat{p}$ is L- bounded, for

$$\hat{p}[\hat{x}](h) = p_h(x_h) \leq p(x)$$

for all $h \in E$, i.e. $\hat{p}[\hat{x}]$ is bounded on E, for every $\hat{x} \in L$, and therefore $\hat{p}$ is L- bounded. From now on we shall make the following

<u>Hypothesis H</u>. There exists a set $\Gamma$ of continuous seminorms over $\mathcal{L}$, which determines the topology of $\mathcal{L}$, such that for every seminorm $p \in \Gamma$, the function $h \mapsto p_h(x_h)$ is upper semicontinuous and null at infinity on E, for every $x \in \mathcal{L}$.

Under the above hypothesis, we may consider the space $LV_\infty$, where $V = \{\hat{p}; p \in \Gamma\}$. The linear map $x \mapsto \hat{x}$ from $\mathcal{L}$ onto $LV_\infty$ is then continuous. In fact given a seminorm $\|\hat{x}\|_{\hat{p}} = \sup \{p_h(x_h) ; h \in E\}$ in $LV_\infty$, we have

$\|\hat{x}\|_{\hat{p}} \leq p(x)$ for all $x \in \mathcal{L}$. The following representation theorem establishes the condition under which $x \mapsto \hat{x}$ is a topological vector isomorphism.

THEOREM 16. A necessary and sufficient condition for the existence of a set $\Gamma$ of seminorms over $\mathcal{L}$, which determines the topology of $\mathcal{L}$, such that $x \mapsto \hat{x}$ is a topological vector isomorphism between $\mathcal{L}$ and $LV_\infty$, where $V = \{\hat{p};\ p \in \Gamma\}$ and $u \mapsto \tilde{u}$ is an isomorphism between $\mathcal{A}$ and A, is that $\mathcal{L}$ be locally convex under $\mathcal{A}$ with respect to the category of all algebras isomorphic to $\mathbb{R}$ (resp. $\mathbb{C}$).

Proof. Suppose that L is locally convex under A with respect to the category $\{\mathbb{R}\}$ of all algebras isomorphic to $\mathbb{R}$ (resp. $\mathbb{C}$). (For definitions, see Nachbin ref. 5). Let $\Gamma$ be the set of all continuous seminorms p over $\mathcal{L}$ which are convex under $\mathcal{A}$ with respect to $\{\mathbb{R}\}$ (resp. $\{\mathbb{C}\}$). For any such seminorm p, we have $p = \sup\{p_{S_h};\ h \in E\}$, (see Nachbin ref. 5) where $p_{S_h}(x) = \inf\{p(x-s);\ s \in S_h\}$. Hence $p_{S_h}(x) = p_h(x_h)$ for all $h \in E$ and $x \in \mathcal{L}$. Therefore $p(x) = \sup\{p_h(x_h);\ h \in E\} = \|\hat{x}\|_{\hat{p}}$ for all $x \in \mathcal{L}$, and $x \mapsto \hat{x}$ is a topological vector isomorphism of $\mathcal{L}$ onto $LV_\infty$, where $V = \{\hat{p};\ p \in \Gamma\}$. To prove $u \mapsto \tilde{u}$ is an isomorphism, just notice that $\tilde{u} = 0$ implies that $u(x)\hat{} = \tilde{u} \cdot \hat{x} = 0$ for all $x \in \mathcal{L}$. Since $x \mapsto \hat{x}$ is an isomorphism it follows that $u(x) = 0$ for all $x \in L$; hence $u = 0$.

Conversely, suppose that there exists a set $\Gamma$ of continuous seminorms over $\mathcal{L}$ such that $x \mapsto \hat{x}$ is a topological vector isomorphism between $\mathcal{L}$ and $LV_\infty$, where $V = \{\hat{p};\ p \in \Gamma\}$. Then the seminorms over $\mathcal{L}$ of the form $\sup\{p_{S_h};h \in E\}$ determine the topology of $\mathcal{L}$. But these are convex under $\mathcal{A}$ with respect to $\{\mathbb{R}\}$ (resp. $\{\mathbb{C}\}$). Hence $\mathcal{L}$ is locally convex under $\mathcal{A}$ with respect to $\{\mathbb{R}\}$ (resp. $\{\mathbb{C}\}$).

## §10. WEIGHTED APPROXIMATION PROBLEM FOR INVARIANT SUBSPACES

Let $\mathcal{L}$ be a locally convex Hausdorff space and $\mathcal{A}$ an algebra of linear operators over $\mathcal{L}$, satisfying Hypothesis H of §9 and such that $\mathcal{L}$ is locally convex under $\mathcal{A}$ with respect to $\{R\}$ (resp. $\{C\}$). If we denote by $\Gamma$ the set of all continuous seminorms over L which are convex under A with respect to $\{R\}$ (resp. $\{C\}$), and $V = \{\hat{p};\ p \in \Gamma\}$, then we can construct $LV_\infty$ as in §9 and by Theorem 16, $\mathcal{L}$ and $LV_\infty$ are topologically and linearly isomorphic under the mapping $x \mapsto \hat{x}$.

If $\mathcal{W} \subset \mathcal{L}$ is a vector subspace of $\mathcal{L}$ invariant under $\mathcal{A}$, then its image W under $x \mapsto \hat{x}$ is an A- submodule of L. The <u>weighted approximation problem</u> consists, then, in finding necessary and sufficient conditions for a given $\mathcal{A}$-invariant subspace to be dense in $\mathcal{L}$, and further in studying when spectral synthesis holds, i.e. when a proper closed $\mathcal{A}$- invariant subspace equals the intersection of the proper closed $\mathcal{A}$- invariant subspace of codimension one containing it.

An $\mathcal{A}$- invariant subspace $\mathcal{W}\subset\mathcal{L}$ is, by definition, said to be <u>localizable under</u> $\mathcal{A}$ in $\mathcal{L}$, if its image W under $x \mapsto \hat{x}$ is localizable under A in $LV_\infty$. It follows that an $\mathcal{A}$- invariant subspace $\mathcal{W}\subset\mathcal{L}$ is localizable under $\mathcal{A}$ in $\mathcal{L}$ if, and only if, its closure in $\mathcal{L}$ consists of all those vectors $x \in \mathcal{L}$ such that, given $\varepsilon > 0$, $p \in \Gamma$ and $h \in E$ arbitrarily, there exists $w \in \mathcal{W}$ and $s \in S_h$ such that $p(x-w-s) < \varepsilon$.

The strict weighted approximation consists, then, in asking for necessary and sufficient conditions for localizability of $\mathcal{A}$- invariant subspaces of $\mathcal{L}$. Using the above definition of localizability and the representation theorem of §9 we can easily establish results similar to those of §6 and §8. As an example we prove a result about spectral synthesis.

THEOREM 17. Let $\mathcal{W} \subset \mathcal{L}$ be a proper closed $\mathcal{A}$- invariant subspace. If $\mathcal{W}$ is localizable under $\mathcal{A}$ in $\mathcal{L}$, then $\mathcal{W}$ is the intersection of all proper closed $\mathcal{A}$- invariant subspaces of codimension one containing it.

Proof. The image W of $\mathcal{W}$ under the mapping $x \to \hat{x}$ is proper and closed in $LV_\infty$. Since $\mathcal{W}$ is localizable under $\mathcal{A}$ in $\mathcal{L}$, the A- submodule W is localizable under A in $LV_\infty$. From Theorem 14 it follows that W is the intersection of all proper closed A- submodules of codimension one which contain it. Since $x \mapsto \hat{x}$ is a topological and linear isomorphism, $\mathcal{W}$ is the intersection of all the $\mathcal{A}$- invariant closed subspaces of codimension one which contain it.

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